TOPICS IN NON COMMUTATIVE ALGEBRAIC GEOMETRY AND CONGRUENT ZETA FUNCTIONS (PART III). SUPPLEMENTARY RESULTS ON COMMUTATIVE ALGEBRAIC GEOMETRY.

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1. quasi coherent sheaves

Definition 1.1. An $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is quasi coherent if there exists an affine open covering

$$\{U_\lambda = \mathrm{Spec}(A_\lambda)\}_{\lambda\in\Lambda}$$

of X such that for each $\lambda \in \Lambda$, $\mathcal{F}|_{X_\lambda}$ is isomorphic to a $\mathcal{O}_{\mathrm{Spec}(A_\lambda)} \otimes_{A_\lambda} M$ for some A-module M.

It is easy to see that

Lemma 1.2. *Let $f : X \to Y$ be a morphism of schemes. For any quasi coherent sheaf $\mathcal{G}$ on Y, $f^*(\mathcal{G})$ is quasi coherent.*

2. K-valued points and fibers

Definition 2.1. Let K be a field. a **K-valued point** P of a scheme X is a morphism

$$\iota_P : \mathrm{Spec}(K) \to X$$

of schemes. Let $\mathcal{F}$ be a quasi coherent $\mathcal{O}_X$-module. Then a **fiber** of $\mathcal{F}$ on a K-valued point P is the pullback $\iota_P^*(\mathcal{F})$. We often identify it with a K-vector space

$$\iota_P^*(\mathcal{F})(K).$$

Example 2.2. Let $X = \mathrm{Spec}(A)$ be an affine scheme. Then a K-valued point P of X is given by a ring homomorphism

$$\mathrm{eval}_P : A \to K.$$

a quasi coherent $\mathcal{O}_X$-module $\mathcal{F}$ is given by an A-module M.

$$\mathcal{F} \cong \mathcal{O}_X \otimes_A M$$

The fiber of $\mathcal{F}$ is then given by

$$\iota_P^*(\mathcal{O}_X \otimes_A M) = K \otimes_A M.$$

In general, we study quasi coherent sheaves on a scheme from three different point of view. Namely, we may study them

(1) section wise,
(2) stalk wise, or
(3) fiber wise.

Each view point is useful.

3. locally free sheaves of finite rank

Definition 3.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ on X is said to be

(1) **free** if it is isomorphic to a direct sum of $\mathcal{O}_X$.
(2) **locally free** if there exists an open covering $\{U_\lambda\}$ of X such that $\mathcal{F}|_{U_\lambda}$ is free for all λ.

Definition 3.2. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a locally free sheaf of rank r on X. By definition, there exists an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of X such that $\mathcal{F}|_{U_\lambda}$ is free for all $\lambda \in Lambda$. In other words, there is an isomorphism

$$\phi_\lambda : \mathcal{F} \cong \mathcal{O}_X^r.$$

Such ϕ_λ is called a **local trivialization** of $\mathcal{F}$.

Given a set of trivializations $\{\phi_\lambda\}_{\lambda \in \Lambda}$ of $\mathcal{F}$, We notice that for any $\lambda, \mu \in \Lambda$ there exists a GL_r-valued function

$$g : U_{\lambda\mu} \to \mathrm{GL}_r$$

such that for any section $s \in \mathcal{F}(U_{\lambda\mu})$, we have

$$\phi_\lambda(s) = g_{\lambda\mu}\phi_\mu(s)$$

We call $\{g_{\lambda\mu}\}$ the **transition functions**.

Lemma 3.3. *The transition functions as in Definition above satisfy the following* **cocycle conditions**.

(1) $g_{\lambda\lambda} = \mathrm{id}$.
(2)
$$g_{\lambda\mu} g_{\mu\nu} = g_{\lambda\nu}.$$

4. Ultra filter

Definition 4.1. A set $\mathfrak{F}$ of subsets of a set X is called a **filter** on X if the following conditions are satisfied.

(1) $\mathfrak{F} \ni A, B \quad \Longrightarrow \quad A \cap B \in \mathfrak{F}$.
(2) $\mathfrak{F} \ni A, \quad A \subset A_1 \subset X \quad \Longrightarrow \quad A_1 \subset \mathfrak{F}$.

Definition 4.2. A maximal filter on a set X is called an **ultra filter** on X.

Those readers who are not familiar with the arguments are invited to read for example [15] or the book of Bourbaki [2].

Lemma 4.3. *Let $\mathfrak{U}$ be a filter on a set X The following statements are equivalent.*

(1) *$\mathfrak{U}$ is an ultrafilter. That means, a maximal filter.*
(2) *for any subset $S \subset X$, we have either $S \in \mathfrak{U}$ or $\complement S \in \mathfrak{U}$*

Definition 4.4. A principal filter on a set X is an ultra filter of the form $\mathfrak{F}_a = \{S \subset X | a \in S\}$ where a is an element of X. A ultrafilter which is not principal filter is called non-principal.

Lemma 4.5. *For any ultrafilter $\mathfrak{U}$, the following statements are equivalent.*

(1) *$\mathfrak{U}$ is principal.*
(2) *$\mathfrak{U}$ is not free. That means, $\cap_{U \in \mathfrak{U}} U \neq \emptyset$.*
(3) *There exists a member E of $\mathfrak{U}$ which is a finite set ($\#E < \infty$).*
(4) *There exists a co-finite subset Y of X (that means, $\#(X \setminus Y) < \infty$,) such that $Y \notin \mathfrak{U}$.*

In particular, if $\mathfrak{U}$ is a non-principal ultrafilter on a set X, then any co-finite subset Y of X of is a member of $\mathfrak{U}$.

An ultrafilter $\mathfrak{U}$ on a set X may be identified with a point of Stone-Čech compactification of (X with discrete topology). A non principal ultrafilter is identified with a boundary point.

Definition 4.6. Let $\mathfrak{K}$ be a number field with the ring of integers $\mathfrak{O}$. Let $\mathfrak{U} = \{U_\lambda\}$ be a non-principal ultrafilter on the set $\operatorname{Spm}(\mathfrak{O})$ of all primes of $\mathfrak{O}$ of height 1.

Let $\mathfrak{I}_\mathfrak{U}$ be an ideal of $\prod_{\mathfrak{p} \in \operatorname{Spm}(\mathfrak{O})} \mathfrak{O}/\mathfrak{p}$ defined as follows:

$$\mathfrak{I}_\mathfrak{U} = \left\{ (f_\mathfrak{p})_{\mathfrak{p} \in \operatorname{Spm}(\mathfrak{O})} \in \prod_{\mathfrak{p} \in \operatorname{Spm}(\mathfrak{O})} (\mathfrak{O}/\mathfrak{p}) \middle| \quad \exists U \in \mathfrak{U} \text{ such that } f_\mathfrak{p} = 0 \quad \text{for } \forall \mathfrak{p} \in U \right\}$$

Then we define a ring $\mathfrak{K}_\mathfrak{U}$ as follows:

$$\mathfrak{K}_\mathfrak{U} = \left(\prod_{\mathfrak{p} \in \operatorname{Spm}(\mathfrak{O})} (\mathfrak{O}/\mathfrak{p}) \right) / \mathfrak{I}_\mathfrak{U}$$

We denote by $\pi_\mathfrak{U}$ the canonical projection from $\prod (\mathfrak{O}/\mathfrak{p})$ to $\mathfrak{K}_\mathfrak{U}$.

Lemma 4.7. *$\mathfrak{K}_\mathfrak{U}$ is a field of characteristic 0.*

Proof. Indeed, let $f = \pi_{\mathfrak{U}}((f_{\mathfrak{p}}))$ be a non zero element in $\mathfrak{K}_{\mathfrak{U}}$. Let $E_1 = \{\mathfrak{p} \in \operatorname{Spm}(\mathfrak{O}); f_{\mathfrak{p}} \neq 0\}$. Then for any $E \in \mathfrak{U}$, intersection $E \cap E_1$ is non empty. Maximality of $\mathfrak{U}$ now implies that E_1 itself is a member of $\mathfrak{U}$. The inverse $g = (g_{\mathfrak{p}})$ of f in $\mathfrak{K}_{\mathfrak{U}}$ is given by the following formula.

$$g_{\mathfrak{p}} = \begin{cases} f^{-1} & \text{if } \mathfrak{p} \in E_1 \\ 0 & \text{otherwise} \end{cases}$$

If $n = 0$ in $\mathfrak{K}_{\mathfrak{U}}$ for a positive integer n, then there exists $E_0 \in \mathfrak{U}$ such that $n \in \cap_{\mathfrak{p} \in E_0} \mathfrak{p}$. On the other hand, as we have mentioned in Lemma 4.5 above, being a member of a non-principal filter $\mathfrak{U}$, E_0 cannot be a finite set. This is a contradiction, since non-zero member n in $\mathfrak{O}$ has only finite "zeros" on the "arithmetic curve" $\operatorname{Spm}(\mathfrak{O})$. Thus the characteristic of $\mathfrak{K}_{\mathfrak{U}}$ is zero.

□

The definition above is partly inspired by works of Kirchberg (See [12] for example.) We would like to give a little explanation on $\pi_{\mathfrak{U}}$. We regard it as a kind of 'limit'. If we are given a member U of $\mathfrak{U}$ and we have an element, say, $h_{\mathfrak{p}}$ of $\mathfrak{O}/\mathfrak{p}$ for each primes $\mathfrak{p} \in U$, then, by assigning arbitrary element to 'exceptional' primes (that means, primes which are not in U), we may interpolate h and consider

$$\pi_{\mathfrak{U}}((h_{\mathfrak{p}})).$$

The element ('limit') does not actually depend on the interpolation. Thus we may refer to the element without specifying the interpolation. In particular, this applies to the case where we have $h_{\mathfrak{p}}$ for almost all primes $\mathfrak{p}$. The same type of argument applies for polynomials. We summarize this in the following Lemma.

Lemma 4.8. *Suppose we have a co-finite subset Y of $\operatorname{Spm}(\mathfrak{O})$ and a collection $\{F_{\mathfrak{p}}\}_{\mathfrak{p} \in Y} \in (\mathfrak{O}/\mathfrak{p})[T_1, T_2, \dots, T_n, U_1, U_2, \dots, U_n]$ of polynomials. Assume we have a bound d for the degrees of the polynomials. That means,*

$$\deg(F_{\mathfrak{p}}) \leq d \qquad (\forall \mathfrak{p} \in Y).$$

Then we may define the 'limit'

$$\pi_{\mathfrak{U}}((F_{\mathfrak{p}}))$$

by taking 'limit' of each of the coefficients. The same arguments also applies for polynomial maps.

For any non-principal ultra filter $\mathcal{U}$ on $P =$ (prime numbers), We may consider the following ring.

$$\mathbb{Q}_{\mathcal{U}}^{(\infty)} = \prod_{p} \mathbb{F}_{p^\infty}/(\mathcal{U} \ni 0)$$

It turns out that,

LEMMA 4.9. (1) $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ *is an algebraically closed field of characteristic* 0.
(2) $\mathbb{Q}_{\mathcal{U}}^{(\infty)}$ *has the same cardinality as* $\mathbb{C}$.

Thus we conclude that

PROPOSITION 4.10. *As an abstract field,*

$$\mathbb{Q}_{\mathcal{U}}^{(\infty)} \cong \mathbb{C}.$$

5. Elementary category theory

We need to develop a fine theory of (n-)category theory in this lecture. But that will take time. Meanwhile, we use an easy part of elementary category theory as a convenient language.

DEFINITION 5.1. A **category** $\mathcal{C}$ is a collection of the following data

(1) A collection $\mathrm{Ob}(\mathcal{C})$ of **objects** of $\mathcal{C}$.
(2) For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$

of **morphisms**.
(3) For each triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map("composition (rule)")

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

satisfying the following axioms

(1) $\mathrm{Hom}(X, Y) \cap \mathrm{Hom}(Z, W) = \emptyset$ unless $(X, Y) = (Z, W)$.
(2) (Existence of an identity) For any $X \in \mathrm{Ob}(\mathcal{C})$, there exists an element $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ such that

$$\mathrm{id}_X \circ f = f, \quad g \circ \mathrm{id}_X = g$$

holds for any $f \in \mathrm{Hom}(S, X), g \in \mathrm{Hom}(X, T)$ $(\forall S, T \in \mathrm{Ob}(\mathcal{C}))$.
(3) (Associativity) For any objects $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$, and for any morphisms $f \in \mathrm{Hom}(X, Y), g \in \mathrm{Hom}(Y, Z), h \in \mathrm{Hom}(Z, W)$, we have

$$(f \circ g) \circ h = f \circ (g \circ h).$$

5.1. **universe.** In order to deal with some set theoretical difficulties, we assume the existence of sufficiently many universes.

DEFINITION 5.2. A **universe** U is a nonempty set satisfying the following axioms:

(1) If $x \in U$ and $y \in x$, then $y \in U$.
(2) If $x, y \in U$, then $\{x, y\} \in U$.
(3) If $x \in U$, then the power set $2^x \in U$.
(4) If $\{x_i | i \in I\}$ is a family of elements of U indexed by an element $I \in U$, then $\cup_{i \in I} x_i \in U$.

LEMMA 5.3. *Let U be an universe. Then the following statements hold.*

(1) *If $x \in U$, then $\{x\} \in U$.*
(2) *If x is a subset of $y \in U$, then $x \in U$.*
(3) *If $x, y \in U$, then the ordered pair $(x, y) = \{\{x, y\}, x\}$ is in U .*
(4) *If $x, y \in U$, then $x \cup y$ and $x \times y$ are in U.*
(5) *If $\{x_i | i \in I\}$ is a family of elements of U indexed by an element $I \in U$, then we have $\prod_{i \in I} x_i \in U$. less than the cardinality of U . In particular, $U \notin U$.*

In this text we always assume the following.

$$\boxed{\text{For any set } S\text{, there always exists a universe } U \text{ such that } S \in U.}$$

The assumption above is related to a “hard part” of set theory. So we refrain ourselves from arguing the “validity” of it.

DEFINITION 5.4. Let U be a universe.

(1) A set S is said to be U**-small** if it is an element of U.
(2) A category $\mathcal{C}$ is said to be U**-small** if
   (a) $\mathrm{Ob}(\mathcal{C})$ is a U-small set.
   (b) For any $X, Y \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}(X, Y)$ is U-small.

Note: The treatment in this subsection owes very much on those of wikipedia:

`http://en.wikipedia.org/wiki/Small_set_(category_theory)`

and planetmath.org:

`http://planetmath.org/encyclopedia/Small.html` but the treatment hear differs a bit from the treatments given there. We also refer to [13] as a good reference.

5.2. **examples of categories.** In this section we fix a sufficiently large universe U. For some of readers it may be happier to neglect the term “U-small”.

Example 5.5. The category (SETS) of U-small sets.

$$\mathrm{Ob(SETS)} = \{U\text{-small sets}\}.$$

For any $X, Y \in \mathrm{Ob(SETS)}$, we put

$$\mathrm{Hom}_{(\mathrm{SETS})}(X, Y) = (\text{the set of all maps from } X \text{ to } Y.)$$

Example 5.6. The category (GROUPS) of U-small groups (that means, groups that are U-small as sets).

$$\mathrm{Ob(GROUPS)} = \{U\text{-small groups}\}.$$

For any $X, Y \in \mathrm{Ob(GROUPS)}$, we put

$$\mathrm{Hom}_{(\mathrm{GROUPS})}(X, Y) = (\text{the set of all group homomorphisms from } X \text{ to } Y.)$$

Likewise, we may easily define categories such as the category (RINGS) of U-small-rings, the category ($R-$ALG) of algebras over a ring R, the category ($k-$ VS) of U-small vector spaces over a field k, and so on.

Example 5.7. The category (TOP) of U-small topological space

$$\mathrm{Ob(GROUPS)} = \{U\text{-small topological space}\}.$$

For any $X, Y \in \mathrm{Ob(TOP)}$, we put

$$\mathrm{Hom}_{(\mathrm{TOP})}(X, Y) = (\text{the set of all continuous maps from } X \text{ to } Y.)$$

One may also consider the category of C^∞-manifolds, the category of C^1-manifolds, and so on.

Of course, the category of schemes (with morphisms the ones we defined in the previous part) is very important category for us.

6. fiber product

6.1. definition of a fiber product.

Definition 6.1. Let $\mathcal{C}$ be a category. Let X, Y, Z be objects of $\mathcal{C}$. Assume that morphisms $f : X \to Z$ and $g : Y \to Z$ are given. Then the **fiber product** $X \times_Z Y$ (more precisely,

$$X \times_{f,Z,g} Y$$

) is defined as an object W together with morphisms

$$p : W \to X, \quad q : W \to Y$$

such that

$$p \circ f = q \circ g$$

which is universal in the following sense.

For any $W_1 \in \mathrm{Ob}(\mathcal{C})$ together with morphisms

$$p_1 : W_1 \to X, \quad q_1 : W_1 \to Y$$

such that

$$p_1 \circ f = q_1 \circ g,$$

there exists a unique morphism $h : W_1 \to W$ such that

$$p \circ h = p_1, q \circ h = q_1$$

holds.

Using the usual universality argument we may easily see that the fiber product is, if exists, unique up to a unique isomorphism.

EXAMPLE 6.2. Fiber products always exists in the category (TOP) of topological spaces. Namely, let $X, Y, Z \in \mathrm{Ob}(\mathrm{TOP})$. Assume that morphisms(=continuous maps) $f : X \to Z$ and $g : Y \to Z$ are given. Then we consider the following subset S of $X \times Y$.

$$\begin{aligned} S =&(f, g)^{-1}(\Delta_Z)(\subset (X \times Y)) \\ =&\{(x, y) \in X \times Y; f(x) = g(y)\}. \end{aligned}$$

We equip the set S with the relative topology. Then S plays the role of the fiber product $X \times_Z Y$. (The morphisms p, q being the (restriction of) projections.)

6.2. **tensor products of algebras over a commutative ring.**

LEMMA 6.3. *Let A be a commutative ring. Let B, C be A-algebras. Then the followings are true.*

(1) *The module*

$$B \otimes_A C$$

carries a natural structure of A-algebra.

(2) *There exits A-algebra homomorphisms*

$$\iota_B : B \ni b \mapsto b \otimes 1 \in B \otimes_A C, \qquad \iota_C : C \ni c \mapsto 1 \otimes c \in B \otimes_A C.$$

(3) *The triple $(B \otimes_A C, \iota_B, \iota C)$ has the following universal property: For any A-algebra D and for any A-algebra homomorphisms $f : B \to D$ and $g : C \to D$, there exists a unique A-algebra homomorphism*

$$h : B \otimes_A C \to D$$

such that $f = h \circ \iota_B$ and $g = h \circ \iota_C$.

PROOF. An easy exercise. $\square$

Note that in the situation of the above Lemma, if A is non commutative, then $B \otimes_A C$ may not have a natural structure of ring. Sooner or later one needs to face this fact.

6.3. **fiber products of schemes.** One may easily see that the definition of the fiber product is the "opposite" of the property of the tensor products shown in the previous subsection. In more accurate terms,

LEMMA 6.4. *Fiber products always exists in the category* (Affine Schemes) *of affine schemes.*

Namely, let $X = \operatorname{Spec}(A_1), Y = \operatorname{Spec}(A_2), Z = \operatorname{Spec}(B)$ *be affine schemes. Assume that morphisms* $f : X \to Z$ *and* $g : Y \to Z$ *are given. Then we have*

$$\operatorname{Spec}(A_1) \times_{f,\operatorname{Spec}(B),g} \operatorname{Spec}(A_2) \cong \operatorname{Spec}(A_1 \otimes_{\Gamma(f),B,\Gamma(g)} A_2)$$

(If the morphisms and homomorphisms involved are clear from the context, we often abbreviate the above equation as:

$$\operatorname{Spec}(A_1) \times_{\operatorname{Spec}(B)} \operatorname{Spec}(A_2) \cong \operatorname{Spec}(A_1 \otimes_B A_2)$$

by the abuse of language.)

REMARK 6.5. By using "gluing lemma" for schemes, we may also prove that fiber products always exists in the category of schemes. We omit the proof. See for example [11] for details. (The author (Tsuchimoto) has often forgot to say (sorry), but of course, Grothendieck's enormous works including EGA1[4],EGA2[5],EGA3[6][7] EGA4[8][9][10] and SGA are the primary source for the whole of this talk.)

So far, we have not developed enough theory of a general schemes except for the affine case. In local theories, the affine case suffices and the generalization to general schemes is fairly easy. Due to the lack of time, we omit detailed arguments. For more detailed account, see EGA or Iitaka [11]

Note that the universality of the fiber product may be interpreted as the following way.

LEMMA 6.6. *Let* $\mathcal{C}$ *be a category. Let* $X, Y, Z \in \operatorname{Ob}(\mathcal{C})$, $f \in \operatorname{Hom}_{\mathcal{C}}(X, Z), g \in \operatorname{Hom}_{\mathcal{C}}(Y, Z)$. *Assume that the fiber product* $X \times_Z Y$ *exists. Then we have*

$$\operatorname{Hom}(W_1, X) \times_{\operatorname{Hom}(W_1,Z)} \operatorname{Hom}(W_1, Y) \cong \operatorname{Hom}(W_1, X \times_Z Y)$$

Corollary 6.7. *Let X, Y, Z be schemes. Let $f : X \to Z, g : Y \to Z$ be morphisms. Then we have*

$$X(K) \times_{Z(K)} Y(K) \cong (X \times_Z Y)(K)$$

for any field K. (Recall that for a scheme X, $X(K)$ denotes the set of K-valued points of K.)

7. flatness

We define here notion of "flatness". For non commutative algebras, we need to distinguish "left-flat" and "right-flat". Ironically, Bourbaki's book "commutative algebra"[1] is one of the best source available on this subject. For the time being, we define flatness for commutative algebras only.

Definition 7.1. Let A be a commutative rings. An A-module M is **flat** if for any exact sequence

$$0 \to N_1 \to N_2 \to N_3 \to 0,$$

a sequence

$$0 \to M \otimes_A N_1 \to M \otimes_A N_2 \to M \otimes_A N_3 \to 0$$

is also exact.

Note that we have the following

Lemma 7.2. *Let A be a commutative rings. Let M be an A-module. Then for any exact sequence*

$$0 \to N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3 \to 0$$

of A-modules, a sequence

$$M \otimes_A N_1 \xrightarrow{\mathrm{id}_M \otimes f_1} M \otimes_A N_2 \xrightarrow{\mathrm{id}_M \otimes f_2} M \otimes_A N_3 \to 0$$

is also exact. Thus M is flat over A if and only if for any module N_2 and for any submodule N_1 of N_2, the map

$$M \otimes_A N_1 \xrightarrow{\mathrm{id}_M \otimes \mathrm{inclusion}} M \otimes_A N_2$$

is injective.

Proof. Given any element $\sum_i m_i \otimes n_i$ of $M \otimes N_3$, we take a lift $\hat{n_i}$ of n_i to N_2. Then we have

$$(\mathrm{id}_M \otimes f_2)(\sum_i m_i \otimes \hat{n_i}) = \sum_i m_i \otimes n_i.$$

Thus the map $\mathrm{id}_M \otimes f_2$ is surjective. To see the exactness in the middle, we first notice that

$$(\mathrm{id}_M \otimes f_2) \circ (\mathrm{id}_M \otimes f_2) = \mathrm{id}_M \otimes (f_2 \circ f_1) = 0.$$

Thus $\mathrm{id} \otimes f_2$ yields an A-module homomorphism

$$\phi : (M \otimes_A N_2)/(M \otimes_A N_1) \to M \otimes_A N_3.$$

On the other hand, for any element $(m, n) \in M \times N_3$, we take a lift $\hat{n}$ of n to N_2 and define

$$\alpha : M \times N_3 \ni (m, n) \mapsto [m \otimes \hat{n}] \in (M \otimes_A N_2)/(M \otimes_A N_1).$$

We may easily check that α is well-defined (independent of the choice of the lift $\hat{n}$ of n) and is A-bilinear. So α defines an A-module homomorphism

$$\psi : M \otimes N_3 \ni (m \otimes n) \mapsto [m \otimes \hat{n}] \in (M \otimes_A N_2)/(M \otimes_A N_1).$$

Then it is easy to show that the homomorphisms ϕ and ψ are inverse to each other.

(See for example [14, Appendix A] or [1].)

□

Definition 7.3. An A-algebra B is **flat** if it is flat as an A-module.

A morphism of affine schemes is **flat** if the corresponding ring homomorphism is flat.

Example 7.4. $\mathbb{Z}/3\mathbb{Z}$ is not flat over $\mathbb{Z}$. Indeed, we consider an exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0.$$

Then by tensoring with $\mathbb{Z}/3\mathbb{Z}$ we obtain a sequence

$$0 \to \mathbb{Z}/3\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$$

which is not exact.

Let us view it as a homomorphism $\phi : \tilde{\mathbb{Z}} \xrightarrow{3} \tilde{\mathbb{Z}}$ of quasi coherent sheaf on $\mathrm{Spec}(\mathbb{Z})$ with the keyword "section wise" and "fiber wise" in mind.

(1) ϕ is injective if and only if it is injective *section wise*. Thus our ϕ is injective.
(2) ϕ is "not injective" at the fiber of the point 3.

Example 7.5. Let k be a field. Then a ring homomorphism $k[X] \to k[X, Y]/(XY, Y^2)$ is not flat. The reason is almost the same with the one above. (We consider an exact sequence

$$0 \to k[X] \xrightarrow{\cdot X} k[X] \to k[X]/Xk[X] \to 0.$$

In this example, an embedded prime (X, Y) of $I = (XY, Y^2)$ in $k[X, Y]$ falls into the zero locus of X.)

8. Various kind of morphisms.

8.1. affine morphisms.

Definition 8.1. Let X be a scheme. Let $\mathcal{A}$ be a quasi coherent sheaf with an $\mathcal{O}_X$-bilinear multiplication so that it is a sheaf of (unital associative) algebras. Then we may construct a scheme $\operatorname{Spec}(\mathcal{A})$ over X. The morphism $\operatorname{Spec}(\mathcal{A}) \to X$ is called an **explicit affine morphism**. A morphism which is isomorphic to an explicit affine morphism so defined is called an **affine morphism**.

8.2. closed immersion.

Definition 8.2. Let X be a scheme. Let $\mathcal{I}$ be a quasi coherent sheaf of ideal of $\mathcal{O}_X$. Then the scheme $\operatorname{Spec}(\mathcal{O}_X/\mathcal{I})$ (which is affine over X) is called a closed subscheme of X. We often call it $V(\mathcal{I})$.

Definition 8.3. A morphism $X \to Y$ of schemes is a **closed immersion** if there exists a sheaf of ideal $\mathcal{I}$ of $\mathcal{O}_Y$ such that f induces an isomorphism $X \to V(\mathcal{I})$ of schemes.

Proposition 8.4. *Affine morphisms and closed immersions are* **stable under base extension**. *That is, if we are given morphisms $f : X \to S$ and $g : T \to S$ of schemes and*

(1) *if f is an affine morphism, then $f_T : X_T = X \times_S T \to T$ ("projection on the second variable") is also an affine morphism.*
(2) *if f is a closed immersion, then $f_T : X_T \to T$ is also a closed immersion.*

Proof. (1): We may assume $X = \operatorname{Spec}(\mathcal{A})$. Then we may verify immediately that $X_T = \operatorname{Spec}(g^*\mathcal{A})$. This argument also proves (2). □

9. differential calculus of schemes

We refer to [3].

9.1. separated morphisms.

The definition of a separated morphism resembles the definition of a Hausdorff space.

Definition 9.1. A morphism $f : X \to S$ of schemes is **separated** if the diagonal

$$\Delta_X \subset X \times_S X$$

(That means, the image of the diagonal map $X \to X \times_S X$) is closed in $X \times_S X$. In other words, f is closed if and only if there exists an ideal sheaf $\mathcal{I}_\Delta$ of $X \times_S X$ such that f induces an isomorphism $X \cong V(\mathcal{I}_\Delta)$ of schemes.

Lemma 9.2. *A morphism* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *of affine schemes is always separated. More generally, an affine morphism is always separated.*

Proof. Let I be the kernel of a ring homomorphism

$$B \otimes_A B \ni b_1 \otimes b_2 \mapsto b_1 b_2 \in B.$$

Then it is easy to see that I gives the defining equation of the diagonal Δ.

For the general affine morphism case, let $X = \mathrm{Spec}(\mathcal{A})$ be a scheme which is affine over Y. Then we have $X \times_Y X = \mathrm{Spec}(\mathcal{A} \otimes \mathcal{O}_Y \mathcal{A})$. We may then see the situation locally and reduce the problem to the first case. □

Lemma 9.3. *Separated morphism is stable under base extension. That is, assume* $f : X \to S$ *be a separated morphism. Let* $g : T \to S$ *be a morphism of schemes. Then* $f_T : X_T \to T$ *is separated.*

Proof.

$$X_T \times_T X_T \cong (X \times_S T) \times_T (T \times_S X) \cong (X \times_S X) \times_S T$$

The diagonal $\Delta_{X/T}$ is isomorphic to $\Delta_{X/S} \times_S T$, and is therefore closed. □

Lemma 9.4. *A composition of separated morphisms is again separated. Namely, if* $f : X \to Y$ $g : Y \to S$ *are separated morphism of schemes, then* $g \circ f : X \to S$ *is also separated.*

Proof. We first claim the following sublemma:

Sublemma 9.5. *Under the assumption of the lemma above, we have*

$$X \times_Y X \cong (X \times_S X) \times_{(Y \times_S Y)} \Delta_{Y/S}$$

The proof of the sublemma above is given by showing that the right hand side satisfies the same universal property as the left hand side.

Now, let us prove the lemma. Since Y is separated over S, we have a closed immersion

$$\Delta_{Y/S} \hookrightarrow Y \times_S Y.$$

By taking a base extension, we obtain a closed immersion

$$(X \times_S X) \times_{(Y \times_S Y)} \Delta_{Y/S} \hookrightarrow X \times_S X.$$

$$X \times_Y X \cong (X \times_S X) \times_{(Y \times_S Y)} \Delta_{Y/S} \hookrightarrow X \times_S X$$

Then $\Delta_{X/S}$ is identified with the composition of closed immersions

$$X \cong \Delta_{X/Y} \hookrightarrow X \times_Y X \hookrightarrow X \times_S X$$

□

PROPOSITION 9.6. *Let $p : X \to S$ be a separated morphism of schemes. Let $q : Y \to S$ be a morphism of schemes. Then any S-morphism $f : X \to Y$ is separated.*

PROOF. Since X is separated over S,

$$X \to \Delta_{X/S} \hookrightarrow X \times_S X$$

is a closed immersion. Now $\Delta_{X/Y} \to X \times_Y X$ may be identified with a pullback of the morphism above

$$\Delta_{X/S} \times_{(X \times_S X)} (X \times_Y X) \to (X \times_Y X)$$

□

9.2. **Linear differential operators.**

9.2.1. *Jets.* Let X be a separated scheme over S. That means, we are given a separated morphism $\varphi : X \to S$. Let $\mathcal{I}_\Delta$ be the defining ideal sheaf of the diagonal Δ in $X \times_S X$. For any positive integer n, we define $\Delta^{(n)}$ to be the closed subscheme of $X \times_S X$ defined by $\mathcal{I}_\Delta^{(n+1)}$.

The sheaf $\mathcal{J}_n = p_{1*}\mathcal{O}_\Delta^{(n+1)}$ on X is called the sheaf of n-jets on X relative to S. There is another description of this sheaf. Let

$$p_1^{(n)} : \Delta^{(n+1)} \to X, \quad p_2^{(n)} : \Delta^{(n+1)} \to X$$

be restrictions of the projections p_1, p_2. Then we have

$$\mathcal{J}_n = (p_1^{(n)})_*(p_2^{(n)})^*\mathcal{O}.$$

For a local section f of $\mathcal{O}_X$, we define the jet ("the Taylor expansion") of f (of order n) by

$$Jet(f) = p_{1*}p_2^* f$$

EXAMPLE 9.7. Let A be any ring in which $n!$ is invertible. Let x be an indeterminate. We put

$$X = \operatorname{Spec}(A[x]), \quad S = \operatorname{Spec}(A),$$

$\varphi : X \to S$ being the canonical projection.

Then we have $X \times_S X = \operatorname{Spec}(A[x, \bar{x}])$. The sheaf of n-jets on X relative to S is

$$A[x, \bar{x}]/(\bar{x} - x)^{n+1}$$

Let us put $h = \bar{x} - x$. Then for any $p = p(x) \in A[x]$, we have

$$Jet(f) = f(\bar{x}) = \sum_{s=0}^{n} \frac{1}{s!} f^{(s)}(x) h^s.$$

When $n!$ is not invertible in A, a similar formula is still valid. The thing is that the operator $1/s!(d/dx)^s$ is defined over $\mathbb{Z}$.

$$(1/s!(d/dx)^s).(\sum_{t=0}^{n} a_t x^t) = \sum_{t=s}^{n} a_t \binom{t}{s} x^{t-s}$$

Like wise, for any quasi coherent sheaf $\mathcal{F}$ on X, we may define the sheaf $\mathcal{J}_n(\mathcal{F})$ of n-jets of $\mathcal{F}$ on X relative to S as

$$\mathcal{J}_n(\mathcal{F}) = p_{1*} p_2^* \mathcal{F}.$$

For any local section f of $\mathcal{F}$, we may define the n-jet of it in the same way as above.

9.2.2. *definition of linear differential operators.* An importance of the sheaves of jets is that they govern linear differential operators.

Definition 9.8. Let $\varphi : X \to S$ be a separated morphism of schemes. Let $\mathcal{F}, \mathcal{G}$ be quasi coherent sheaves on X. Then a **linear differential operator** of n-th order from $\mathcal{F}$ to $\mathcal{G}$ on X relative to S is a composition of an element of

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{J}_n(\mathcal{F}), \mathcal{G}).$$

with the Taylor expansion.

9.2.3. *local description of differential operators.* Differential operators are defined locally. Thus we may restrict ourselves to the affine case and look them carefully by the language of algebras and modules.

Lemma 9.9. *Let $X = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = Y$ be a morphism of affine scheme. Then $X \times_Y X = \mathrm{Spec}(B \otimes_A B)$. $\Delta_{X/Y}$ corresponds to an ideal $I_{B/A}$ of $B \otimes_A B$ generated by*

$$\{f \otimes 1 - 1 \otimes f; f \in B\}$$

Proof. In $B \otimes_A B / I_{B/A}$, every class $[\sum_j f_j \otimes g_j]$ of an element $\sum_j f_j \otimes g_j \in B \otimes_A B$ is equal to

$$[\sum_j f_j \otimes g_j] = \sum_j [f_j \otimes 1][1 \otimes g_j] = \sum_j [1 \otimes f_j][1 \otimes g_j] = [1 \otimes \sum_j f_j g_j]$$

$\square$

Lemma 9.10 (criterion for being a differential operator). *Let $X = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = Y$ be an morphism of schemes. Let M, N be B-modules. Then an n-th order differential operator P from $\mathcal{F} = \mathcal{O}_X \otimes_B M$ to $\mathcal{G} = \mathcal{O}_X \otimes_B N$ is identified with an A-linear homomorphism*

$$\Gamma(P) : M \to N.$$

An A-linear homomorphism $\phi : M \to N$ corresponds to an n-th order differential operator if and only if for any elements $f_1, f_2, \dots, f_n, f_{n+1}$ of A and for any element $m \in M$, a relation

$$\begin{aligned} &(\mu_N \circ (id_B \otimes \phi))((\prod_{i=1}^{n}(f_i \otimes 1 - 1 \otimes f_i))m) \\ &(= \sum_{I \subset \{1,2,3,\dots,n+1\}} (-1)^{|I|} f^{\complement I} \phi(f^I)) \\ = 0 & \end{aligned}$$

holds.

□

Corollary 9.11. *A first order differential operator $P : \mathcal{O}_X \to \mathcal{G}$ on a scheme X relative to S corresponds to an $\mathcal{O}_S$-module homomorphism $P : \mathcal{O}_X \to \mathcal{G}$ such that for any local section $f, g \in \mathcal{O}_S$, we have*

$$P(fg) = fP(g) + gP(f) - P(1)fg.$$

□

Using the Lemma of criterion for being a differential operator, We deduce the following useful lemma.

Lemma 9.12. *Let $X = \mathrm{Spec}(B) \to \mathrm{Spec}(A) = Y$ be an morphism of schemes. A-linear homomorphism $\phi : M \to N$ corresponds to an n-th order differential operator if and only if for any $f \in B$, the "commutator"*

$$[\phi, f] = \phi(f\bullet) - f\phi(\bullet)$$

corresponds to an $n-1$-th order differential operator.

□

Corollary 9.13. *A composition of an n-th order differential operator $P : \mathcal{F} \to \mathcal{G}$ and an m-th order differential operator $Q : \mathcal{G} \to \mathcal{H}$ is a differential operator of $(n+m)$-th order.*

Proof. We note that for any local regular function f, w

$$[QP, f] = [Q, f]P + Q[P, f]$$

holds. Then we may easily verify the statement by using induction.

□

DEFINITION 9.14. For any separable scheme X over S, we denote the sheaf of n-th linear differential operators on X from a quasi coherent sheaf $\mathcal{F}$ to a quasi coherent sheaf $\mathcal{G}$ relative to S by

$$\mathcal{D}iff^n_{X/S}(\mathcal{F}, \mathcal{G}).$$

The inductive limit

$$\mathcal{D}iff_{X/S}(\mathcal{F}, \mathcal{G}) = \varinjlim_n \mathcal{D}iff^n_{X/S}(\mathcal{F}, \mathcal{G})$$

is called the sheaf of linear differential operators on X relative to S.

We use the following abbreviational symbols.

$$\mathcal{D}iff^n_{X/S}(\mathcal{F}) = \mathcal{D}iff_{X/S}(\mathcal{F}, \mathcal{F})^n, \qquad \mathcal{D}iff^n_{X/S} = \mathcal{D}iff^n_{X/S}(\mathcal{O}_X),$$

$$\mathcal{D}iff_{X/S}(\mathcal{F}) = \mathcal{D}iff_{X/S}(\mathcal{F}, \mathcal{F}), \qquad \mathcal{D}iff_{X/S} = \mathcal{D}iff_{X/S}(\mathcal{O}_X).$$

Note that $\mathcal{D}iff_{X/S}$ is a sheaf of algebras over X. It is an important example of an object which is a "non-commutative algebras glued together".

9.3. **The sheaf of differential 1-forms.** Let X be a separated S-scheme. For each $n \in \mathbb{N}$, there exists a natural projection map.

$$\pi_n : \mathcal{J}_{n+1} \to \mathcal{J}_n$$

Let us restrict ourselves to the case where $n = 0$. $\mathcal{J}_0$ is equal to $\mathcal{O}_X$ and π_0 splits in a natural way.

$$\pi_0 : \mathcal{J}_1 \to \mathcal{O}_X \qquad \text{(split)}$$

which yields an decomposition

$$\mathcal{J}_1 \cong \mathcal{O}_X \oplus \Omega^1_{X/S}$$

for a unique quasi coherent sheaf $\Omega^1_{X/S}$.

DEFINITION 9.15. The sheaf $\Omega^1_{X/S}$ is called the **sheaf of 1-forms** on X relative to S.

9.3.1. *derivations.*

LEMMA 9.16. *For any separated scheme X over S, and for any quasi coherent sheaf $\mathcal{F}$ on X, An inclusion*

$$\mathcal{F} \cong \mathcal{D}iff^0_{X/S}(\mathcal{O}_X, \mathcal{F}) \to \mathcal{D}iff^1_{X/S}(\mathcal{O}_X, \mathcal{F})$$

Admits a section. Namely, "evaluation by 1*.*

$$\mathcal{F} \cong \mathcal{D}iff^1_{X/S}(\mathcal{O}_X, \mathcal{F}) \overset{eval_1}{\to} \mathcal{F}$$

DEFINITION 9.17. The kernel of the evaluation map in the lemma above is called the sheaf of **derivations** on X relative to S. We denote it by $\mathscr{D}er_{X/S}(\mathcal{O}_X, \mathcal{F})$.

It is easy to see that

LEMMA 9.18. *$\mathscr{D}er_{X/S}(\mathcal{O}_X, \mathcal{F})$ is a quasi coherent sheaf on X. Its section consists of $\mathcal{O}_S$-linear maps $D : \mathcal{O}_X \to \mathcal{F}$ which satisfy*

$$D(fg) = fD(g) + gD(f)$$

for any local regular functions f, g.

9.3.2. *The sheaf of differential 1-forms as the universal derivation.* Let X be a separated S-scheme. We define derivative

$$d : \mathcal{O}_X \to \Omega^1_{X/S}$$

as follows

$$\mathcal{O}_X \stackrel{jet_1}{\to} \mathcal{J}_1 \to \Omega^1_{X/S}$$

PROPOSITION 9.19. *For any sheaf homomorphism $\varphi : \Omega^1_{X/S}, \mathcal{F})$,*

$$f \mapsto \varphi(df)$$

is a derivation from $\mathcal{O}_X \to \mathcal{F}$ relative to S. This assignment yields a isomorphism of $\mathcal{O}_X$-module

$$\mathscr{D}er_{X/S}(\mathcal{O}_X, \mathcal{F}) \cong \mathscr{H}om(\Omega^1_{X/S}, \mathcal{F}).$$

9.3.3. *first properties of 1-forms.*

PROPOSITION 9.20. *Let $p : X \to S$ and $q : Y \to S$ be separated morphisms. Let $f : X \to Y$ be a separated morphism of schemes such that $q \circ f = p$. Then we have a following exact sequence.*

$$f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

PROOF. Since the question is local on X, Y, S, we may assume that these schemes are affine. Then the claim is deduced by the following arguments (Corollary 9.22), and a Yoneda-type argument. □

LEMMA 9.21. *Let A, B, C be (unital commutative associative) rings. Assume we are given homomorphisms $A \to B \to C$ of rings. Then for any A-module M, we have an exact sequence*

$$(*_M) \qquad 0 \to \mathrm{Der}_{A/B}(A, M) \stackrel{\alpha_M}{\to} \mathrm{Der}_{A/C}(A, M) \stackrel{\beta_M}{\to} \mathrm{Der}_{B/C}(B, M)$$

The sequence is **natural** *in the sense that if we have another A-module N and an A-module homomorphism $\varphi : M \to N$, then we have a* *commutative diagram.*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Der}_{A/B}(A,M) & \xrightarrow{\alpha_M} & \mathrm{Der}_{A/C}(A,M) & \xrightarrow{\beta_M} & \mathrm{Der}_{B/C}(B,M) \\
 & & \circ\varphi \big\downarrow & & \circ\varphi \big\downarrow & & \circ\varphi \big\downarrow \\
0 & \longrightarrow & \mathrm{Der}_{A/B}(A,N) & \xrightarrow{\alpha_N} & \mathrm{Der}_{A/C}(A,N) & \xrightarrow{\beta_N} & \mathrm{Der}_{B/C}(B,N)
\end{array}$$

PROOF. Any A-derivation $D : A \to M$ over B may be regarded as a derivation over C which we denote by $\alpha_M(D)$.

Any A-derivation $D : A \to M$ over C defines by restriction a B-derivation $B \to M$ over C which we denote by $\beta_M(D)$.

The rest is easy observation. □

With the help of universality of d, we obtain the following corollary.

COROLLARY 9.22. *Let A, B, C be (unital commutative associative) rings. Assume we are given homomorphisms $A \to B \to C$ of rings. Then for any A-module M, we have an exact sequence*
$(**_M)$

$$0 \to \mathrm{Hom}_A(\Omega^1_{B/A}, M) \overset{\alpha_M}{\to} \mathrm{Hom}_A(\Omega^1_{A/C}, M) \overset{\beta_M}{\to} \mathrm{Hom}_A(A \otimes_B \Omega^1_{B/C}, M)$$

It is natural in a sense similar to the Lemma above. □

PROPOSITION 9.23 (A Yoneda type argument). *Let A be a commutative associative ring.*

(1) *Let M_1, M_2 be A-modules. Assume for each A-module M, we are given a homomorphism*

$$\alpha_M : \mathrm{Hom}_A(M_1, M) \to \mathrm{Hom}_A(M_2, M).$$

We assume that the assignment $M \mapsto \alpha_M$ is natural. That means, for any A-modules M, N and for any A-module homomorphism $\phi : M \to N$, we have the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{Hom}_A(M_1, M) & \xrightarrow{\alpha_M} & \mathrm{Hom}_A(M_2, M) \\
\phi\circ \big\downarrow & & \phi\circ \big\downarrow \\
\mathrm{Hom}_A(M_1, N) & \xrightarrow{\alpha_N} & \mathrm{Hom}_A(M_2, N)
\end{array}$$

In other words, we have

$$\phi \circ \alpha_M(\psi) = \alpha_N(\phi \circ \psi)$$

for any $\psi \in \mathrm{Hom}_A(M_1, M)$. Then α is "representable". That means, there exists a unique $f_\alpha \in \mathrm{Hom}_A(M_2, M_1)$ such that

$$\alpha_M(\psi) = \psi \circ f_\alpha$$

holds for all $\psi \in \operatorname{Hom}_A(M_1, M)$.

(2) *Let* M_1, M_2, M_3 *be* A*-modules. Assume for each* A*-module* M*, we are given homomorphisms*

$$\alpha_M : \operatorname{Hom}_A(M_1, M) \to \operatorname{Hom}_A(M_2, M),$$
$$\beta_M : \operatorname{Hom}_A(M_2, M) \to \operatorname{Hom}_A(M_3, M).$$

We assume that the assignments α, β *is natural. Assume furthermore that for any* A*-module* M*, a sequence*

$$(***_M) \qquad 0 \to \operatorname{Hom}_A(M_1, M) \xrightarrow{\alpha_M} \operatorname{Hom}_A(M_2, M) \xrightarrow{\beta_M} \operatorname{Hom}_A(M_3, M)$$

is always exact. Then the corresponding sequence

$$M_3 \xrightarrow{f_\beta} M_2 \xrightarrow{f_\alpha} M_1 \to 0$$

(which arises due to the claim above) is also exact.

Proof. (1) Put

$$f_\alpha = \alpha_{M_1}(\mathrm{id}_{M_1}).$$

Then for any $\psi \in \operatorname{Hom}_A(M_1, M)$, we have

$$\alpha_M(\psi) = \alpha_M(\psi \circ \mathrm{id}_{M_1}) = \psi \circ \alpha_{M_1}(\mathrm{id}_{M_1}) = \psi \circ f_\alpha.$$

(2) Since $\beta_M \circ \alpha_M = 0$, we deduce that $f_\alpha \circ f_\beta = 0$ using the uniqueness of the homomorphism which represents $\beta \circ \alpha$.

For surjectivity of f_α, we use the sequence $(***_M)$ for $M = M_1/f_\alpha(M_2)$. For the exactness at the middle term, we use the sequence $(***_M)$ for $M = M_2/f_\beta(M_3)$. We leave the detail as an easy exercise. □

10. Étale morphism

10.1. morphism of finite type.

Definition 10.1. Let X, Y be schemes, and let $f : X \to Y$ be a morphism. We say that f is of **finite type** if there exists an open cover $\{U_i\}$ of Y by affine schemes and a finite open cover $\{V_{ij}\}$ of each $f^{-1}(U_i)$ by affine schemes such that $f_{ij} = f|_{V_{ij}}$ is "a morphism of finite type" for every i and j. That means, if we put

$$\Gamma_{f_{ij}} : A_i = \Gamma(\mathcal{O}_{U_i}) \to \Gamma(\mathcal{O}_{V_{ij}}) = B_{ij}$$

Then B_{ij} is finitely generated algebra over A_i.

10.2. **Unramified morphism.**

DEFINITION 10.2. A separated morphism $\phi : X \to Y$ of finite type is said to be **unramified** if $\Omega^1_{X/Y} = 0$.

LEMMA 10.3. *Let A be a B-algebra. Assume A is generated by $\{x_\lambda\}_{\lambda\in\Lambda}$ as an A-algebra. Then $I_{A/B} = \operatorname{Ker}(A \otimes_B A \to A)$ is generated by*

$$S = \{x_\lambda \otimes 1 - 1 \otimes x_\lambda; \quad \lambda \in \Lambda\}.$$

as an ideal of $A \otimes_B A$.

PROOF. Let us denote by J the ideal of $A \times_B A$ generated by S. Then we define a subset T of A as follows.

$$T = \{x \in A; \quad x \otimes 1 - 1 \otimes x \in J\}$$

Now we claim the following facts.

(1) T is closed under addition.
(2) T is stable under multiplication by any element of B.
(3) $1 \in T$.
(4) $x_\lambda \in T \qquad (\forall \lambda \in \Lambda)$.
(5) T is closed under multiplication.

The only (5) may require proof. For any elements $t_1, t_2 \in T$, we have

$$\begin{aligned} & t_1 t_2 \otimes 1 - 1 \otimes t_1 t_2 \\ =& (t_1 \otimes 1)(t_2 \otimes 1 - 1 \otimes t_2) - (t_1 \otimes 1 - 1 \otimes t_1)(1 \otimes t_2) \in J. \end{aligned}$$

So the subset T is a B-subalgebra of A containing the generators $\{x_\lambda\}$ of A. Thus we have $T = A$.

□

PROPOSITION 10.4. *A separated morphism $\phi : X \to Y$ of finite type is unramified if and only if the diagonal map $X \cong \Delta_{X/Y} \hookrightarrow X \otimes_Y X$ is an open immersion.*

PROOF. Let us first prove the "if" part. Assume $\Delta_{X/Y}$ is open. then $\Delta_{X/Y}$ is a clopen ("closed and open") subset of $X \otimes_Y X$. Namely,

$$X \otimes_Y X = (\Delta_{X/Y}) \cup (\complement \Delta_{X/Y})$$

is a decomposition of the scheme $X \otimes_Y X$ into two Zariski open set. Thus we have

$$\mathcal{O}_{X \otimes_Y X} = \mathcal{I}_{\Delta_{X/Y}} \oplus I_{\complement \Delta_{X/Y}}.$$

We then note in particular that $\mathcal{I}_{\Delta_{X/Y}}$ has a distinguished global section ("the identity") u defined by

$$u = \begin{cases} 1 & \text{on } \Delta_{X/Y} \\ 0 & \text{on } \complement\Delta_{X/Y}. \end{cases}$$

Then we see that

$$\mathcal{I}_{\Delta_{X/Y}} = u\mathcal{I}_{\Delta_{X/Y}} \subset \mathcal{I}^2_{\Delta_{X/Y}}.$$

So we have

$$\Omega^1_{X/Y} = \mathcal{I}_{\Delta_{X/Y}}/\mathcal{I}^2_{\Delta_{X/Y}} = 0$$

as required.

Let us now prove the "only if" part. The question is local on X and on Y. So we may assume that f is of the form

$$f : X = \mathrm{Spec}(A) \to Y = \mathrm{Spec}(B)$$

where A is a finitely generated algebra over B. Let $I = I_{\Delta_{X/Y}}$ be the ideal of definition of the diagonal. The previous Lemma tells us that I is finitely generated over $A \otimes_B A$. By the assumption we have

$$I/I^2 = 0.$$

Now we use the Nakayama's lemma (theorem below) to find an element $c \in I$ such that

$$cx = x \qquad (\forall x \in I).$$

Then it is easy to see that c is an idempotent and that $I = c(A \otimes_B A)$ is its range. □

10.2.1. *NAK.*

THEOREM 10.5 (Nakayama's lemma, or NAK). *Let A be a commutative ring. Let M be an A-module. We assume that M is finitely generated (as a module) over A. That means, there exists a finite set of elements $\{m_i\}_{i=1}^t$ such that*

$$M = \sum_{i=1}^{t} Am_i$$

holds. If an ideal I of A satisfies

$$IM = M \quad (\textit{that is, } M/IM = 0),$$

then there exists an element $c \in I$ such that

$$cm = m \qquad (\forall m \in M)$$

holds. If furthermore I is contained in the nilradical of A, then we have $M = 0$.

PROOF. Since $IM = M$, there exists elements $b_{il} \in I$ such that

$$a_i = \sum_{l=1}^{t} b_{il} a_l \qquad (1 \le i \le t)$$

holds. In a matrix notation, this may be rewritten as

$$v = Bv$$

with $v =^t (m_1, \dots, m_n)$, $B = (b_{ij}) \in M_t(I)$. Using the unit matrix $1_t \in M_t(A)$ one may also write :

$$(1_t - B)v = 0.$$

Now let R be the adjugate matrix of $1_t - B$. In other words, it is a matrix which satisfies

$$R(1_t - B) = (1_t - B)R = (\det(1_t - B))1_t.$$

Then we have

$$\det(1_t - B) \cdot v = R(1_t - B)v = 0.$$

On the other hand, since $1_t - B = 1_t$ modulo I, we have $\det(1_t - B) = 1 - c$ for some $c \in I$. This c clearly satisfies

$$v = cv.$$

□

Let us interpret the claim of the above theorem in terms of a sheaf $\mathcal{F} = \mathcal{O}_X \otimes_A M$ on $X = \operatorname{Spec}(A)$. M is assumed to be finitely generated over A. Note that this in particular means that every fiber of $\mathcal{F}$ on a K-valued point (for each field K) is finite dimensional K-vector space. In other words, it is "a pretty little(=finite dimensional) vector spaces in a row."

The next assumption simply means that $\mathcal{F}$ restricted to $V(I)$ is equal to zero. So $\mathcal{F}$ sits somewhere other than $V(I)$.

The claim of the theorem (NAK) is that one may choose a regular function c which "distinguishes $V(I)$ and "the support of $\mathcal{F}$". c is equal to 0 on $V(I)$ and is equal to 1 where $\mathcal{F}$ sits.

10.3. **Étale morphism.**

DEFINITION 10.6. A separated morphism $\phi : X \to Y$ of finite type is said to be **étale** if it is flat and unramified.

10.4. **Smooth morphism.** For any non negative integer n and for any scheme S, we put

$$\mathbb{A}^n_S = \operatorname{Spec}\mathbb{Z}[x_1, x_2, \dots, x_n] \times_{\operatorname{Spec}\mathbb{Z}} S$$

and $\pi_S : \mathbb{A}^n_S \to S$ the standard projection.

A smooth scheme over S is a scheme which ”étale locally look like” $\mathbb{A}^n_S$.

DEFINITION 10.7. A separated morphism $\phi : X \to S$ of finite type is smooth of relative dimension n if for any point x on X, there exists an open neighborhood U of X and an 'etale morphism $\psi : U \to \mathbb{A}^n_S$ such that

$$\phi = \psi \circ \pi_S$$

holds.

Let us close this section by quoting the following fundamental result.

THEOREM 10.8 (SGA1,Éxpose II,Corollary4.6). *Let $f : X \to Y$ be a morphism of smooth S-schemes. then f is étale at $x \in X$ if and only if $f^*(\Omega^1_{Y/S}) \to \Omega^1_{X/S}$ is isomorphism at x.*

11. APPENDIX

11.1. exact sequence.

DEFINITION 11.1. Let A be a ring. Then a sequence

$$M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$$

is **exact** if condition

$$\operatorname{Image}(f_1) = \operatorname{Ker}(f_2)$$

holds.

We also use the notion of exact sequences for sheaves on schemes. It is also defined likewise. This could be summarized in the theory of abelian categories. We postpone the precise argument to a near future.

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