# $\mathbb{Z}_p$, $\mathbb{Q}_p$, AND THE RING OF WITT VECTORS

No.06: ring of Witt vectors (2)

6.1. $\Lambda(A)$ **for arbitrary commutative ring** $A$**.** In the previous lecture we defined the ring structure on $\Lambda(A)$ for $A = \Omega$, a field of characteristic 0. Now we want to define the structure for arbitrary commutative ring $A$. Note that addition is already known:

$$(f)_W + (g)_W = (fg)_W$$

We would like to know the product $(f)_W(g)_W$. Before doing that, we consider "universal" power serieses:

$$a(T) = 1 + a_1 T + a_2 T^2 + a_3 T^3 + \dots,$$
$$b(T) = 1 + b_1 T + b_2 T^2 + b_3 T^3 + \dots,$$

with $a_1, a_2, \dots, b_1, b_2, b_3, \dots$ be all independent variables. We need a fairly large field $\Omega$, namely,

$$\Omega = \overline{\mathbb{Q}(a_1, a_2, \dots, b_1, b_2, \dots)},$$

the algebraic closure of an infinite trancendent extension of $\mathbb{Q}$. We find:

$$(a(T))_W(b(T))_W = (m_{a,b}(T))W$$

where

$$m_{a,b}(T) = 1 + m_{a,b;1}T + m_{a,b;2}T^2 + m_{a,b;3}T^3 + \dots$$

with $m_{a,b;k} \in \Omega$.

We also see:

- For fixed $a$, $m_{a,b,k}$ only depend on $b_1, b_2, b_3, \dots, b_k$. (In other words, it is an element of $\overline{\mathbb{Q}(a_1, \dots, a_k, b_1, b_2, \dots, b_k)}$.
- By using a Galois-theoretic arguments (or by using arguments on symmetric polynomials,) we see that $m_{a,b,k}$ actually lie in $\mathbb{Q}(a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k)$.
- $m_{a,b,k}$ is integral over the polynomial ring $\mathbb{Z}[a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k]$. It is thus itself belongs to the ring $\mathbb{Z}[a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k]$.
- The fact that $\Lambda(\Omega)$ obeys each of the axioms of ring, such as

  $$((a)_W(b)_W)(c)_W = (a)_W((b)_W(c)_W)$$

  (associativity), gives a set of polynomial identities in $a, b, c, m_{ab;k}, m_{bc,k}$. Such identities in term guarantees that for any ring $A$, $\Lambda(A)$ satisfy such axiom.

Proposition 6.1. *For any commutative ring $A$, $\Lambda(A)$ carries the structure of a ring.*

6.2. **Yet another way to deal with the multiplication of** $\Lambda(A)$**.**

Proposition 6.2. *$\Lambda(A)$ is generated by $\{(1 - cT^n)_W; c \in A, n \in \mathbb{N}\}$ as a topological additve group.*

Proof. Induction. (We leave it as Exercise 6.1) □

PROPOSITION 6.3. *Let* $a, b \in A$. *Assume* $n, m \in \mathbb{Z}_{>0}$ *such that* $\gcd(n, m) = d$, $n = n_1 d, m = n_2 d$. *Then:*

$$(1 - aT^n)_W (1 - bT^m)_W = (1 - a^{m_1} b^{n_1} T^{n_1 m_1 d})^d$$

(Exercise 6.2)* Note: The answer can be somewhat different than that in the statement. Sorry about that.

COROLLARY 6.4. *The multiplication of* $\Lambda(A)$ *surely remain in* $\Lambda(A)$ *as it should be.*