

# CONGRUENT ZETA FUNCTIONS. NO.4

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## 4.1. Definition of congruent Zeta function.

DEFINITION 4.1. Let  $q$  be a power of a prime. Let  $\{f_1, f_2, \dots, f_m\}$  be a set of polynomial equations in  $n$ -variables over  $\mathbb{F}_q$ . Recall that we have defined in section 2 the affine variety  $V = \text{Spec}(\mathbb{F}_q[X_1, \dots, X_n]/(f_1, \dots, f_m))$ . We may identify  $V(\mathbb{F}_{q^s})$  with the set of solutions of  $\{f_1, \dots, f_m\}$  in  $(\mathbb{F}_{q^s})^n$ . That means,

$$V(\mathbb{F}_{q^s}) = \{x \in (\mathbb{F}_{q^s})^n; f_1(x) = 0, f_2(x) = 0, \dots, f_m(x) = 0\}.$$

Then we define

$$Z(V/\mathbb{F}_q, T) = \exp\left(\sum_{s=1}^{\infty} \left(\frac{1}{s} \#V(\mathbb{F}_{q^s}) T^s\right)\right).$$

EXERCISE 4.1. Compute congruent zeta function for  $V = \text{Spec}(\mathbb{F}_q[X, Y]/(XY))$ .

EXERCISE 4.2. Compute congruent zeta function for  $V = \text{Spec}(\mathbb{F}_q[X, Y]/(X^2 + Y^2 - 1))$ .

4.2. **First properties of congruent Zeta function.** Let us first recall an elementary formula

LEMMA 4.2.

$$\sum_{k=1}^{\infty} \frac{1}{k} T^k = -\log(1 - T)$$

DEFINITION 4.3. Let  $\mathbb{k}$  be a ring. We define  $\mathbb{A}^n$  as the affine spectrum of the polynomial ring  $\mathbb{k}[X_1, \dots, X_n]$ . For any field (or ring)  $L$  over  $\mathbb{k}$ , we have

$$\mathbb{A}^n(L) = \{(x_1, x_2, \dots, x_n); x_1, x_2, \dots, x_n \in L\}.$$

PROPOSITION 4.4.

$$Z(\mathbb{A}^n/\mathbb{F}_q, T) = \frac{1}{1 - q^n T}$$

PROPOSITION 4.5. *Let  $V, W, W_1, W_2$  be affine varieties.*

- (1) *If  $\#V(\mathbb{F}_{q^s}) = \#W(\mathbb{F}_{q^s})$  for any  $s$ , then  $Z(V/\mathbb{F}_q, T) = Z(W/\mathbb{F}_q, T)$ .*
- (2) *If  $\#V(\mathbb{F}_{q^s}) = \#W_1(\mathbb{F}_{q^s}) + \#W_2(\mathbb{F}_{q^s})$  for any  $s$ , then:*

$$Z(V/\mathbb{F}_q, T) = Z(W_1/\mathbb{F}_q, T)Z(W_2/\mathbb{F}_q, T).$$

PROPOSITION 4.6. *Let  $f \in \mathbb{F}_q[X]$  be an irreducible polynomial in one variable of degree  $d$ . Let us consider  $V = \text{Spec}(\mathbb{k}[X]/(f))$ . Then:*

(1)

$$V(\mathbb{F}_{q^s}) = \begin{cases} d & \text{if } d|s \\ 0 & \text{otherwise} \end{cases}$$

(2)

$$Z(V/\mathbb{F}_q, T) = \frac{1}{1 - T^d}$$

EXERCISE 4.3. Describe what happens when we omit the assumption of  $f$  being irreducible in Proposition 4.6.