ZETA FUNCTIONS. NO.9

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In this lecture we make use of a scheme X of finite type over $\mathbb{Z}$, It is a patchwork of affine schemes of finite type over $\mathbb{Z}$,

An affine scheme X of finite type over $\mathbb{Z}$, in turn, is related to a set $f_1, \dots f_m$ of polynomial equations of coefficients in $\mathbb{Z}$, and is written as $\mathrm{Spec}(A)$ for a ring $A = \mathbb{Z}[X_1, \dots X_n]/(f_1, \dots f_m)$ of finite type over $\mathbb{Z}$, We consider quasi coherent sheaves over these objects. When X is affine ($X = \mathrm{Spec}(A)$), the category of quasi coherent sheaves over X is equivalent to the category of A-modules.

For any scheme X of finite type over $\mathbb{Z}$, we put

$$\zeta(X, s) = \prod_{x \in \mathrm{Spm}(X)} (1 - (Nx)^{-s})^{-1}$$

It is equal to the zeta function of the category of quasi coherent sheaves on X.

Recall we have defined the congruent zeta function as

$$Z(X, t) = \exp(\sum_{m=1}^{\infty} \frac{\#X(\mathbb{F}_{q^m})}{m} t^m).$$

PROPOSITION 9.1. *Let X be a scheme of finite type over $\mathbb{F}_q$. Then we have*

$$\zeta(X, s) = Z(X, q^{-s}).$$

PROOF.

$$
\begin{aligned}
&\log(\zeta(X,s))\\
&=-\sum_{\mathfrak{m}\in \mathrm{Spm}(X)} \log(1-N(\mathfrak{m})^{-s})\\
&=\sum_{\mathfrak{m}}\sum_{r=1}^{\infty}\frac{N(\mathfrak{m})^{-rs}}{r}\\
&=\sum_{u=1}^{\infty}\sum_{\substack{\mathfrak{m}\\ [R/\mathfrak{m}:\mathbb{F}_q]=u}}\sum_{r=1}^{\infty}\frac{N(\mathfrak{m})^{-rs}}{r}\\
&=\sum_{u=1}^{\infty}\frac{\#X(\mathbb{F}_{q^u})_*}{u}\sum_{r=1}^{\infty}\frac{q^{-urs}}{r}=(\heartsuit).
\end{aligned}
$$

where we put

$$
X(\mathbb{F}_{q^u})_* = X(\mathbb{F}_{q^u})\setminus \cup_{t<u}(X(\mathbb{F}_{q^t})).
$$

Now let us put $t=q^{-s}$ and proceed further.

$$
\begin{aligned}
(\heartsuit) &= \sum_{u=1}^{\infty}\sum_{r=1}^{\infty}\frac{\#X(\mathbb{F}_{q^u})_*}{ur}t^{ur}\\
&=\sum_{m}\frac{t^m}{m}\sum_{ur=m}\#X(\mathbb{F}_{q^u})_*\\
&=\sum_{m}\frac{t^m}{m}\#X(\mathbb{F}_{q^m})\\
&=-\log(Z(X,t))
\end{aligned}
$$

□

PROPOSITION 9.2. *Let X be a scheme of finite type over $\mathbb{Z}$. then we have*

$$
\zeta(X,s)=\prod_{p:\text{prime}}\zeta((X \mod p),s).
$$

Where we define $X \mod p$ as a fiber product $X\times_{\mathrm{Spec}\,\mathbb{Z}}\mathrm{Spec}\,\mathbb{F}_p$.

LEMMA 9.3.

$$
Z(X\times\mathbb{A}^1,t)=Z(X,qt).
$$

$$
\zeta(X\times\mathbb{A}^1,s)=\zeta(X,s-1)
$$