COHOMOLOGIES.

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04. projective and injective modules

LEMMA 4.1. *Let R be a (unital associative but not necessarily commutative) ring. Then for any R-module M, the following conditions are equivalent.*

(1) *M is a direct summand of free modules.*
(2) *M is projective*

COROLLARY 4.2. *For any ring R, the category* (R-modules) *of R-modules* **have enough projectives.** *That means, for any object $M \in$* (R-modules)*, there exists a projective object P and a surjective morphism $f : P \to M$.*

DEFINITION 4.3. Let R be a commutative ring. We assume R is a domain (that means, R has no zero-divisors except for 0.)

An R-module M is said to be **divisible** if for any $r \in R \setminus \{0\}$, the multplication map

$$M \overset{r\times}{\to} M$$

is surjective.

DEFINITION 4.4. Let R be a commutative ring. We assume R is a domain (that means, R has no zero-divisors except for 0.)

An R-module M is said to be **divisible** if for any $r \in R \setminus \{0\}$, the multplication map

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LEMMA 4.5. *Let R be a (commutative) principal ideal domain (PID). Then an R-module I is injective if and only if it is divisible.*

PROPOSITION 4.6. *For any (not necessarily commutative) ring R, the category* (R-modules) *of R-modules* **has enough injectives.** *That means, for any object $M \in$* (R-modules)*, there exists an injective object I and an monic morphism $f : M \to I$.*

For the proof of the proposition above, we need the followin lemmas.

LEMMA 4.7. *For any $\mathbb{Z}$-module M, let us denote by $\hat{M}$ the module* $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{T}_1)$ *where $T_1 = \mathbb{R}/\mathbb{Z}$. Then:*

(1) *For any free $\mathbb{Z}$-module F, $\hat{F}$ is divisible (hence is $\mathbb{Z}$-injective).*
(2) *For any $\mathbb{Z}$-module M, there is a canonical injective $\mathbb{Z}$-homomorphism $M \to \widehat{(\hat{M})}$.*
(3) *Any Z-module M may be embeded in a divisible module T.*

LEMMA 4.8. *Let T be a divisible module. Then for any ring A,*

$$\mathrm{Hom}_{\mathbb{Z}}(A, T)$$

is A-injective.