No.9:

$\boxed{\text{The ring of Witt vectors when } A \text{ is a ring of characteristic } p \neq 0.}$

9.1. **Idempotents.**

We are going to decompose the ring of Witt vectors $\mathcal{W}_1(A)$. Before doing that, we review facts on idempotents. Recall that an element x of a ring is said to be **idempotent** if $x^2 = x$.

THEOREM 9.1. *Let R be a commutative ring. Let $e \in R$ be an idempotent. Then:*

1. *$\tilde{e} = 1 - e$ is also an idempotent. (We call it the* **complementary idempotent** *of e.)*
2. *$e, \tilde{e}$ satisfies the following relations:*
$$e^2 = 1, \quad \tilde{e}^2 = 1, \quad e\tilde{e} = 0.$$
3. *R admits an direct product decomposition:*
$$R = (Re) \times (R\tilde{e})$$

DEFINITION 9.2. For any ring R, we define a partial order on the idempotents of if as follows:

$$e \succeq f \iff ef = f$$

It is easy to verify that the relation $\succeq$ is indeed a partial order. We note also that, having defined the order on the idempotents, for any given family $\{e_\lambda\}$ of idempotents we may refer to its "supremum" $\vee e_\lambda$ and its"infimum" $\wedge e_\lambda$. (We are not saying that they always exist: they may or may not exist.) When the ring R is topologized, then we may also discuss them by using limits,

9.2. **Playing with idempotents in the ring of Witt vectors.**

DEFINITION 9.3. Let A be a commutative ring. For any $a \in A$, we denote by $[a]$ the element of $\mathcal{W}_1(A)$ defined as follows:

$$[a] = (1 - aT)_W$$

We call $[a]$ *the "Teichmüller lift" of a.*

LEMMA 9.4. *Let A be a commutative ring. Then:*

1. *$\mathcal{W}_1(A)$ is a commutative ring with the zero element $[0]$ and the unity $[1]$.*
2. *For any $a, b \in A$, we have*
$$[a] \cdot [b] = [ab]$$

□

PROPOSITION 9.5. *Let A be a commutative ring. If n is a positive integer which is invertible in A, then n is invertible in $\mathcal{W}_1(A)$. To be more precise, we have*

$$\frac{1}{n} \cdot [1] = \left((1 - T)^{\frac{1}{n}}\right)_W = \left((1 + \sum_{j=1}^{\infty} \binom{\frac{1}{n}}{j} (-T)^j\right)_W.$$

PROOF. It is easy to find out, by using iterative approximation, an element x of $A[[T]]$ such that

$$(1+x)^n = (1-T).$$

It also follows from the next lemma. □

LEMMA 9.6. *Let n be a positive integer. Let k be a non negative integer. Then we have always*

$$\binom{\frac{1}{n}}{k} \in \mathbb{Z}\left[\frac{1}{n}\right].$$

PROOF.

$$\begin{aligned}\binom{\frac{1}{n}}{k} &= \frac{\frac{1}{n}(\frac{1}{n}-1)\cdots(\frac{1}{n}-(k-1))}{k!} \\ &= \frac{1}{n^k}\frac{(1(1-n)(1-2n)\dots(1-(k-1)n)}{k!}\end{aligned}$$

So the result follows from the next sublemma. □

SUBLEMMA 9.7. *Let n be a positive integer. Let k be a non negative integer. Let $\{a_j\}_{j=1}^k \subset \mathbb{Z}$ be an arithmetic progression of common difference n. Then:*

(1) *For any positive integer m which is relatively prime to n, we have*

$$\#\{j;\ m|a_j\ \} \geq \left\lfloor \frac{k}{m} \right\rfloor$$

(2) *For any prime p which does not divide n, let us define*

$$c_{k,p} = \sum_{i=1}^{\infty} \lfloor \frac{k}{p^i} \rfloor$$

(which is evidently a finite sum in practice.) Then

$$p^{c_{k,p}} | \prod_{j=1}^{k} a_j$$

(3)

$$p^{c_{k,p}}|k!, \qquad p^{c_{k,p}+1} \nmid k!$$

(4)

$$\frac{\prod_{j=1}^k a_j}{k!} \in \mathbb{Z}_{(p)}$$

PROOF. (1) Let us put $t = \lfloor \frac{k}{m} \rfloor$. Then we divide the set of first kt-terms of the sequence $\{a_j\}$ into disjoint sets in the following way.

$$\begin{aligned}&S_0 = \{a_1, a_2, \dots, a_m\}, \\ &S_1 = \{a_{m+1}, a_{m+2}, a_{m+m}\}, \\ &S_2 = \{a_{2m+1}, a_{2m+2}, a_{2m+m}\}, \\ &\dots \\ &S_{t-1} = \{a_{(t-1)m+1}, a_{(t-1)m+2}, \dots, a_{(t-1)m+m}\}\end{aligned}$$

Since m is coprime to n, we see that each of the S_u gives a complete representative of $\mathbb{Z}/n\mathbb{Z}$.

(2): Apply (1) to the cases where $m = p, p^2, p^3, \dots$ and count the powers of p which appear in $\prod a_j$.

(3): Easy. (4) is a direct consequence of (2),(3). □

DEFINITION 9.8. For any positive integer n which is invertible in a commutative ring A, we define an element e_n as follows:

$$e_n = \frac{1}{n} \cdot (1 - T^n)_W.$$

LEMMA 9.9. *Let A be a commutative ring. Then for any positive integer n which is invertible in A, we have:*

(1) *e_n is an idempotent.*
(2)
$$e_n = (1 - \frac{1}{n}T^n + \textit{(higher order terms)})_W$$
(3) *If $n|m$, with m invertible in A, then $e_n \geq e_m$ in the order of idempotents.*

PROOF. if $n|m$, then we have

$$e_n \cdot e_m = e_m.$$

$\square$

It should be important to note that the range of the projection e_n is easy to describe.

PROPOSITION 9.10. *Let n be an integer invertible in A.* $e_n \cdot \mathcal{W}_1(A) = \{(f)_W | f \in 1 + T^n A[[T^n]]\}$

PROOF. Easy. Compare with Lemma 9.20 below.

$\square$

9.3. **The ring of p-adic Witt vectors (when the characteristic of the base ring A is p).** Before proceeding further, let me illustrate the idea. Proposition 9.5 tells us an existence of a set $\{e_n; n \in \mathbb{Z}_{>0}, p \nmid n\}$ of idempotents in $\mathcal{W}_1(A)$ such that its order structure is somewhat like the one found on the set $\{n\mathbb{N}; n \in \mathbb{Z}_{>0}, p \nmid n\}$. Knowing that the idempotents correspond to decompositions of $\mathcal{W}_1(A)$, we may ask:

PROBLEM 9.11. What is the partition of $\mathbb{Z}_{>0}$ generated by the subsets $\{n\mathbb{N}; n \in \mathbb{Z}_{>0}\}$?

To answer this problem, it would probably be better to find out, for given positive number n which is coprime to p, what the set

$$S_{n;p} = n\mathbb{N} \setminus (\bigcup_{\substack{n|m \\ n<m \\ p|m}} m\mathbb{N})$$

should be. The answer is given by a fact which we know very well: every positive integer may uniquely be written as

$$p^s k \quad (s \in \mathbb{Z}_{\geq 0}, \quad k \in \mathbb{Z}_{>0}, \quad \gcd(p, k) = 1),$$

Knowing that, we see that the set $S_{n;p}$ as above is equal to

$$\{p^s n; s \in \mathbb{Z}_{\geq 0}\}.$$

The answer to the problem is now given as follows:

$$\mathbb{Z}_{>0} = \coprod_{p \nmid n} \{p^s n; s \in \mathbb{Z}_{\geq 0}\}.$$

The same story applies to the ring $\mathcal{W}_1(A)$ of universal Witt vectors for a ring A of characteristic p. We should have a direct product expansion

$$\mathcal{W}_1(A) = \prod_{p \nmid n} e_{n;p} \mathcal{W}_1(A)$$

where the idempotent $e_{n;p}$ is defined by

$$e_{n;p} = e_n - \bigvee_{\substack{n|m \\ n<m \\ p \nmid m}} e_m$$

Of course we need to consider infimum of infinite idempotents. We leave it to an exercise:

EXERCISE 9.1. Show that the supremum

$$\bigvee_{\substack{n|m \\ n<m \\ p \nmid m}} e_m = e_n - \prod_{\substack{n|m \\ n<m \\ p \nmid m}} (e_n - e_m)$$

exists. In other words, show that the right hand side converges.

PROPOSITION 9.12. *Let p be a prime. Let A be an integral domain of characteristic p. Let us define an idempotent f of $\mathcal{W}_1(A)$ as follows.*

$$f = \bigvee_{\substack{n>1 \\ p \nmid n}} e_n (= [1] - \prod_{\substack{p \nmid n \\ n>1}} ([1] - e_n))$$

Then f defines a direct product decomposition

$$\mathcal{W}_1(A) \cong (f \cdot \mathcal{W}_1(A)) \times (([1] - f) \cdot \mathcal{W}_1(A)) .$$

We call the factor algebra $([1] - f) \cdot \mathcal{W}_1(A)$ **the ring $\mathcal{W}^{(p)}(A)$ of p-adic Witt vectors**.

The following proposition tells us the importance of the ring of p-adic Witt vectors.

PROPOSITION 9.13. *Let p be a prime. Let A be a commutative ring of characteristic p. For each positive integer k which is not divisible by p, let us define an idempotent f_k of $\mathcal{W}_1(A)$ as follows.*

$$f_k = \bigvee_{\substack{p \nmid n \\ n>1}} e_{kn} (= e_k - \prod_{\substack{p \nmid n \\ n>1}} (e_k - e_{kn}))$$

Then f_k defines a direct product decomposition

$$e_k \mathcal{W}_1(A) \cong (f_k \cdot \mathcal{W}_1(A)) \times ((e_k - f_k) \cdot \mathcal{W}_1(A)) .$$

Furthermore, the factor algebra $(e_k - f_k) \cdot \mathcal{W}_1(A)$ is isomorphic to the ring $\mathcal{W}^{(p)}(A)$ of p-adic Witt vectors. Thus we have a direct product decomposition

$$\mathcal{W}_1(A) \cong \mathcal{W}^{(p)}(A)^{\mathbb{N}}.$$

9.4. **The ring of p-adic Witt vectors for general A.** In the preceding subsection we have described how the ring $\mathcal{W}_1(A)$ of universal Witt vectors decomposes into a countable direct sum of the ring of p-adic Witt vectors. In this subsection we show that the ring $W^{(p)}(A)$ can be defined for any ring A (that means, without the assumption of A being characteristic p).

We need some tools.

DEFINITION 9.14. Let A be any commutative ring. Let n be a positive integer. Let us define additive operators V_n, F_n on $\mathcal{W}_1(A)$ by the following formula.

$$V_n((f(T))_W) = (f(T^n))_W.$$

$$F_n((f(T))_W) = (\prod_{\zeta \in \mu_n} f(\zeta T^{1/n}))_W$$

(The latter definition is a formal one. It certainly makes sense when A is an algebra over $\mathbb{C}$. Then the definition descends to a formal law defined over $\mathbb{Z}$ so that F_n is defined for any ring A. In other words, F_n is actually defined to be the unique continuous additive map which satisfies

$$F_n((1-aT^l)_) = ((1-a^{m/l}T^{m/n})^{ln/m})_W \qquad (m = \mathrm{lcm}(n,l)).$$

)

LEMMA 9.15. *Let p be a prime number. Let A be a commutative ring of characteristic p. Then:*

(1) *We have*

$$F_p(f(T)) = (f(T^{1/p}))^p \qquad (\forall f \in \mathcal{W}_1(A)).$$

in particular, F_p is an algebra endomorphism of $\mathcal{W}_1(A)$ in this case.

(2)

$$V_p(F_p((f)_W) = F_p(V_p((f)_W)) = (f(T)^p)_W = p \cdot (f(T))_W$$

DEFINITION 9.16. Let A be any commutative ring. Let p be a prime number. We denote by

$$\mathcal{W}^{(p)}(A) = A^{\mathbb{N}}.$$

and define

$$\pi_p : \mathcal{W}_1(A) \to \mathcal{W}^{(p)}(A)$$

by

$$\pi_p\left(\sum_{j=1}^{\infty}(1 - x_j T^j)\right) = (x_1, x_p, x_{p^2}, x_{p^3} \dots).$$

LEMMA 9.17. *Let us define polynomials $\alpha_j(X,Y) \in \mathbb{Z}[X,Y]$ by the following relation.*

$$(1-xT)(1-yT) = \prod_{j=1}^{\infty}(1 - \alpha_j(x,y)T^j).$$

Then we have the following rule for "carry operation":

$$(1-xT^n)_W + (1-yT^n)_W = \sum_{j=1}^{\infty}(1-\alpha_j(x,y)T^{jn}).$$

PROPOSITION 9.18. *There exist unique binary operators $+$ and $\cdot$ on $\mathcal{W}^{(p)}(A)$ such that the following diagrams commute.*

$$\begin{tikzcd}
\mathcal{W}_1(A) \times \mathcal{W}_1(A) \arrow[r, "+"] \arrow[d, "\pi_p"'] & \mathcal{W}_1(A) \arrow[d, "\pi_p"'] \\
\mathcal{W}^{(p)}(A) \times \mathcal{W}^{(p)}(A) \arrow[r, "+"] & \mathcal{W}^{(p)}(A)
\end{tikzcd}$$

$$\begin{array}{ccc} \mathcal{W}_1(A) \times \mathcal{W}_1(A) & \xrightarrow{\cdot} & \mathcal{W}_1(A) \\ {\scriptstyle \pi_p}\downarrow & & {\scriptstyle \pi_p}\downarrow \\ \mathcal{W}^{(p)}(A) \times \mathcal{W}^{(p)}(A) & \xrightarrow{\cdot} & \mathcal{W}^{(p)}(A) \end{array}$$

PROOF. Using the rule as in the previous lemma, we see that addition descends to an addition of $\mathcal{W}^{(p)}(A)$. It is easier to see that the multiplication also descends.

□

DEFINITION 9.19. For any commutative ring A, elements of $W^{(p)}(A)$ are called **p-adic Witt vectors** over A. The ring $(W^{(p)}(A), +, \cdot)$ is called **the ring of p-adic Witt vectors** over A.

LEMMA 9.20. *Let p be a prime number. Let A be a ring of characteristic p. Then for any n which is not divisible by p, the map*

$$\frac{1}{n} \cdot V_n : \mathcal{W}_1(A) \to \mathcal{W}_1(A)$$

is a "non-unital ring homomorphism". Its image is equal to the range of the idempotent e_n. That means,

$$\text{Image}(\frac{1}{n} \cdot V_n) = e_n \cdot \mathcal{W}_1(A) = \{\sum_j (1 - y_j T^{nj})_W ; y_j \in A\}.$$

PROOF. V_n is already shown to be additive. The following calculation shows that $\frac{1}{n} \cdot V_n$ preserves the multiplication: for any positive integer a, b with lcm m and for any element $x, y \in A$, we have:

$$\begin{aligned} &(\frac{1}{n} \cdot V_n((1 - xT^a)_W)) \cdot (\frac{1}{n} \cdot V_n((1 - yT^b)_W)) \\ =&(\frac{1}{n} \cdot (1 - xT^{an})_W) \cdot (\frac{1}{n} \cdot (1 - yT^{bn})_W) \\ =&\frac{1}{n^2} \cdot \frac{an \cdot bn}{nm} \left((1 - x^{m/a} y^{m/b} T^{nm})^d\right)_W \\ =&\frac{1}{n} \cdot V_n(((1 - xT^a)_W \cdot (1 - yT^b)_W) \end{aligned}$$

We then notice that the image of the unit element [1] of the Witt algebra is equal to $\frac{1}{n} V_n([1]) = e_n$ and that $\frac{1}{n} V(e_n f) = e_n f$ for any $f \in \mathcal{W}_1(A)$. The rest is then obvious. □

In preparing from No.7 to No.10 of this lecture, the following reference (especially its appendix) has been useful:

`http://www.math.upenn.edu/~chai/course_notes/cartier_12_2004.pdf`