

CONGRUENT ZETA FUNCTIONS. NO.1

YOSHIFUMI TSUCHIMOTO

In this lecture we define and observe some properties of congruent zeta functions.

existence of finite fields.

For a good brief account of finite fields, consult Chapter I of a book [?] of Serre.

LEMMA 1.1. *For any prime number p , $\mathbb{Z}/p\mathbb{Z}$ is a field. (We denote it by \mathbb{F}_p .)*

Funny things about this field are:

LEMMA 1.2. *Let p be a prime number. Let R be a commutative ring which contains \mathbb{F}_p as a subring. Then we have the following facts.*

(1)

$$\underbrace{1 + 1 + \cdots + 1}_{p\text{-times}} = 0$$

holds in R .

(2) *For any $x, y \in R$, we have*

$$(x + y)^p = x^p + y^p$$

We would like to show existence of “finite fields”. A first thing to do is to know their basic properties.

LEMMA 1.3. *Let F be a finite field (that means, a field which has only a finite number of elements.) Then we have,*

(1) *There exists a prime number p such that $p = 0$ holds in F .*

(2) *F contains \mathbb{F}_p as a subfield.*

(3) *$q = \#(F)$ is a power of p .*

(4) *For any $x \in F$, we have $x^q - x = 0$.*

(5) *The multiplicative group $(F_q)^\times$ is a cyclic group of order $q - 1$.*

The next task is to construct such field. An important tool is the following lemma.

LEMMA 1.4. *For any field K and for any non zero polynomial $f \in K[X]$, there exists a field L containing L such that f is decomposed into polynomials of degree 1.*

To prove it we use the following lemma.

LEMMA 1.5. *For any field K and for any irreducible polynomial $f \in K[X]$ of degree $d > 0$, we have the following.*

- (1) $L = K[X]/(f(X))$ is a field.
- (2) Let a be the class of X in L . Then a satisfies $f(a) = 0$.

Then we have the following lemma.

LEMMA 1.6. *Let p be a prime number. Let $q = p^r$ be a power of p . Let L be a field extension of \mathbb{F}_p such that $X^q - X$ is decomposed into polynomials of degree 1 in L . Then*

- (1)

$$L_1 = \{x \in L; x^q = x\}$$
 is a subfield of L containing \mathbb{F}_p .
- (2) L_1 has exactly q elements.

Finally we have the following lemma.

LEMMA 1.7. *Let p be a prime number. Let r be a positive integer. Let $q = p^r$. Then we have the following facts.*

- (1) *There exists a field which has exactly q elements.*
- (2) *There exists an irreducible polynomial f of degree r over \mathbb{F}_p .*
- (3) *$X^q - X$ is divisible by f .*
- (4) *For any field K which has exactly q -elements, there exists an element $a \in K$ such that $f(a) = 0$.*

In conclusion, we obtain:

THEOREM 1.8. *For any power q of p , there exists a field which has exactly q elements. It is unique up to an isomorphism. (We denote it by \mathbb{F}_q .)*

The relation between various \mathbb{F}_q 's is described in the following lemma.

LEMMA 1.9. *There exists a homomorphism from \mathbb{F}_q to $\mathbb{F}_{q'}$ if and only if q' is a power of q .*

EXERCISE 1.1. Compute the inverse of 113 in the field \mathbb{F}_{359} .