ZETA FUNCTIONS. NO.3

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In this lecture we define and observe some properties of conguent zeta functions.

3.1. **Definition of congruent Zeta function.**

Definition 3.1. Let q be a power of a prime. Let $V = \{f_1, f_2, \dots, f_m\}$ be a set of polynomial equations in n-variables over $\mathbb{F}_q$. We denote by $V(\mathbb{F}_{q^s})$ the set of solutions of V in $(\mathbb{F}_{q^s})^n$. That means,

$$V(\mathbb{F}_{q^s}) = \{x \in (\mathbb{F}_{q^s})^n; f_1(x) = 0, f_2(x) = 0, \dots, f_m(x) = 0\}.$$

Then we define

$$Z(V/\mathbb{F}_q, T) = \exp(\sum_{s=1}^{\infty}(\frac{1}{s}\#V(\mathbb{F}_{q^s})T^s)).$$

Exercise 3.1. Compute congruent zeta function for $V = \{XY\}$ (an equation on two variables).

Exercise 3.2. Compute congruent zeta function for $V = \{X^2 + Y^2 - 1\}$ (an equation on two variables).

3.2. **First properties of congruent Zeta function.**

Let us first recall an elementary formula

Lemma 3.2.

$$\sum_{k=1}^{\infty} \frac{1}{k} T^k = -\log(1 - T)$$

Definition 3.3. We denote by $\mathbb{A}_n$ the "void set of equation" in n-variables. That means, for any field (or ring) k, we put

$$\mathbb{A}_n(k) = \{x \in k^n\}.$$

PROPOSITION 3.4.

$$Z(\mathbb{A}_n/\mathbb{F}_q, T) = \frac{1}{1 - q^n T}$$

PROPOSITION 3.5. *Let V, W, W_1, W_2 be sets of equations.*

(1) *If $\#V(\mathbb{F}_{q^s}) = \#W(\mathbb{F}_{q^s})$ for any s ,then $Z(V/\mathbb{F}_q, T) = Z(W/\mathbb{F}_q, T)$.*
(2) *If $\#V(\mathbb{F}_{q^s}) = \#W_1(\mathbb{F}_{q^s}) + \#W_2(\mathbb{F}_{q^s})$ for any s ,then:*

$$Z(V/\mathbb{F}_q, T) = Z(W_1/\mathbb{F}_q, T) Z(W_2/\mathbb{F}_q, T).$$

PROPOSITION 3.6. *Let $f \in \mathbb{F}_q[X]$ be an irreducible polynomial in one variable of degree d. Let us consider $V = \{f\}$, an equation in one variable. Then:*

(1)
$$V(\mathbb{F}_{q^s}) = \begin{cases} d & \text{if } d|s \\ 0 & \text{otherwise} \end{cases}$$
(2)
$$Z(V/\mathbb{F}_q, T) = \frac{1}{1 - T^d}$$

EXERCISE 3.3. Describe what happens when we omit the assumption of f being irreducible in Proposition 3.6.