Dolbeault complex of non-commutative projective varieties.

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Motivation

- To understand a symmetry of $H^{k,l} = H^l(X, \Omega^k)$

$$\tilde{H}^{k,l} \cong H^{l,k}$$

over fields of positive characteristics.

- Deligne Illusie theory: $\bar{\partial}$ “resolution” of $\Omega^k$ is quasi isomorphic to the Frobenius “pullback” (somehow) of $\Omega^{k,l}$.

- Cartier operators are in action.

- To obtain a lot of examples of non commutative objects.
Weyl algebras, Clifford algebras

\( \mathbb{k} \): commutative field, \( \text{char } \mathbb{k} = p \gg 0 \), \( \text{char } \mathbb{k} \neq 0 \).

\( h, k, C \): variables which commute with other variables ...
Weyl algebras, Clifford algebras

$k$: comutative field, char $k = p \gg 0$, char $k \neq 0$.
$h, k, C$: variables which commute with other variables

Weyl algebra:

\[ \text{Weyl}^{(h, C)}_{n+1} = k[h, C, X_0, X_1, \ldots, X_n, \bar{X}_0, \bar{X}_1, \ldots, \bar{X}_n] \]

relation (CCR): \[ [\bar{X}_i, X_j] = hC \delta_{ij}. \]

Clifford algebra

\[ \text{Cliff}^{(h, C, k)}_{n+1} = k[h, C, k, E_0, \ldots, E_n, \bar{E}_0, \ldots, \bar{E}_n] \]

relation (CAR): \[ [\bar{E}_i, E_j]_+ = Chk \delta_{ij}. \]
Weyl-Clifford algebras

$$\text{WC}_{n+1}^{(h, C, k)} = \text{Weyl}_{n+1}^{(h, C)} \otimes_{\mathbb{k}[h, C]} \text{Cliff}_{n+1}^{(h, C, k)}$$

$$= \mathbb{k}[h, C, k, X_0, \ldots, X_n, \bar{X}_0, \ldots, \bar{X}_n, E_0, \ldots, E_n, \bar{E}_0, \ldots, \bar{E}_n]$$

Existence of odd derivations $\partial, \bar{\partial}$: ...
Weyl-Clifford algebras

$$WC_{n+1}^{(h,C,k)} = \text{Weyl}^{(h,C)}_{n+1} \otimes _{\mathbb{K}[h,C]} \text{Cliff}^{(h,C,k)}_{n+1}$$

$$= \mathbb{K}[h, C, k, X_0, \ldots, X_n, \bar{X}_0, \ldots, \bar{X}_n, E_0, \ldots, E_n, \bar{E}_0, \ldots, \bar{E}_n]$$

Existence of odd derivations $\partial, \bar{\partial}$:

$$\partial : \begin{cases} 
  X_i \mapsto E_i \\
  \bar{X}_i \mapsto 0 \\
  E_i \mapsto 0 \\
  \bar{E}_i \mapsto k\bar{X}_i.
\end{cases}$$

$$\bar{\partial} : \begin{cases} 
  X_i \mapsto 0 \\
  \bar{X}_i \mapsto \bar{E}_i \\
  E_i \mapsto -kX_i \\
  \bar{E}_i \mapsto 0.
\end{cases}$$

$$E_i = \bar{\partial}X_i, \quad \bar{E}_i = \bar{\partial}\bar{X}_i.$$
things to note:

\[ WC_{n+1} \cong WC_1 \otimes WC_1 \otimes \cdots \otimes WC_1 \]
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- Logically by definition \( X \) and \( \bar{X} \) are independent variables.
Presence of $k$

\[ [\bar{E}_i, E_i]_+ = Chk \]

\[ [\bar{\partial}, \partial]_+ f = -k \text{sdeg}_\mu(f)f. \]

\text{sdeg}_\mu:

\[ X \mapsto 1, \quad E \mapsto 1, \quad \bar{X} \mapsto -1, \quad \bar{E} \mapsto -1. \]

...
Presence of $k$

$$[\tilde{E}_i, E_i]_+ = Chk$$

$$[\bar{\partial}, \partial]_+ f = -k \text{sdeg}_\mu(f)f.$$

\[\text{sdeg}_\mu : \]

\[X \mapsto 1, \quad E \mapsto 1, \quad \bar{X} \mapsto -1, \quad \bar{E} \mapsto -1.\]

...For a plain $\mathbb{A}^{2n}$, $k$ is not such a very good boy.
Before doing anything else, please keep in mind that we will use “super” notations. We define a signature of elements of WC: 

\( X_i, \bar{X}_i \): even.
\( E_i, \bar{E}_i \): odd.

The symbol \([a, b]\) will be used to mean the super commutator instead of usual commutator.

\[
[a, b] = ab - (-1)^{\hat{a} \cdot \hat{b}} ba
\]

\( \hat{a}, \hat{b} \): signature of \( a, b \).
$WC_1$ (revisited)

\[ WC_1 = \mathbb{K}[h, k, C, X, \tilde{X}, E, \tilde{E}] \]

\[ [\tilde{X}, X] = \tilde{X}X - X\tilde{X} = Ch \]

\[ [\tilde{E}, E] = \tilde{E}E + E\tilde{E} = Chk \]

\[ E^2 = 0, \quad \tilde{E}^2 = 0 \]

“$X$-variables” $(X, \tilde{X})$ and “$E$-variables” $(E, \tilde{E})$ commute:

\[ [X, E] = 0, \quad [X, \tilde{E}] = 0, \quad [\tilde{X}, E] = 0, \quad [\tilde{X}, \tilde{E}] = 0. \]
Let us denote \( d = \partial + \bar{\partial} \): \( E = dX, \bar{E} = d\bar{X} \).

\[
WC_1 = \mathbb{K}[h, k, C, X, \bar{X}, dX, d\bar{X}]
\]

\[
[\bar{X}, X] = \bar{X}X - X\bar{X} = Ch
\]

\[
[\bar{d}X, dX] = Chk
\]

\[
(dX)^2 = 0, (d\bar{X})^2 = 0.
\]

“\( X \)-variables” \( (X, \bar{X}) \) and “\( d \bullet \)-variables” \( (dX, \bar{d}X) \) commute. \( \partial, \bar{\partial} \) are computed in the same way as usual except:

\[
\partial(d\bar{X}) = -kX, \quad \bar{\partial}(dX) = kX.
\]
The Weyl algebra is a simple algebra when the base field \( \mathbb{k} \) is of characteristic zero.

When \( \text{char}(\mathbb{k}) \neq 0 \) (as we always assume in this talk,) the Weyl algebra has a fairly large center.

\( \text{Weyl}^{(h, C)}_{n+1} \) corresponds to a coherent sheaf of algebras on \( \mathbb{A}^{n+1}_k[h, C] \).

We may obtain results over fields of characteristic 0 by using “ultra filters” on \( \text{Spm}(\mathbb{Z}) \).
To do:

1. Construct a sheaf $\mathcal{A}$ of super algebras on $\mathbb{P}^n \times \mathbb{P}^n$.
2. See that $\mathcal{A}$ is a double complex with respect to $\partial, \bar{\partial}$.
3. $(\mathcal{A}, \bar{\partial})$ is quasi isomorphic to another sheaf of algebras on $\mathbb{P}^n \times \mathbb{P}^n$.
5. Mimic Deligne-Illusie theory.
6. Comparison to the commutative theory by taking the limit $h \to 0$.
7. Watch $\partial \leftrightarrow \bar{\partial}$ symmetry.
$A^{\text{pre}}$ (constraint with $\mu_R = 0$)

$WC = \mathbb{K}[h, C, k, X_0, \ldots, X_n, \bar{X}_0, \ldots, \bar{X}_n, E_0, \ldots, E_n, \bar{E}_0, \ldots, \bar{E}_n]$

$$\mu_R = \sum_i (X_i \bar{X}_i k + E_i \bar{E}_i) - RkC$$

$$[\mu_R, f] = \text{sdeg}_\mu(f)f.$$ 

$(WC)_0 \overset{\text{def}}{=} \{ x \in WC; \text{sdeg}_\mu(x) = 0 \}$
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$$(WC)_0 \overset{\text{def}}{=} \{x \in WC; \text{sdeg}_\mu(x) = 0\}$$

$$A^{\text{pre}} = (WC)_0/\langle \mu_R \rangle$$

Marsden-Weinstein quotient.
Throw away torsions

\[ A = \text{Image}(A^{\text{pre}} \to A^{\text{pre}}[\frac{1}{k}]). \]

\[ \mu_R = \sum_i (X_i \bar{X}_i k + E_i \bar{E}_i) - RkC = 0 \quad \text{in } A \]

\[ \implies m := - \sum_i X_i \bar{X}_i = \frac{1}{k} \sum_i E_i \bar{E}_i \quad \text{in } A \]

\[ \implies m(m - Ch)(m - 2Ch) \cdots (m - (n + 1)Ch) = 0 \quad \text{in } A. \]

(Note that \((E_i \bar{E}_i)^2 = khE_i \bar{E}_i\) holds.)

(Secretly changed the sign of \(m\) compared to my november talk at MSJ.) (Oct.29: Secretly corrected the equation. We forgot to put some \(C\)'s here.)
Dolbeault complex

1. We define the sheaf of super algebras $\mathcal{A}$ on $\mathbb{P}^n \times \mathbb{P}^n$ as the sheaf corresponding to $A$.

2. $\mathcal{A}$ is a double complex with respect to $\partial$, $\bar{\partial}$. (In particular,

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3. We need to find a sheaf quasi isomorphic to $(\mathcal{A}, \bar{\partial})$. 
Projective coordinate ring where $X_0 \neq 0$

$$A^\lozenge = A_{\{x_0 \neq 0\}}$$

$$= \mathbb{k}[h, k, C, x_1, \ldots, x_n, x'_1, \ldots, x'_n, e_0, \ldots, e_n, e'_0, \ldots, e'_n, m]$$

$$x_i = X_i X_0^{-1}, x'_i = X_0 \bar{X}_i, e_i = E_i X_0^{-1}, e'_i = X_0 \bar{E}_i$$

$$m = \frac{1}{k} \left( \sum_{i=0}^{n} e_i e'_i \right).$$

$$x_0 = 1, x'_0 = -\sum_{i=1}^{n} x_i x'_i - m$$
Ring structure of the projective coordinate ring where $X_0 \neq 0$

$A^\heartsuit$

$= \mathbb{k}[h, k, C, x_1, \ldots, x_n, x'_1, \ldots, x'_n, e_0, \ldots, e_n, e'_0, \ldots, e'_n, m]$

$[x'_i x_j] = h C \delta_{ij}$
$[x_i, x_j] = 0, [x'_i, x'_j] = 0$
$[e'_i, e_i]( = [e'_i, e_i]_+) = Chk \delta_{ij}$
$e_i^2 = 0, (e'_i)^2 = 0$
$m = \frac{1}{k} \sum_{i=0}^{n} e_i e'_i \quad (\text{in } A^\heartsuit[\frac{1}{k}])$
In short, $A^\heartsuit$ is an algebra by adjoining $e_0, e'_0, m$ to the Weyl-Clifford algebra $WC_n$

Essentially (probably up to “Morita equivalence”), we come back to our original $WC_n$.

Note our covering $\bigcup_j \{X_j \neq 0\}$ of (non-commutative) $\mathbb{P}^n \times \mathbb{P}^n$ is only good for $\overline{\partial}$-action and is no good for $\partial$-action.
freeness of $A^\heartsuit$

- To concentrate on $x, x', e, e'$-variables, we denote
  $$\mathbb{k}_3 = \mathbb{k}[h, C, k]$$

- $$A^\heartsuit \cong \mathbb{k}_3[x, x'] \otimes_{\mathbb{k}_3} \mathbb{k}_3[e, e', m]$$

- $\mathbb{k}_3[e, e', m]$ is a free finite module over $\mathbb{k}_3$

It follows that $A^\heartsuit$ corresponds to a finite free $\mathcal{O}$ module over $\mathbb{A}^n \times \mathbb{P}^n$
Freeness of $M = \mathbb{k}_3[e, e', m]$ (normal ordering)

(This slide is for the completeness sake only.)

- By using suitable commutation relations,

\[
M = \sum \mathbb{k}_3 e' m^{[l]} (e')^J
= \sum \mathbb{k}_3 e' \frac{l!}{k!} \sum_{|K|=l} e^K (e')^K (e')^J
\]

The last module is isomorphic to a submodule $M_1$ of the exterior algebra

\[
\mathbb{k}_3 \left[ \frac{1}{k} \right] (\wedge (\oplus_{i=0}^n Ke_i)) \otimes (\wedge (\oplus_{i=0}^n Ke'_i))
\]

$M_1$ is of the form $\mathbb{k} [h, k, C] \otimes_{\mathbb{k}[k]} M_0$ for some torsion free $\mathbb{k}[k]$-module $M_0$. By using a general theory of modules over PID, we see that $M_0$ is free. We may thus see that $M$ is free.
Local quasi isomorphism

Theorem

$A \bigodot$ is quasi isomorphic to the following graded super subalgebra as a graded $\bar{\partial}$-complex.

\[
\mathbb{k}[h, k, C, x_1, \ldots, x_n, \beta_1, \ldots, \beta_n, \\
(x'_1)^p, \ldots, (x'_n)^p, (x'_1)^{p-1}e'_1, \ldots, (x'_n)^{p-1}e'_n, \\
\epsilon - RCe_0]
\]

where

\[
\beta_i = e_i - x_i e_0 \quad (i = 1, 2, \ldots, n)
\]

\[
\epsilon = \sum_{i=0}^{n} x'_i e_i,
\]
One can think of \( \mathbb{k}[h, k, C, x_1, \ldots, x_n, \beta_1, \ldots, \beta_n, (x'_1)^p, \ldots, (x'_n)^p, (x'_1)^{p-1}e'_1, \ldots, (x'_n)^{p-1}e'_n, \epsilon - RCe_0] \) as the ring of differentiable forms of (an affine piece of) \( \mathbb{P}^n \).
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$$\mathbb{k}[h, k, C, x_1, \ldots, x_n, \beta_1, \ldots, \beta_n, (x'_1)^p, \ldots, (x'_n)^p, (x'_1)^{p-1}e'_1, \ldots, (x'_n)^{p-1}e'_n, \epsilon - RCe_0]$$

as the ring of differentiable forms of (an affine piece of ) $\mathbb{P}^n$ twisted by Frob. Let us denote it by $\Omega_{\text{sparse}, \mathbb{P}^n}$. 
Conclusion:

There exists a $\mathcal{O}_{\mathbb{P}^n}$-algebra $\mathcal{B}$ on $\mathbb{P}^n$ such that

1. $\mathcal{B}$ is an $\Omega_{\mathbb{P}^n}$-algebra.
2. $\mathcal{B}$ is free of rank two as an $\Omega_{\mathbb{P}^n}$-module.
3. $(\mathcal{A}, \bar{\partial})$ is quasi-isomorphic to $(\mathcal{B} \boxtimes \Omega_{\text{sparse, } \mathbb{P}^n}, 0)$.
4. 

$$R\pi_{2*}\mathcal{A} \simeq \mathcal{B} \boxtimes \bigoplus_{j} R\Gamma(\mathbb{P}^n, \Omega^j)$$
what about varieties:

$V \subset \mathbb{P}^n$: algebraic variety

\[ \implies \text{One can consider } A/(I^p_V + \bar{I}^p_V). \]

This suggests some type of symmetry in cohomologies.