COHOMOLOGIES.

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09. Cohomology of groups

Let G be a group. Let us consider a functor

$$F^G : M \mapsto M^G = \{m \in M; \quad g.m = m (\forall g \in G)\}$$

The functor is left-exact. The derived functor of this functor

$$H^j(G, M) = R^j F^G(M)$$

is called **the j-th cohomology of G with coefficients in M**. Let us consider $\mathbb{Z}$ as a G-module with trivial G-action. Then we may easily verify that

$$F^G(M) = M^G \cong \mathrm{Hom}_G(\mathbb{Z}, M).$$

Thus we have

$$H^j(G, M) = \mathrm{Ext}^j_G(\mathbb{Z}, M).$$

To compute cohomologies of G, it is useful to use $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$. For any tuples $g_0, g_1, g_2, \dots, g_t$ of G, we introduce a symbol

$$[g_0, g_1, g_2, \dots, g_t]$$

and we consider the following sequence

$(*_G)$

$$0 \leftarrow \mathbb{Z} \overset{d}{\leftarrow} \bigoplus_{g_0 \in G} \mathbb{Z} \cdot [g_0] \overset{d}{\leftarrow} \bigoplus_{g_0, g_1 \in G} \mathbb{Z} \cdot [g_0, g_1] \overset{d}{\leftarrow} \bigoplus_{g_0, g_1, g_2 \in G} \mathbb{Z} \cdot [g_0, g_1, g_2] \overset{d}{\leftarrow} \dots$$

where ϵ, d are determined by the following rules.

$$d([g_0]) = 1$$
$$d([g_0, g_1]) = [g_1] - [g_0]$$
$$d([g_0, g_1, g_2]) = [g_1, g_2] - [g_0, g_2] + [g_0, g_1]$$
$$d([g_0, g_1, g_2, g_3]) = [g_1, g_2, g_3] - [g_0, g_2, g_3] + [g_0, g_1, g_3] - [g_0, g_1, g_2]$$
$$\dots$$

To see that the sequence $*_G$ is acyclic, we consider a homotopy

$$h([g_0, g_1, \dots, g_t]) = [1, g_0, g_1, \dots, g_t]$$

EXERCISE 9.1. Show that $h \circ d + d \circ h = \mathrm{id}$

LEMMA 9.1. (1) *Each of the modules that appears in the sequence $*_G$ admits an action of G determined by*

$$g.[g_0, g_1, g_2, \dots, g_t] = [g \cdot g_0, g \cdot g_1, g \cdot g_2, \dots, g \cdot g_t]$$

(2)

$$C_t = \bigoplus_{g_0, g_1, g_2, \dots g_t \in G} \mathbb{Z} \cdot [g_0, g_1, g_2, \dots, g_t]$$

is $\mathbb{Z}[G]$-free

There are several choices for the $\mathbb{Z}[G]$-basis of C_t. One such is clearly

$$\{[1, g_1, g_2, g_3, \ldots, g_t]; g_1, g_2, \ldots, g_t \in G\}.$$

It is traditional (and probably useful) to use another basis

$$\{\langle g_1, g_2, g_3, \ldots, g_t\rangle; g_1, g_2, \ldots, g_t \in G\}.$$

where

$$\langle g_1, g_2, g_3 \ldots g_t\rangle = [1,\ g_1,\ g_1g_2,\ g_1g_2g_3,\ \ldots,\ g_1g_2g_3\ldots g_t].$$

Conversely we have

$$[1, a_1, a_2, \ldots, a_t] = \langle a_1, a_1^{-1}a_2, a_2^{-1}a_3, \ldots, a_{t-1}^{-1}a_t\rangle.$$

DEFINITION 9.2. For any group G, the derived functor of a functor

$$F_G : (G - modules) \to (modules)$$

defined by

$$M \mapsto M_G = M/(\mathbb{Z} - \mathrm{span}\{g.m - m; g \in G, M \in M\})$$

is called the homology of G with coefficients in M. We denote the homology group $L_jF_G(M)$ by $H_j(G;M)$.

LEMMA 9.3.

$$H_j(G;M) \cong \mathrm{Tor}_j^{\mathbb{Z}[G]}(\mathbb{Z}, M)$$