AFFINE GROUP SCHEMES

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Definition 2.1. A **category** $\mathcal{C}$ is a collection of the following data

(1) A collection $\mathrm{Ob}(\mathcal{C})$ of **objects** of $\mathcal{C}$.
(2) For each pair of objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set
$$\mathrm{Hom}_{\mathcal{C}}(X, Y)$$
of **morphisms**.
(3) For each triple of objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map("composition (rule)")
$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

satisfying the following axioms

(1) $\mathrm{Hom}(X, Y) \cap \mathrm{Hom}(Z, W) = \emptyset$ unless $(X, Y) = (Z, W)$.
(2) (Existence of an identity) For any $X \in \mathrm{Ob}(\mathcal{C})$, there exists an element $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ such that
$$\mathrm{id}_X \circ f = f, \quad g \circ \mathrm{id}_X = g$$
holds for any $f \in \mathrm{Hom}(S, X), g \in \mathrm{Hom}(X, T)$ $(\forall S, T \in \mathrm{Ob}(\mathcal{C}))$.
(3) (Associativity) For any objects $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$, and for any morphisms $f \in \mathrm{Hom}(X, Y), g \in \mathrm{Hom}(Y, Z), h \in \mathrm{Hom}(Z, W)$, we have
$$(f \circ g) \circ h = f \circ (g \circ h).$$

Definition 2.2. A functor F from a category $\mathcal{C}_1$ to a category $\mathcal{C}_2$ is a collection of the following data

(1) a map sending each object $X \in \mathrm{Ob}(\mathcal{C}_1)$ to an object $F(X) \in \mathrm{Ob}(\mathcal{C}_2)$
(2) a map sending each morphism $f : X \to Y$ in $\mathrm{Hom}_{\mathcal{C}_1}(X, Y)$ to a morphism $F(f) : F(X) \to F(Y)$

satisfying the following axioms.

(1) F preserves composition: $F(g \circ f) = F(g) \circ F(f)$ whenever the left-hand side is well-defined,
(2) F preserves identity morphisms: for each object $X \in \mathrm{Ob}(\mathcal{C}_1)$, $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Definition 2.3. An affine group scheme is a representable functor F: $(rings) \to (groups)$.

(Recall that a functor F: $(rings) \to (groups)$ is said to be representable if there exists a ring A such that
$$F(R) = \mathrm{Hom}(A, R)$$
).)

Examples $\mathrm{GL}_2()$, $\mathrm{SL}_2()$, $\mathrm{GL}_n()$, $\mathrm{SL}_n()$ are affine group schemes.

Definition 2.4. A homomorphism of affine groups chemes is a natural map $\varphi : G \to H$ for which each $\varphi_R : G(R) \to H(R)$ is a homomorphism.

DEFINITION 2.5. Let $\psi : H' \to G$ be a homomorphism. If the corresponding algebra hom $B' \to A$ is surjective, we call ψa closed embedding.

DEFINITION 2.6. Assume A: a Hopf algebra. Then I is a Hopf ideal of A if A/I is a Hopf algebra such that the quotient map $A \to A/I$ corresponds to a closed embedding.