COMMUTATIVE ALGEBRA

YOSHIFUMI TSUCHIMOTO

03. Algebraic geometry of affine schemes

Definition 3.1. For any commutative ring A, we define its **spectrum** as

$$\operatorname{Spec}(A) = \{\mathfrak{p} \subset A; \mathfrak{p} \text{ is a prime ideal of } A.$$

For any subset S of A we define

$$V(S) = V_{\operatorname{Spec} A}(S) = \{\mathfrak{p} \in \operatorname{Spec} A; \mathfrak{p} \supset S\}$$

Then we may topologize $\operatorname{Spec}(A)$ in such a way that the closed sets are sets of the form $V(S)$ for some S. Namely,

$$F : \text{closed} \iff \exists S \subset A(F = V(S))$$

We refer to the topology as the **Zariski topology**.

Lemma 3.2. *For any ring A, the following facts holds.*

(1) *For any subset S of A, we have*

$$V(S) = \bigcap_{s \in S} V(\{s\}).$$

(2) *For any subset S of A, let us denote by $\langle S \rangle$ the ideal of A generated by S. then we have*

$$V(S) = V(\langle S \rangle)$$

Proposition 3.3. *For any ring homomorphism $\varphi : A \to B$, we have a map*

$$\operatorname{Spec}(\varphi) : \operatorname{Spec}(B) \ni \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p}) \in \operatorname{Spec}(A).$$

It is continuous with respect to the Zariski topology.

Proposition 3.4. *For any ring A, the following statements hold.*

(1) *For any ideal I of A, let us denote by $\pi_I : A \to A/I$ the canonical projection. Then $\operatorname{Spec}(\pi_I)$ gives a homeomorphism between $\operatorname{Spec}(A/I)$ and $V_{\operatorname{Spec} A}(I)$.*
(2) *For any element s of A, let us denote by $\iota_s : A \to A[s^{-1}]$ be the canonical map. Then $\operatorname{Spec}(\iota_s)$ gives a homeomorphism between $\operatorname{Spec}(A[s^{-1}])$ and $\complement V_{\operatorname{Spec}_A}(\{s\})$.*

Proposition 3.5. *Let A, B be a ring. Let $\varphi : A \to B$ be a ring homomoprhism. We regard B as an A module via φ. If B is a finite A-module, then*

$$\operatorname{Spec}(\varphi) : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$$

is a closed map with respect to the Zariski topology.

Definition 3.6. Let X be a topological space. A closed set F of X is said to be **reducible** if there exist closed sets F_1 and F_2 such that

$$F = F_1 \cup F_2, \quad F_1 \neq F, F_2 \neq F$$

holds. F is said to be **irreducible** if it is not reducible.

DEFINITION 3.7. Let I be an ideal of a ring A. Then we define its **radical** to be

$$\sqrt{I} = \{x \in A; \exists N \in \mathbb{Z}_{>0} \text{ such that } x^N \in I\}.$$

PROPOSITION 3.8. *Let A be a ring. Then;*

(1) *For any ideal I of A, we have $V(I) = V(\sqrt{I})$.*
(2) *For two ideals I, J of A, $V(I) = V(J)$ holds if and only if $\sqrt{I} = \sqrt{J}$.*
(3) *For an ideal I of A, $V(I)$ is irreducible if and only if $\sqrt{I}$ is a prime ideal.*

DEFINITION 3.9. We define a dimension of a topological space X as a supremum of the length of sequences

$$Y_1 \supsetneq Y_2 \supsetneq Y_3 \supsetneq \cdots \supsetneq Y_s$$

of irreducible subsets of X.

We define the Krull dimension of a ring A as the dimension of $\mathrm{Spec}(A)$.