

CONGRUENT ZETA FUNCTIONS. NO.11

YOSHIFUMI TSUCHIMOTO

Let f be a self map $f : M \rightarrow M$ of a set M . It defines a (discrete) **dynamical system** (M, f) .

To explain the basic idea, we first examine the case where M is a finite set.

We put $A = C(M, \mathbb{C})$, the set of \mathbb{C} -valued functions on M .

f defines a pull-back of functions:

$$f^*(a)(x) = a(f(x)) \quad (a \in A)$$

and push-forward:

$$f_*(a)(x) = \sum_{f(y)=x} a(y) \quad (a \in A).$$

(It might be better to treat the push-forward as above as a push-forward of measures.)

We note also that any element of A admits an integration

$$\int_M a = \sum_{x \in M} a(x) \quad (a \in A)$$

(which is a integration with respect to the counting measure.)

PROPOSITION 11.1. *We have*

$$\int_M (f^* a) b = \int_M a (f_* b)$$

In other words, f_ is the adjoint of f^* .*

PROPOSITION 11.2. *Let us put $M = \{1, 2, \dots, n\}$. Let e_1, \dots, e_n be the indicators of elements of M . Then $\{e_1, \dots, e_n\}$ forms a basis of A . f^* is represented by a matrix $P_f = (\delta_{f(i)j})$. f_* is represented by a matrix ${}^t P_f = (\delta_{if(j)})$.*

DEFINITION 11.3. We define the set $\text{Fix}(f)$ as the set of fixed points of f . Namely,

$$\text{Fix}(f) = \{x \in M; f(x) = x\}.$$

PROPOSITION 11.4. $\text{tr}(f^*) = \text{tr}(f_*) = \# \text{Fix}(f)$.

It should be noted that $\text{tr}((f^k)^*)$ may be comuted using a “path-integral”-like formula.

$$\text{tr}((f^k)^*) = \sum_{\alpha \in M^k} P_{\alpha_1 \alpha_2} P_{\alpha_2 \alpha_3} \cdots P_{\alpha_{k-1} \alpha_k} P_{\alpha_k \alpha_1}$$

DEFINITION 11.5. We define the Artin-Mazur zeta function of a dynamical system (M, f) as

$$Z((M, f), T) = \exp\left(\sum_{j=1}^{\infty} \frac{\# \text{Fix}(f^j) T^j}{j}\right)$$

PROPOSITION 11.6.

$$Z((M, f), T) = \frac{1}{\det(1 - T f^*)}$$

11.1. **Congruent zeta as a zeta of a dynamical system.** The definition of Artin Mazur zeta function is valid without assuming the number of the base space M to be a finite set.

DEFINITION 11.7. Let M be a set. Let $f : M \rightarrow M$ be a map such that $\# \text{Fix}(f^n)$ is finite for any $n > 0$. We define the Artin-Mazur zeta function of a dynamical system (M, f) as

$$Z((M, f), T) = \exp\left(\sum_{j=1}^{\infty} \frac{\# \text{Fix}(f^j) T^j}{j}\right)$$

Let q be a power of a prime p . We may consider an automorphism Frob_q of $\bar{\mathbb{F}}_q$ over \mathbb{F}_q by

$$\text{Frob}_q(x) = x^q$$

PROPOSITION 11.8. $\text{Frob}_q : \bar{\mathbb{F}}_q \rightarrow \bar{\mathbb{F}}_q$ is an automorphism of order r . It is a generator of the Galois group $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$.

For any projective variety X defined over \mathbb{F}_q , we may define a Frobenius action Frob_q on $X(\bar{\mathbb{F}}_q)$:

$$\text{Frob}_q([x_0 : x_1 : \dots : x_N]) = ([x_0^q : x_1^q : \dots : x_N^q]).$$

For any $\bar{\mathbb{F}}_q$ -valued point $x \in X(\bar{\mathbb{F}}_q)$, We have

$$\text{Frob}_q^r(x) = x \iff x \in X(\mathbb{F}_{q^r}).$$

PROPOSITION 11.9. The Artin Mazur zeta function of the dynamical system $(X(\bar{\mathbb{F}}_q), \text{Frob}_q)$ coincides with the congruent zeta function $Z(X/\mathbb{F}_q, t)$.