ZETA FUNCTIONS. NO.4

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PROPOSITION 4.1. *Let $f \in \mathbb{F}_q[X]$ be an irreducible polynomial in one variable of degree d. Let us consider $V = \{f\}$, an equation in one variable. Then:*

(1)
$$V(\mathbb{F}_{q^s}) = \begin{cases} d & \textit{if } d|s \\ 0 & \textit{otherwise} \end{cases}$$

(2)
$$Z(V/\mathbb{F}_q, T) = \frac{1}{1-T^d}$$

$\boxed{\text{projective space and projective varieties.}}$

DEFINITION 4.2. Let R be a ring. A polynomial $f(X_0, X_1, \dots, X_n) \in R[X_0, X_1, \dots, X_n]$ is said to be **homogenius** of degree d if an equality

$$f(\lambda X_0, \lambda X_1, \dots, \lambda X_n) = \lambda^d f(X_0, X_1, \dots, X_n)$$

holds as a polynomial in $n+2$ variables $X_0, X_1, X_2, \dots, X_n, \lambda$.

DEFINITION 4.3. Let k be a field.

(1) We put
$$\mathbb{P}^n(k) = (k^{n+1} \setminus \{0\})/k^\times$$
and call it (the set of k-valued points of) the **projective space**. The class of an element $(x_0, x_1, \dots, x_n)$ in $\mathbb{P}^n(k)$ is denoted by $[x_0 : x_1 : \cdots : x_n]$.

(2) Let $f_1, f_2, \dots, f_l \in k[X_0, \dots, X_n]$ be homogenious polynomials. Then we set
$$V_h(f_1, \dots, f_l) = \{[x_0 : x_1 : x_2 : \dots x_n]; f_j(x_0, x_1, x_2, \dots, x_n) = 0 \qquad (j = 1, 2, 3, \dots, l)\}.$$
and call it (the set of k-valued point of) the **projective variety** defined by $\{f_1, f_2, \dots, f_l\}$.

(Note that the condition $f_j(x) = 0$ does not depend on the choice of the representative $x \in k^{n+1}$ of $[x] \in \mathbb{P}^n(k)$.)

LEMMA 4.4. *We have the following picture of $\mathbb{P}^2$.*

(1)
$$\mathbb{P}^2 = \mathbb{A}^2 \coprod \mathbb{P}^1.$$
That means, $\mathbb{P}^2$ *is divided into two pieces* $\{Z \neq 0\} = \complement V_h(Z)$ *a nd* $V_h(Z)$*.*

(2)
$$\mathbb{P}^2 = \mathbb{A}^2 \cup \mathbb{A}^2 \cup \mathbb{A}^2.$$
That means, $\mathbb{P}^2$ *is covered by three "open sets"* $\{Z \neq 0\}, \{Y \neq 0\}, \{X \neq 0\}$*. Each of them is isomorphic to the plane (that is, the affine space of dimension 2).*

We quote the famous $\boxed{\text{Weil conjecture}}$

CONJECTURE 4.5 (Now a theorem [1]). Let X be a projective smooth variety of dimension d. Then:

W1. (Rationality)
$$Z(X,T) = \frac{P_1(X,T)P_3(X,T)\dots P_{2d-1}(X,T)}{P_0(X,T)P_2(X,T)\dots P_{2d}(X,T)}$$

W2. (Integrality) $P_0(X,T) = 1 - T$, $P_{2d}(X,T) = 1 - q^d T$, and for each r, P_r is a polynomial in $\mathbb{Z}[T]$ which is factorized as
$$P_r(X,T) = \prod (1 - a_{r,i}T)$$
where $a_{r,i}$ are algebraic integers.

W3. (Functional Equation)
$$Z(X, 1/q^d T) = \pm q^{d\chi/2} T^\chi Z(t)$$
where $\chi = (\Delta.\Delta)$ is an integer.

W4. (Rieman Hypothesis) each $a_{r,i}$ and its conjugates have absolute value $q^{r/2}$.

W5. If X is the specialization of a smooth projective variety Y over a number field, then the degeee of $P_r(X,T)$ is equal to the r-th Betti number of the complex manifold $Y(\mathbb{C})$. (When this is the case, the number χ above is equal to the "Euler characteristic" $\chi = \sum_i (-1)^i b_i$ of $Y(\mathbb{C})$.)

It is a profound theorem, relating rational points $X(\mathbb{F}_q)$ of X over finite fields and topology of $Y(\mathbb{C})$.

The following proposition (which is a precursor of the above conjecture) is a special case

[1] There are a lot of people who contributed. See the references.

PROPOSITION 4.6 (Weil). *Let E be an elliptic curve over $\mathbb{F}_q$. Then we have*

$$Z(E/\mathbb{F}_q, T) = \frac{1 - aT + qT^2}{(1-T)(1-qT)}.$$

where a is an integer which satisfies $|a| \leq 2\sqrt{q}$.

Note that for each E we have only one unknown integer a to determine the Zeta function. So it is enough to compute $\#E(\mathbb{F}_q)$. to compute the Zeta function of E. (When $q = p$ then one may use the result in the preceding section.)

For a further study we recommend [1, Appendix C],[2].

References

[1] R. Hartshorne, *Algebraic geometry*, Springer Verlag, 1977.
[2] J. S. Milne, *Étale cohomology*, Princeton University Press, 1980.