CONGRUENT ZETA FUNCTIONS. NO.4

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4.1. **First properties of congruent Zeta function.** Let us first recall an elementary formula

LEMMA 4.1.

$$\sum_{k=1}^{\infty} \frac{1}{k} T^k = -\log(1-T)$$

DEFINITION 4.2. We denote by $\mathbb{A}_n$ the "void set of equation" in n-variables. That means, for any field (or ring) k, we put

$$\mathbb{A}_n(k) = \{x \in k^n\}.$$

PROPOSITION 4.3.

$$Z(\mathbb{A}_n/\mathbb{F}_q, T) = \frac{1}{1-q^n T}$$

PROPOSITION 4.4. *Let V, W, W_1, W_2 be sets of equations.*

(1) *If $\#V(\mathbb{F}_{q^s}) = \#W(\mathbb{F}_{q^s})$ for any s ,then $Z(V/\mathbb{F}_q, T) = Z(W/\mathbb{F}_q, T)$.*
(2) *If $\#V(\mathbb{F}_{q^s}) = \#W_1(\mathbb{F}_{q^s}) + \#W_2(\mathbb{F}_{q^s})$ for any s ,then:*

$$Z(V/\mathbb{F}_q, T) = Z(W_1/\mathbb{F}_q, T)Z(W_2/\mathbb{F}_q, T).$$

PROPOSITION 4.5. *Let $f \in \mathbb{F}_q[X]$ be an irreducible polynomial in one variable of degree d. Let us consider $V = \{f\}$, an equation in one variable. Then:*

(1)
$$V(\mathbb{F}_{q^s}) = \begin{cases} d & \text{if } d|s \\ 0 & \text{otherwise} \end{cases}$$

(2)
$$Z(V/\mathbb{F}_q, T) = \frac{1}{1-T^d}$$

EXERCISE 4.1. Describe what happens when we omit the assumption of f being irreducible in Proposition 4.5.