COMMUTATIVE ALGEBRA

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$\boxed{\text{01.Review of elementary definitions on modules.}}$

Definition 1.1. A (unital associative) **ring** is a set R equipped with two binary operations (addition ("+") and multiplication ("·")) such that the following axioms are satisfied.

(Ring-1) R is an additive group with respect to the addition.
(Ring-2) distributive law holds. Namely, we have

$$a(b+c) = ab + bc, \quad (a+b)c = ac + bc \qquad (\forall a, \forall b, \forall c \in R).$$

(Ring-3) The multiplcation is associative.
(Ring-4) R has a multiplicative unit.

In this lectuer we are mainly interested in **commutative rings**, that means, rings on which the multiplication satisfies the commutativity law.

For any ring R, we denote by 0_R (respectively, 1_R) the zero element of R (respectively, the unit element of R). Namely, 0_R and 1_R are elements of R characterized by the following rules.

- $a + 0_R = a, \quad 0_R + a = a \ \forall a \in R.$
- $a \cdot 1_R = a, \quad 1_R \cdot a = a \ \forall a \in R.$

When no confusion arises, we omit the subscript '$_R$' and write $0, 1$ instead of $0_R, 1_R$.

Definition 1.2. A map $R \to S$ from a unital associative ring R to another unital associative ring S is said to be **ring homomorphism** if it satisfies the following conditions.

(Ringhom-1) $f(a+b) = f(a) + f(b)$
(Ringhom-2) $f(ab) = f(a)f(b)$
(Ringhom-3) $f(1_R) = 1_S$

Definition 1.3. Let R be a unital associative ring. An R**-module** M is an additive group M with R-action

$$R \times M \to M$$

which satisfies

(Mod-1) $(r_1 r_2).m = r_1.(r_2.m) \quad (\forall r_1, \forall r_2 \in R, \forall m \in M)$
(Mod-2) $1.m = m \quad (\forall m \in M)$
(Mod-3) $(r_1 + r_2).m = r_1.m + r_2.m \quad (\forall r_1, \forall r_2 \in R, \forall m \in M).$
(Mod-4) $r.(m_1 + m_2) = r.m_1 + r.m_2 \quad (\forall r \in R, \forall m_1, \forall m_2 \in M).$

Example 1.4. Let us give some examples of R-modules.

(1) If k is a field, then the concepts "k-vector space" and "k-module" are identical.
(2) Every abelian group is a module over the ring of integers $\mathbb{Z}$ in a unique way.

DEFINITION 1.5. Let M, N be modules over a ring R. Then a map $f : M \to N$ is called an **R-module homomorphism** if it is additive and preserves the R-action.

The set of all module homomorphisms from M to N is denoted by $\mathrm{Hom}_R(M, N)$. It has an structure of an module in an obvious manner. Furthermore, when R is a commutative ring, then it has a structure of an R-module.

DEFINITION 1.6. An subset M of an R-module N is said to be an **R-submodule** of N if M itself is an R-module and the inclusion map $j : M \to N$ is an R-module homomorphism.

DEFINITION 1.7. An subset N of an R-module M is said to be an **R-submodule** of M if N itself is an R-module and the inclusion map $j : N \to M$ is an R-module homomorphism.

DEFINITION 1.8. Let R be a ring. Let N be an R-submodule of an R-module M. Then we may define the **quotient** M/N by

$$M/N = M/\sim_N$$

where the equivalence relation $\sim_N$ is defined as follows:

$$m_1 \sim_N m_2 \quad \iff \quad m_1 - m_2 \in N.$$

It may be shown that the quotient M/N so defined is actually an R-module and that the natural projection

$$\pi : M \to M/N$$

is an R-module homomorphism.

DEFINITION 1.9. Let $f : M \to N$ be an R-module homomorphism between R-modules. Then we define its **kernel** as follows.

$$\mathrm{Ker}(f) = f^{-1}(0) = \{m \in M; f(m) = 0\}.$$

The kernel and the image of an R-module homomorphism f are R-modules.

THEOREM 1.10. *Let $f : M \to N$ be an R-module homomorphism between R-modules. Then*

$$M/\mathrm{Ker}(f) \cong f(N).$$

DEFINITION 1.11. Let R be a ring. An "sequence"

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

is said to be **an exact sequence of R-modules** if the following conditions are satisfied

(Exact1) M_1, M_2 are R-modules.
(Exact2) f, g are R-module homomorphisms.
(Exact3) $\mathrm{Ker}(g) = \mathrm{Image}(f)$.

For any R-submodule N of an R-module M, we have the following exact sequence.

$$0 \to N \to M \to M/N \to 0$$

EXERCISE 1.1. Compute the following modules.

(1) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z})$.
(2) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$.

(3) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}/5\mathbb{Z})$.

Definition 1.12. Let A be an associative unital (but not necessarily commutative) ring. Let L be a right A-module. Let M be a left A-module. For any ($\mathbb{Z}$-)module N, an map

$$\varphi : L \times M \to N$$

is called an **A-balanced biadditive map** if

(1) $\varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y) \quad (\forall x_1, \forall x_2 \in L, \forall y \in M)$.
(2) $\varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2) \quad (\forall x \in L, \forall y_1, \forall y_2 \in M)$.
(3) $\varphi(xa, y) = \varphi(x, ay) \quad (\forall x \in L, \forall y \in M, \forall a \in A)$.

Proposition 1.13. *Let A be an associative unital (but not necessarily commutative) ring. Then for any right A-module L and for any left A-module M, there exists a ($\mathbb{Z}$-)module $X_{L,M}$ together with a A-balanced map*

$$\varphi_0 : L \times M \to X_{L,M}$$

which is universal amoung A-balanced maps.

Definition 1.14. We employ the assumption of the proposition above. By a standard argument on universal objects, we see that such object is unique up to a unique isomorphism. We call it the **tensor product** of L and M and denote it by

$$L \otimes_A M.$$

Lemma 1.15. *Let A be an associative unital ring. Then:*

(1) $A \otimes_A M \cong M$.
(2) $(L_1 \oplus L_2) \otimes_A M \cong (L_1 \otimes M) \oplus (L_2 \otimes_A M)$.
(3) *For any left A-module M, the functor $L \mapsto L \otimes_A M$ is a right exact functor. Namely, for any exact sequence*

$$0 \to L_1 \to L_2 \to L_3 \to 0,$$

the sequence

$$L_1 \otimes_A M \to L_2 \otimes_A M \to L_3 \otimes_A M \to 0,$$

is also exact.
(4) *For any right ideal J of A and for any A-module M, we have*

$$(A/J) \otimes_A M \cong M/J.M$$

In particular, if the ring A is commutative, then for any ideals I, J of A, we have

$$(A/I) \otimes_A (A/J) \cong A/(I + J)$$

Exercise 1.2. Compute $(\mathbb{Z}/3\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z})$ and $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z})$.

Definition 1.16. A left A-module M is said to be **flat** if $L \mapsto L \otimes_A M$ is an exact functor. Namely, for any exact sequence

$$0 \to L_1 \to L_2 \to L_3 \to 0,$$

of left A-modules, the sequence

$$0 \to L_1 \otimes_A M \to L_2 \otimes_A M \to L_3 \otimes_A M \to 0,$$

is also exact.

**The following two facts may give some intuitive idea of what flatness means.

THEOREM 1.17. *If A is a Noetherian ring and M is a finitely-generated R-module, then M is flat over A if and only if the associated sheaf $\tilde{M}$ on* $\operatorname{Spec}(A)$ *is locally free.*

THEOREM 1.18. [1, Theorem 23.1+Theorem 15.1] *Let $(A, \mathfrak{m}_A)$ be a regular local ring. Let $(B, \mathfrak{m}_B)$ be a Cohen-Macaulay local ring. Let $\varphi : A \to B$ be a local ring homomorphism. We set*

$$F = B \otimes_A k(\mathfrak{m}_A) = B/\mathfrak{m}_A B$$

for the fiber ring of φ over $\mathfrak{m}_A$. Then an equality

$$\dim B = \dim A + \dim F$$

holds if and only if B is flat over A.

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References

[1] H. Matsumura, *Commutative ring theory*, Cambridge university press, 1986.