COMMUTATIVE ALGEBRA

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$\boxed{\text{Smoothness and finiteness properties}}$

DEFINITION 9.1. Let A be a ring. an A-module P is said to be **projective** if it satisfies the following condition: For any A-module morphism $f : P \to N$ and for any surjective A-module homomorphism $\pi : M \to N$, f "lifts" to an A-module morphism $\hat{f} : M \to I$.

$$\begin{tikzcd} P \arrow[r, "\hat{f}"] \arrow[d, equal] & M \arrow[d, "\pi"] \\ P \arrow[r, "f"] & N \end{tikzcd}$$

LEMMA 9.2. *An A-module M is projective if and only if it is a direct summand of a free A-module.*

PROPOSITION 9.3. *Let B be a* 0*-smooth algebra over a ring A. Then $\Omega^1_{B/A}$ is projective.*

PROOF. Let us express the algebra B as a quotient C/I where C is a polynomial algebra and I is an ideal of C. Then by Theorem 08.5, we know that

$$0 \to I/I^2 \to \Omega^1_C \otimes_C B \to \Omega^1_B \to 0$$

is split exact. So Ω^1_B is a direct sum of $\Omega^1_C \otimes_C B$. On the other hand, Ω^1_C is free C-module so that $\Omega^1_C \otimes_C B$ is also a free B-module.

□

We would like to define "smoothness" as a something good. Especially, we would expect "smooth algebras" to be flat. But that is not always true if we regard "smoothness" as 0-smoothness. The following example is an easy case of [1, example 7.2].

EXAMPLE 9.4. Let us put $A = \mathbb{C}[\{\sqrt[2^n]{T}\}_{n=1}^{\infty}] = \mathbb{C}[T, \sqrt{T}, \sqrt[2]{T}, \sqrt[4]{T}, \dots,]$ and put

$$I = \{f \in A; f(0) = 0\} = \sum_{n=1}^{\infty} \sqrt[2^n]{T} A.$$

Then we see that $I^2 = I$. Thus A/I is 0-smooth over A. where as A/I is not flat over A.

DEFINITION 9.5. Let A be a ring.

(1) An A-algebra B is said to be **finitely generated** over A if B is generated by a finite set as an A-algebra. In other words, it is finitely generated if there exists a surjective A-algebra homomorphism from a finitely generated polynomial ring $A[X_1, X_2, \dots, X_r]$ to B.
(2) An A-algebra B is said to be **finitely presented** over A if there exists a surjective A-algebra homomorphism φ from a finitely generated polynomial ring $P = A[X_1, X_2, \dots, X_r]$ to B such that its kernel is a finitely generated ideal of P. is a finitely generated ideal of P.

Definition 9.6. Let A be a ring. An A-algebra B is said to be **smooth** over A if it is 0-smooth and finitely presented over A.

We may define unramified/étale algebras in a same manner.

Let us recall the definition of Noetherian ring.

Definition 9.7. A ring is called **Noetherian** if its ideals are always finitely generated.

Proposition 9.8. *If A is Noetherian, then:*

(1) *Any of its quotient ring is Noetherian.*
(2) *The polynomial ring $A[X]$ is Noetherian.*

It follows that any finitely generated A-algebra B is also Noetherian. We note also that B is finitely presented over A in this case.

References

[1] M. Maltenfort, *A new look at smoothness*, Expo. Math. (2002), 59–93.