COHOMOLOGIES.

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$\boxed{\text{07. Ext as a derived functor}}$ We recommend the book of Lang [1] as a good reference. The treatment here follows the book for the most part.

THEOREM 7.1. *Let $\mathcal{C}_1$ be an abelian category with enough injectives, and let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a covariant additive left functor to another abelian category $\mathcal{C}_2$. Then:*

(1) $F \cong R^0 F$.

(2) *For each short exact sequence*

$$0 \to M' \to M \to M'' \to 0$$

and for each $n \geq 0$ there is a natural homomorphism

$$\delta^n : R^n F(M'') \to R^{n+1} F(M)$$

such that we obtain a long exact sequence

$$\cdots \to R^n F(M') \to R^n F(M) \to R^n F(M'') \xrightarrow{\delta^n} R^{n+1} F(M') \to \ldots.$$

(3) *δ is natural. That means, for a morphism of short exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

the δ's give a commutative diagram:

$$\begin{array}{ccc} R^n F(M'') & \xrightarrow{\delta^n} & R^{n+1} F(M') \\ \downarrow & & \downarrow \\ R^n F(N'') & \xrightarrow{\delta^n} & R^{n+1} F(N') \end{array}$$

(4) *For each injective objective object I of A and for each $n > 0$ we have $R^n F(I)$.*

The collection $\{R^j F\}$ of functors $R^j F$ is a "universal delta functor". See [1].

LEMMA 7.2. *Under the assumption of the previous theorem, for any exact sequence $0 \to M' \to M \to M'' \to 0$ of objects in $\mathcal{C}_1$, there exists injective resolutions $I_{M'}, I_M, I_{M''}$ of M', M, M'' respectively and a commutative diagram*

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_{M'} & \longrightarrow & I_M & \longrightarrow & I_{M''} & \longrightarrow & 0 \end{array}$$

such that the diagram of resolutions is exact. Thus we obtain a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(M') & \longrightarrow & F(M) & \longrightarrow & F(M'') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(I_{M'}) & \xrightarrow{\alpha} & F(I_M) & \xrightarrow{\beta} & F(I_{M''}) & \longrightarrow & 0
\end{array}$$

such that each row in the last line is exact.

Note that j-th cohomology of the complex $F(I_M)$ (respectively, $F(I_{M'}), F(I_{M''})$) gives the $R^jF(M)(respectively, (R^jF(M'), R^jF(M'').)$ Using the resolution given in the lemma above, we may prove Theorem 7.1. Let us describe the map δ in more detail when $\mathcal{C}_2$ is a category of modules by "diagram chasing". Namely, for $x \in R^n(M'')$, let us show how to compute $\delta(x)$.

(1) $x \in R^n(M'')$ may be represented as a class $[c_x]$ of a cocycle $c_x \in \operatorname{Ker}(d : F(I^n_{M''}) \to F(I^{n+1}_{M''}))$.
(2) We take a "lift" $\tilde{c}_x \in F(I^n_M)$ such that $\beta^n(\tilde{c}_x) = c_x$. Note that $\tilde{c}_x$ is no longer a cocycle.
(3) Consider $e_x = d\tilde{c}_x \in F(I^{n+1}M)$. It is a coboundary and we have $\beta(e_x) = 0$.
(4) There thus exists an element $a_x \in F(I^n_{M'})$ such that $\alpha(a_x) = e_x$. a_x is no longer a coboundary but it is a cocycle.
(5) The cohomology class $[a_x]$ of a_x is the required $\delta(x)$.

Such computation appears frequently when we deal with cohomologies.

Definition 7.3. Let A be a ring. Let M, N be A-modules. Then an **extension** of N by M is a module L with a exact sequence

(E) $$0 \to N \xrightarrow{\alpha} L \xrightarrow{\beta} M \to 0.$$

of A-modules. Let

$$0 \to N \xrightarrow{\alpha'} L' \xrightarrow{\beta'} M \to 0.$$

be another extension. Then the two extensions are said to be isomorphic if there exists a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{\alpha} & L & \xrightarrow{\beta} & M & \longrightarrow & 0 \\
 & & {\scriptstyle =}\downarrow & & \downarrow & & {\scriptstyle =}\downarrow & & \\
0 & \longrightarrow & N & \xrightarrow{\alpha'} & L' & \xrightarrow{\beta'} & M & \longrightarrow & 0.
\end{array}$$

Proposition 7.4. *There exists a bijection between the isomorphism classs of the extensions and elements of the* $\operatorname{Ext}^1_A(M, N)$. *The bijection is given by corresponding the extension (E) to the class* $\delta(1_N) \in \operatorname{Ext}^1(M, N)$ *of the identity map* 1_N *by* δ *associated to the exact sequence (E).*

See [1, XX,Exercise 27]

References

[1] S. Lang, *Algebra (graduate texts in mathematics)*, Springer Verlag, 2002.