COMMUTATIVE ALGEBRA

YOSHIFUMI TSUCHIMOTO

02. Localization

DEFINITION 2.1. Let A be a commutative ring. Let S be its subset. We say that S is multiplicative if

(1) $1 \in S$
(2) $x, y \in S \implies xy \in S$

holds.

DEFINITION 2.2. Let S be a multiplicative subset of a commutative ring A. Then we define $A[S^{-1}]$ as

$$A[\{X_s; s \in S\}]/(\{sX_s - 1; s \in S\})$$

where in the above notation X_s is a indeterminate prepared for each element $s \in S$.) We denote by ι_S a canonical map $A \to A[S^{-1}]$.

LEMMA 2.3. *Let S be a multiplicative subset of a commutative ring A. Then the ring $B = A[S^{-1}]$ is characterized by the following property:*

Let C be a ring, $\varphi : A \to C$ be a ring homomorphism such that $\varphi(s)$ is invertible in C for any $s \in S$. Then there exists a unique ring homomorphism $\psi = \phi[S^{-1}] : B \to C$ such that

$$\varphi = \psi \circ \iota_S$$

holds.

COROLLARY 2.4. *Let S be a multiplicative subset of a commutative ring A. Let I be an ideal of A given by*

$$I = \{x \in I; \exists s \in S \text{ such that } sx = 0\}$$

Then I is an ideal of A. Let us put $\bar{A} = A/I$, $\pi : A \to \bar{A}$ the canonical projection. Then:

(1) *$\bar{S} = \pi(S)$ is multiplicatively closed.*
(2) *We have*
$$A[S^{-1}] \cong \bar{A}[\bar{S}^{-1}]$$
(3) *$\iota_{\bar{S}} : \bar{A} \to \bar{A}[\bar{S}^{-1}]$ is injective.*

There is another description of $A[S^{-1}]$. Namely, We consider an equivalence relateion $\sim_S$ on a set $S \times A$ by

$$(s_1, a_1) \sim_S (s_2, a_2) \iff t(s_1 a_2 - s_2 a_1) = 0 (\exists t \in S)$$

We call the quotient space space $S \times A/\sim_S$ as $S^{-1}A$. The equivalence class of $(s, a) \in S \times A$ in $S^{-1}A$ is denoted by $s^{-1}a$. Then it is easy to introduce a ring structure of $S^{-1}A$ and see that $S^{-1}A$ actually satisfies the universal property of $A[S^{-1}]$. We thus have a canonical isomorphism $S^{-1}A \cong A[S^{-1}]$.

EXAMPLE 2.5. $A_f = A[S^{-1}]$ for $S = \{1, f, f^2, f^3, f^4, \dots\}$. The total ring of quotients $Q(A)$ is defined as $A[S^{-1}]$ for

$$S = \{x \in A; x \text{ is not a zero divisor of A}\}.$$

When A is an integral domain, then $Q(A)$ is the field of quotients of A.

DEFINITION 2.6. Let A be a commutative ring. Let $\mathfrak{p}$ be its prime ideal. Then we define the localization of A with respect to $\mathfrak{p}$ by

$$A_{\mathfrak{p}} = A[(A \setminus \mathfrak{p})^{-1}]$$

DEFINITION 2.7. Let S be a multiplicative subset of a commutative ring A. Let M be an A-module we may define $S^{-1}M$ as

$$\{(m/s); m \in M, s \in S\}/\sim$$

where the equivalence relation $\sim$ is defined by

$$(m_1/s_1) \sim (m_2/s_2) \iff t(m_1 s_2 - m_2 s_1) = 0 \quad (\exists t \in S).$$

We may introduce a $S^{-1}A$-module structure on $S^{-1}M$ in an obvious manner.

$S^{-1}M$ thus constructed satisfies an universality condition which the reader may easily guess.

By a universality argument, we may easily see the following proposition.

PROPOSITION 2.8. *Let A be a commutative ring. Let S be a multiplicative subet of A. Let M be an A-module. Then we have an isomorphism*

$$S^{-1}A \otimes_A M \cong S^{-1}M$$

of $S^{-1}A$-modules.

PROPOSITION 2.9. *Let A be a commutative ring. Let S be a multiplicative subet of A. Then the natural homomorphism $A \to S^{-1}A$ is flat.*

2.1. **local rings.**

DEFINITION 2.10. A commutative ring A is said to be a local ring if it has only one maximal ideal.

EXAMPLE 2.11. We give examples of local rings here.

- Any field is a local ring.
- For any commutative ring A and for any prime ideal $\mathfrak{p} \in \mathrm{Spec}(A)$, the localization $A_{\mathfrak{p}}$ is a local ring with the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.

LEMMA 2.12. (1) *Let A be a local ring. Then the maximal ideal of A coincides with $A \setminus A^{\times}$.*
(2) *A commutative ring A is a local ring if and only if the set $A \setminus A^{\times}$ of non-units of A forms an ideal of A.*

PROOF. (1) Assume A is a local ring with the maximal ideal $\mathfrak{m}$. Then for any element $f \in A \setminus A^{\times}$, an ideal $I = fA + \mathfrak{m}$ is an ideal of A. By Zorn's lemma, we know that I is contained in a maximal ideal of A. From the assumption, the maximal ideal should be $\mathfrak{m}$. Therefore, we have

$$fA \subset \mathfrak{m}$$

which shows that

$$A \setminus A^{\times} \subset \mathfrak{m}.$$

The converse inclusion being obvious (why?), we have

$$A \setminus A^{\times} = \mathfrak{m}.$$

(2) The "only if" part is an easy corollary of (1). The "if" part is also easy.

□

COROLLARY 2.13. *Let A be a commutative ring. Let $\mathfrak{p}$ its prime ideal. Then $A_{\mathfrak{p}}$ is a local ring with the only maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$.*

DEFINITION 2.14. Let A, B be local rings with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$ respectively. A local homomorphism $\varphi : A \to B$ is a homomorphism which preserves maximal ideals. That means, a homomorphism φ is said to be loc al if

$$\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$$

EXAMPLE 2.15 (of NOT being a local homomorphism).

$$\mathbb{Z}_{(p)} \to \mathbb{Q}$$

is not a local homomorphism.

In the argument above, we have used the following lemma.

LEMMA 2.16 (Zorn's lemma). *Let $\mathcal{S}$ be a partially ordered set. Assume that every totally ordered subset of $\mathcal{S}$ has an upper bound in $\mathcal{S}$. Then $\mathcal{S}$ has at least one maximal element.*

We prove here another consequence of the lemma.

PROPOSITION 2.17. *Let A be a commutative ring. let I be an ideal of A such that $A \neq I$. Then there exists a maximal ideal $\mathfrak{m}$ of A which contains I.*