CONSTRUCTIONS ON NON COMMUTATIVE SCHEMES

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1. Introduction

In this paper we consider two constructions in non commutative schemes. One is a gluing of two categories at a common localization and the other is a crossed product. The latter, of course, is the more important topic here. Crossed products are originally defined and extensively used in the theory of operator algebras [1]. They not only give examples of important classes of operator algebras, but many algebras are studied by decomposing them into certain crossed products. We believe that the crossed products also play equally an important role in the theory of non commutative schemes. As an example, we define in this paper a skew projective line associated to an automorphism of a category. This gives an example of "non commutative deformation" of a scheme. We hope many other deformations are realized in a similar way.

Our approach here is based on a definition of non commutative schemes given in a paper [3] of Rosenberg, where schemes are presented by "the category of quasi-coherent sheaves on them". One of the advantages of using his framework is that we may construct crossed product globally, (that is, without "cutting a scheme into affine pieces and glue them together again.") This enables us to obtain a good perspective, which makes it easier to develop our theory.

The structure of the paper is as follows: in Section 2, we review some of the basic definitions and results in the paper [3] of Rosenberg, in Section 3, we show how to glue two categories in a special case where the intersection is a localization of the two, in Section 4, we define a crossed product, and in Section 5, we give as an example of the previous construction a skew projective line associated to an automorphism of a category.

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2. A review of Rosenberg's definition of schemes.

In this section we give a brief review of definition of schemes described in [3]. The reader is encouraged to read the original paper for a detail.

Definition 2.1. [3] Let $\mathcal{C}_1, \mathcal{C}_2$ be abelian categories. A morphism f from $\mathcal{C}_1$ to $\mathcal{C}_2$ (in the sense of Rosenberg) is an isomorphness class of right exact additive functors from $\mathcal{C}_2$ to $\mathcal{C}_1$. A representative of the class is called an inverse image functor. And if we made a choice of one such, we denote it by f^*. f is said to be continuous if the inverse image functor f^* has a right adjoint (called a direct image functor f_* of f).

Notice: In [3], the concept of morphism is (probably) defined for a larger class of categories. For the sake of simplicity we concentrate ourselves on abelian categories. Note also that in this case the right-exactness of f^* is readily verified if we have a right adjoint f_* of f^*. We shall mainly consider continuous morphisms in this paper.

Example . Let X_1, X_2 be schemes in the usual (commutative) sense. Let $\mathrm{Qcoh}(X_i)$ be the category of quasi-coherent $\mathcal{O}_{X_i}$-modules ($i = 1, 2$). Then for any morphism $f : X_1 \to X_2$ in the usual sense gives a continuous morphism $\hat{f} : \mathrm{Qcoh}(X_1) \to \mathrm{Qcoh}(X_2)$ (in the sense of Rosenberg) with the usual inverse image and direct image functors: $\tilde{f}^* = f^*, \tilde{f}_* = f_*$.

For any homomorphism $\phi : A \to B$ of rings, one has a morphism $\tilde{\phi} : (B\text{-mod}) \to (A\text{-mod})$ associated to ϕ. When the algebras A, B are commutative, this example may be regarded as a special case of the example above.

Lemma 2.1. *Let k be a ring, R_1, R_2 be k-algebras. Let $\rho_i : (R_i\text{-mod}) \to (k\text{-mod})$ be structure morphisms (that is, morphism defined by the structure homomorphisms $k \to R_i$) ($i = 1, 2$). We define an equivalence of homomorphisms from R_1 to R_2 by saying that two homomorphisms $\phi, \psi : R_1 \to R_2$ are equivalent if and only if there exists an invertible element u of R_2 such that $u\phi u^{-1} = \psi$. Then there is a bijection between the set of equivalence classes $\mathrm{Hom}_k(R_1, R_2)/\sim$ of k-algebra homomorphism and the set of morphisms $f : (R_2\text{-mod}) \to (R_1\text{-mod})$ which satisfies $f \circ \rho_1 = \rho_2$.*

Note:

1. The "structure morphism" in the above lemma is necessary since we cannot distinguish between Morita-equivalent rings otherwise.

2. For general schemes, a morphism in the sense of Rosenberg does not necessarily come from a usual morphism of schemes. In fact, for

any proper noetherian scheme X over $\mathbb{C}$ and for any coherent sheaf $\mathcal{G}$ on X, there exists an morphism $f : (\mathbb{C}\text{-mod}) \to \mathrm{Qcoh}(X)$ such that $f_*(V) = V \otimes_{\mathbb{C}} \mathcal{G}$, $f^*(\mathcal{F}) = (\mathrm{Hom}_X(\mathcal{F}, \mathcal{G}))^*$ (a suitable "topological dual" of $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$).

Definition 2.2. [3] A morphism f (in the sense of Rosenberg) between categories is said to be flat if the inverse image functor f^* is exact. f is called a flat localization if f is flat and has a fully faithful direct image functor.

Example . A morphism $(B\text{-mod}) \to (A\text{-mod})$ associated to a ring homomorphism $\phi : A \to B$ is a flat localization if and only if ϕ is (left-)flat and the multiplication map $B \otimes_A B \to B$ gives an isomorphism. In particular, if the algebras A, B are commutative, any localization in the usual sense yields a flat localization of the corresponding category.

Definition 2.3. [3] A continuous morphism f (in the sense of Rosenberg) between categories is said to be almost affine if the direct image functor f_* is exact and faithful. f is said to be affine if f_* is faithful and has a right adjoint.

There is a nice description of an almost affine morphism which uses a concept of monad. Recall that a monad $F = (F, \mu^{(F)}, \eta^{(F)})$ on a category $\mathcal{C}$ is a functor $F : \mathcal{C} \to \mathcal{C}$ with a natural transformations $\mu^{(F)} : F^2 \to F$ ("multiplication") and $\eta^{(F)} : \mathrm{id} \to F$ ("unity") which satisfies certain axioms ("associativity" and ""unity" being unity")[2]. We denote by $(F\text{-mod})$ the category of F-modules (F-algebra in the language of Mac Lane[2]). By definition, an F-module is a pair (M, α) of an object M of the category $\mathcal{C}$ and an arrow $\alpha : FM \to M$ ("action") which satisfies "axioms of action".

Lemma 2.2. [3] *Let $\mathcal{C}_2$ be an abelian category. For any right-exact monad F on $\mathcal{C}_2$, we have a morphism $f : (F\text{-mod}) \to \mathcal{C}_2$. A continuous morphism $f : \mathcal{C}_1 \to \mathcal{C}_2$ is almost affine if and only if $\mathcal{C}_1$ is equivalent over $\mathcal{C}_2$ to $(F\text{-mod})$ for some right-exact monad F on $\mathcal{C}_2$.*

Example . Let A be an algebra and B an A-algebra. Then a functor $F : \mathcal{C} = (A\text{-mod}) \to \mathcal{C}$ given by tensor products $F(M) = B \otimes_A M$ is a monad. The category $(F\text{-mod})$ is isomorphic to the category $(B\text{-mod})$ of B-modules. The morphism $f : (F\text{-mod}) \to \mathcal{C}$ in this case is identified with the morphism associated to the structure morphism $A \to B$.

In many situations, we may deal a monad in an analogous way as the example above. For instance, the following lemma holds.

Lemma 2.3. *Let $\mathcal{C}$ be an abelian category. Let F, G be monads on $\mathcal{C}$. Let $\theta : F \to G$ be a morphism of monads [2]. Then there exists a* *morphism (in the sense of Rosenberg) such that its direct- and inverse image functors are given as follows.*

$$f_*(y,\beta) = (y, \beta \circ \theta_y)$$
$$f^*(x,\alpha) = (Gx/\langle(\eta_x^{(G)} \circ \alpha - \theta_x)Fx\rangle_G, \mu^{(G)})$$

where the symbol $\langle m\rangle_G$ denotes a "G-submodule of $(Gx, \mu^{(G)})$ generated by a sub object m of Gx". That is,

$$\langle m\rangle_G = \mathrm{Image}(\mu^{(G)} \circ G\lambda : Gm \to Gx) \quad (\lambda : m \to Gx \textit{ is the inclusion})$$

A quasi-scheme (in the sense of Rosenberg) is defined to be an abelian category which is "locally almost affine".

Definition 2.4. [3] A set of flat localizations $\{f_i : \mathcal{C}_i \to \mathcal{C}\}$ is said to be a Zariski cover of $\mathcal{C}$ if any arrow s of $\mathcal{C}$ such that $f_i^*(s)$ is invertible for all i is invertible. A continuous morphism $f : \mathcal{A} \to \mathcal{C}$ is said to be a quasi-scheme over $\mathcal{C}$ if there exists a Zariski cover $\{u_i : \mathcal{A}_i \to \mathcal{A}\}$ such that $f \circ u_i$ is almost affine for each i.

3. Gluing two categories at a common flat localization.

In this section we show that we may "glue" two categories at a common flat localization.

Lemma 3.1. *Let U, V, W be abelian categories. Let $i : W \to U$, $j : W \to V$ be flat localizations. We define X by*

$$X = \{(M, N, \phi) | M \in \mathrm{Ob}(U), N \in Ob(V), \phi : i^*M \cong j^*N\}$$

$$\pi : U \times V \to X$$
$$\pi^*(M, N, \phi) = (M, N)$$
$$\pi_*((M, N)) = (M \oplus i_*j^*N, N \oplus j_*i^*M, \phi_{M,N})$$

where $\phi_{M,N}$ is defined as follows

$$\begin{aligned} i^*(M \oplus i_*j^*N) &\overset{\cong}{\to} i * M \oplus i^*i_*j^*N \\ &\overset{\cong}{\to} i^*M \oplus j^*N \qquad (\because i\text{:}\textit{localization}) \\ &\overset{\cong}{\to} j^*j_*i^*M \oplus j^*N \quad (\because j\text{:}\textit{localization}) \\ &\overset{\cong}{\to} j^*(j_*i^*M \oplus N) \\ &\overset{\cong}{\to} j^*(N \oplus j_*i^*M). \end{aligned}$$

Then the following statements hold.

1. *X is an abelian category*
2. *π is a continuous morphism*

3. *Restrictions* $\pi|_U, \pi|_V$ *of* π *are flat localization.*
4. π *is a covering.*
5. *If the categories* U, V, W *are almost affine schemes over an abelian category* Z*, and if the morphisms* i, j *are morphisms over* Z *(that is, if they commute with "structure morphisms"), then* X *is a scheme over* Z*.*

PROOF.

1.: Clear.

2.: We need to show that π_* is an adjoint of π^*. For any objects $(M, N, \phi) \in \mathrm{Ob}(X)$, $(M_1, N_1) \in \mathrm{Ob}(U \times V)$, we have an isomorphism

$$\begin{aligned}\mathrm{Hom}_{U\times V}(\pi^*(M,N,\phi),(M_1,N_1)) &\cong \mathrm{Hom}_{U\times V}((M,N),(M_1,N_1))\\ &\cong \mathrm{Hom}_U(M,M_1)\oplus \mathrm{Hom}_V(N,N_1).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\mathrm{Hom}_X((M,N,\phi),\pi_*(M_1,N_1)))\\ =&\mathrm{Hom}_X((M,N,\phi),(M_1\oplus i_*j^*N_1, N_1\oplus j_*i^*M_1,\phi_{M_1,N_1}))\\ =&\left\{(\alpha,\beta,\gamma,\delta);\begin{array}{c}\alpha\in\mathrm{Hom}(M,M_1),\beta\in\mathrm{Hom}(M,i_*j^*N_1)\\ \gamma\in\mathrm{Hom}(N,N_1),\delta\in\mathrm{Hom}(N,j_*i^*M_1)\\ j^*\gamma\circ\phi=i^*\beta, i^*\alpha\circ\phi^{-1}=j^*\delta\quad(*)\end{array}\right\}.\end{aligned}$$

The condition $(*)$ above (and the fact that i, j are localizations) implies that the arrows β, δ are determined uniquely by the arrows α, γ. We thus have a natural isomorphism

$$\mathrm{Hom}_{U\times V}(\pi^*(M,N,\phi),(M_1,N_1))) \cong \mathrm{Hom}_X((M,N,\phi),\pi_*(M_1,N_1))).$$

3.:It is clear to see that π^* is exact. That is, π is flat. For any object M of U, we have,

$$\pi_1^*\pi_{1*}(M) = \pi_1^*((M, j_*i^*M, \phi_{M,0})) = M.$$

This implies that π_1 is a localization.

4.,5.: Clear. □

4. CROSSED PRODUCT

In this section we deal with one of the most useful kind of construction of non commutative geometry, namely a crossed product. The theory of crossed product is developed and used as one of the basic tools in the theory of operator algebras [1]

We first need to define an "action" of a unital semi group on a category. It turns out that, along with the expected set $\{f_s\}$ of morphisms, we need a cocycle of natural isomorphisms.

Definition 4.1. Let S be a semi group with the unity e. Let $\mathcal{C}$ be an abelian category. We say an action (f,η) of S on $\mathcal{C}$ is given if we are given the following data 1.,2. which satisfy conditions 3.,4.,5.,6.

1. A continuous morphism (in the sense of Rosenberg) $f_s : \mathcal{C} \to \mathcal{C}$ for each element s of S.
2. A natural isomorphism $\eta_{s,t} : f_{s*}f_{t*} \to f_{st*}$ for each pair s,t of elements of S,
3. $f_e = \mathrm{id}$. $f_{e*} = \mathrm{id}$.
4. $\eta_{st,u} \circ (\eta_{s,t} f_{u*}) = \eta_{s,tu} \circ (f_{s*}\eta_{t,u})$ for $\forall s,t,u \in S$.
5. $\eta_{e,s} = \mathrm{id}, \eta_{s,e} = \mathrm{id}$ for $\forall s \in S$.
6. f_{s*} has a right adjoint (hence is exact and commutes with inductive limits) for $\forall s \in S$.

Note that the condition 6. above is satisfied if each morphism f_s is affine (in particular, if each f_s is invertible.) Note also that the definition above uses direct image functors and tells nothing explicitly about inverse image functors. If however the semi group S is in fact a group, then we may choose f_s^* as $f_{s^{-1}*}$ with adjunction given by $\eta_{s,s^{-1}}$ so that we may obtain a similar relation as above for the inverse image functors.

Definition 4.2. Two actions $(f,\eta),(\bar{f},\bar{\eta})$ of a unital semi-group S on an abelian category $\mathcal{C}$ are said to be equivalent if there exists an natural isomorphism $\lambda_s : f_{s*} \to \bar{f}_{s*}$ for each $s \in S$ such that the following identity hold.

$$\bar{\eta}_{s,t} = \lambda_{st} \circ \eta_{s,t} \circ (\lambda_s \bar{f}_{t*} \circ f_{s*}\lambda_t)^{-1} \quad (s,t \in S)$$

Example . Assume a unital semi group S acts on a scheme $(X, \mathcal{O}_X)$ in the usual sense. Then S acts on $\mathrm{Qcoh}(X)$.

The following easy fact is used to prove the next lemma.

Sublemma 4.1. *Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be categories. Let $F_1, G_1 : \mathcal{C}_1 \to \mathcal{C}_2$ $F_2, G_2 : \mathcal{C}_2 \to \mathcal{C}_3$ be functors and Let $\eta_1 : F_1 \to G_1$, $\eta_2 : F_2 \to G_2$ be natural morphisms of functors. Then we have an identity*

$$\eta_2 G_1 \circ F_2\eta_1 = G_2\eta_1 \circ \eta_2 F_1.$$

Lemma 4.1. *Let $h : \mathcal{C} \to \mathcal{C}$ be a continuous morphism in the sense of Rosenberg. If h^* is a category equivalence, then there exists an action (f,η) of $\mathbb{Z}$ on $\mathcal{C}$ such that $f_1 = h$.*

PROOF. We choose h^* and then take h_* as its adjoint. We take adjunctions

$$h^*h_* \xrightarrow{\epsilon} \mathrm{id}, \mathrm{id} \xrightarrow{\eta} h_*h^*$$

and note that, since h is invertible, ϵ, η are isomorphisms. We put

$$f_{n*} = \begin{cases} (h_*)^n & (n > 0) \\ \mathrm{id} & (n = 0) \\ (h^*)^{-n} & (n < 0) \end{cases}$$

and define $\eta_{n,m}$ by "canceling h^* and h_* by using adjunctions ϵ, η". Then the triangular identities [2] for the adjunctions ϵ, η tell us that the relation 5. of Definition 4.1 holds for $s, t, u = 1, -1, 1$. General cases of relation 5. of Definition 4.1 may be verified by using this special case and Sublemma 4.1 It is then easy to verify the rest of the properties in the Definition 4.1. □

Definition 4.3. The action (f, η) of $\mathbb{Z}$ on $\mathcal{C}$ constructed in the proof of the above lemma is called the action generated by the automorphism h.

Definition 4.4. Let (f, η) be an action of a semi group with the unity e on an abelian category $\mathcal{C}$. Then we define the crossed product $(S \ltimes \mathcal{C})$ of $\mathcal{C}$ by S as follows

$$\mathrm{Ob}(S \ltimes \mathcal{C}) = \left\{ (M, \alpha); \begin{array}{l} M \in \mathrm{Ob}\mathcal{C} \\ \alpha = \{\alpha_s\}_{s \in S} \\ \alpha_s \in \mathrm{Hom}(f_{s*}M, M) \\ \alpha_{st} \circ \eta_{s,t} = \alpha_s \circ (f_{s*}\alpha_t) \quad (\forall s, \forall t \in S) \\ \alpha_e = \mathrm{id} \end{array} \right\}$$

$$\mathrm{Hom}((M, \alpha), (N, \beta)) = \{\phi \in \mathrm{Hom}(M, N) | \phi \circ \alpha_g = \beta_g \circ f_{g*}\phi \text{ for } \forall g \in S\}$$

It is easy to see that equivalent actions yield the same crossed product.

Example . Let S be a semi group with the unity e, A be a ring. Suppose we have an anti-action σ of S on A. That means, we have an semi group anti-homomorphism $\sigma : S \to \mathrm{End}_{\mathrm{ring}}(A, A)$ which preserves the unity. Then S acts on $\mathcal{C} = (A\text{-mod})$. The crossed product $S \ltimes \mathcal{C}$ is isomorphic to the category $(S \ltimes A\text{-mod})$ of $S \ltimes A$-modules, where the crossed product $S \ltimes A$ is a ring generated by elements of A and symbols $\{X_s\}_{s \in S}$ with the following relations and the original summation and multiplication rules for A.

$$X_{s_1} X_{s_2} = X_{s_1 s_2}, \quad f X_s = X_s \sigma(s)(f) \quad (s, s_1, s_2 \in S, f \in A)$$

Note that the commutation relation above gives a direct sum decomposition

$$S \ltimes A = \oplus_{s \in S} X_s A$$

as a right A-module.

Remark. The definition of the crossed product $S \ltimes \mathcal{C}$ (as a category over $\mathcal{C}$) indeed depend on the choice of η, as the following example indicates.

Example . Take $\mathcal{C} = (\mathbb{R}\text{-mod}), S = \mathbb{Z}/2\mathbb{Z}$, $f_{s*} = \mathrm{id}(s = \bar{0}, \bar{1})$, $\eta_{\bar{a},\bar{b}} = (-1)^{ab}\mathrm{id}$. Then an object of the crossed product $S \ltimes \mathcal{C}$ is given by a pair (M, T) of an $\mathbb{R}$-vector space M with an automorphism T of M whose square equals to -1. There is no object (M, T) of $S \ltimes \mathcal{C}$ with one dimensional M. (Compare this with the case where η is trivial.)

Proposition 4.1. *Let $S, \mathcal{C}, (f, \eta)$ as in definition 4.4. Assume furthermore that $\mathcal{C}$ is closed under taking direct sums of $\#S$ elements. Then we may define a monad F in the following way.*

$$F = \oplus_{s \in S} f_{s*}$$

$$\mu^{(F)} : F^2 M = \oplus_{s,t} f_{s*} f_{t*} M \overset{\oplus \eta_{s,t}}{\to} \oplus_{s,t} f_{st*} M \overset{st=u}{\to} \oplus f_{u*} M = FM$$

$$\eta^{(F)} : M = f_{e*} M \overset{s=e}{\to} \oplus_{s \in S} f_{s*} M$$

Then we have the following isomorphism of categories over $\mathcal{C}$.

$$S \ltimes \mathcal{C} \cong (F\text{-mod})$$

In particular, the crossed product $S \ltimes \mathcal{C}$ is an almost affine scheme over the base category $\mathcal{C}$.

Lemma 4.2. *Let S_1, S_2 be unital semi groups. Let $\mathcal{C}$ be an abelian category with direct sums of arbitrary many objects. Let $\phi : S_1 \to S_2$ be a homomorphism which preserves the unity. Assume there exists an action (f, η) of S_2 on $\mathcal{C}$. Then*

1. *There exists an action $(\bar{f}, \bar{\eta})$ of S_1 on $\mathcal{C}$, where $\bar{f}, \bar{\eta}$ are defined as follows.*
$$\bar{f}_{s_1} = f_{\phi(s_1)}, \quad \bar{\eta}_{s_1,s_2} = \eta_{\phi(s_1),\phi(s_2)}$$
2. *There exists an almost affine morphism $\tilde{\phi} : S_2 \ltimes \mathcal{C} \to S_1 \ltimes \mathcal{C}$ such that $\tilde{\phi}_*((M, \alpha)) = (M, \alpha \circ \phi)$.*

Lemma 4.3. *Let $h : \mathcal{C}_1 \to \mathcal{C}_2$ be an morphism between two abelian categories $\mathcal{C}_1, \mathcal{C}_2$. Let $(f^{(i)}, \eta^{(i)})$ be actions of a group G on $\mathcal{C}_i$ $(i = 1, 2)$. If h commutes with G action, that means, if there exists a natural isomorphism*

$$\lambda_s : f^{(2)}_{s*} \circ h_* \overset{\cong}{\to} h_* \circ f^{(1)}_{s*}$$

for each element s of S such that equation

$$(h_* \eta^{(1)}_{s,t}) \circ (\lambda_s f^{(1)}_{t*}) \circ (f^{(2)}_{s*} \lambda_t) = \lambda_{st} \circ (\eta^{(2)}_{s,t} h_*)$$

holds for each $s, t \in G$, then for any unital subsemigroup S of G, there exists a morphism $S \ltimes h : S \ltimes \mathcal{C}_1 \to S \ltimes \mathcal{C}_2$ whose direct image functor is given as follows.

$$(S \ltimes h)_*(M, \alpha) = (h_*(M), h_*(\alpha))$$

If the morphism h is a flat localization, then so is $S \ltimes h$

Corollary 4.1. *Let $\mathcal{C}$ be a quasi-scheme over an abelian category $\mathcal{C}_0$. Assume that there is given an action of a group G on the category $\mathcal{C}$. If there exists a Zariski covering $\{\mathcal{C}_i \to \mathcal{C}\}$ such that each $\mathcal{C}_i$ is "G-invariant", that is, if there exist actions of G on $\mathcal{C}_i$ which commute with the covering map, then for any unital subsemigroup S of G, $S \ltimes \mathcal{C}$ is a quasi-scheme over $\mathcal{C}_0$.*

Lemma 4.4. *Let S be a semi group with unity e. Let $\mathcal{C}$ be an abelian category with direct sums of $\#S$-elements. Assume there exists an action (f, η) of S on $\mathcal{C}$. If $s \in S$ is invertible (that is, $\exists s^{-1} \in S$ such that $s^{-1}s = ss^{-1} = e$), then we have the following*

1. *Each functor $f_{s*} : \mathcal{C} \to \mathcal{C}$ is an equivalence of categories. In particular, for any pair M, N of objects of $\mathcal{C}$, we have an isomorphism*
$$\mathrm{Hom}(M, N) \cong \mathrm{Hom}(f_{s*}M, f_{s*}N).$$
2. *For any object (M, α) of $S \ltimes \mathcal{C}$, we have*
   (a) α_s*: invertible.*
   (b) $\eta_{s^{-1},s} = \alpha_{s^{-1}} \circ f_{s^{-1}*}\alpha_s$
   (c) $\eta_{s,s^{-1}} = \alpha_s \circ f_{s*}\alpha_{s^{-1}}$
   (d) $\alpha_{s^{-1}} = \eta_{s^{-1},s} \circ (f_{s^{-1}*}\alpha_s)^{-1} = \eta_{s^{-1},s} \circ f_{s^{-1}*}(\alpha_s^{-1})$

Lemma 4.5. *Let (f, η) be an action on an abelian category $\mathcal{C}$. Assume the category $\mathcal{C}$ has countable direct sums. Let $\phi : \mathbb{N} \to \mathbb{Z}$ be the inclusion, where $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of natural numbers. Then the morphism*

$$\tilde{\phi} : \mathbb{Z} \ltimes \mathcal{C} \to \mathbb{N} \ltimes \mathcal{C}$$

defined as in Lemma 4.2 is a flat localization. (That is, $\tilde{\phi}$ is flat and the functor $\tilde{\phi}_*$ is fully faithful.)

Proof. We first show that $\tilde{\phi}_*$ is fully faithful. For objects $(M, \alpha), (N, \beta)$ of $(\mathbb{Z} \ltimes \mathcal{C})$, we have

$$\begin{aligned}\mathrm{Hom}(\tilde{\phi}_*(M, \alpha), \tilde{\phi}_*(N, \beta)) &= \mathrm{Hom}((M, \alpha|_{\mathbb{N}}, (N, \beta|_{\mathbb{N}})) \\ &= \{\lambda \in \mathrm{Hom}(M, N); \lambda \circ \alpha_n = \beta_n \circ (f_{n*}\lambda) \text{ for } \forall n \in \mathbb{N}\}\end{aligned}$$

Applying f_{-n*} to the last line above, we see that an arrow λ ("an $\mathbb{N}$-homomorphism") above satisfies the following equation.

$$f_{-n*}\lambda \circ f_{-n*}\alpha_n = f_{-n*}\beta_n \circ f_{-n*}f_{n*}\lambda$$

Then by using the equation $f_{-n*}f_{n*}\lambda = \eta^{-1}_{-n,nN} \circ \lambda \circ \eta_{-n,nM}$ and Lemma 4.4, we see that the arrow λ is a "$\mathbb{Z}$-homomorphism". That is, $\lambda \circ \alpha_n = \beta_n \circ (f_{n*}\lambda)$ for $\forall n \in \mathbb{Z}$. This implies that there exists an natural isomorphism

$$\mathrm{Hom}(\tilde{\phi}_*(M,\alpha), \tilde{\phi}_*(N,\beta)) \cong \mathrm{Hom}((M,\alpha),(N,\beta)).$$

To prove the flatness of $\tilde{\phi}$, we we use Lemma 2.3 and see that the inverse image functor $\tilde{\phi}^*$ is given by an inductive limit

$$\tilde{\phi}^*(M,\alpha) = (\varinjlim{}_k f_{-k*}M, \tilde{\alpha})$$

for a suitable choice of "connecting homomorphisms" and "a $\mathbb{Z}$-action" $\tilde{\alpha}$. □

5. Skew projective line over a category.

As an application of previous two constructions, we define a "skew projective line" on a category. It is an projective line "skewed" by an automorphism of $\mathcal{C}$.

Proposition 5.1. *Let $\mathcal{C}$ be an abelian category with direct sums of countable objects. Let $h : \mathcal{C} \to \mathcal{C}$ be a morphism in the sense of Rosenberg which is invertible. Consider an action of $\mathbb{Z}$ on $\mathcal{C}$ generated by h (Definition 4.3). Then we may define a "skew projective line" $\mathbb{P}(h)$ associated to h as follows*

$$\mathbb{P}(h) = (\mathbb{N} \ltimes \mathcal{C}) \cup_{\mathbb{Z}\ltimes\mathcal{C}} ((-\mathbb{N}) \ltimes \mathcal{C})$$

it is a quasi scheme on $\mathcal{C}$. If $\mathcal{C}$ is a quasi scheme over a base scheme $\mathcal{C}_0$ and there exists a h-invariant Zariski cover of $\mathcal{C}$, then $\mathbb{P}(h)$ is a quasi scheme over $\mathcal{C}_0$.

Proof. We note that there are flat localizations $\mathbb{Z} \ltimes \mathcal{C} \to \pm\mathbb{N} \ltimes \mathcal{C}$ (Lemma 4.5). The existence of the union is proved in Lemma 3.1. The last statement of the proposition follows from Corollary 4.1 □

It is easy to verify that if $\mathcal{C} = (\mathcal{O}_X\text{-mod})$ and $h : \mathcal{C} \to \mathcal{C}$ is an identity then the skew projective line $\mathbb{P}(h)$ is "isomorphic to" $\mathbb{P}^1 \times X$ That means,

$$\mathbb{P}(h) \cong \mathrm{Qcoh}(\mathbb{P}^1 \times X)$$

Example . Let E be a torus defined over $\mathbb{C}$. For each $\mathbb{C}$-valued point t of E, we have an automorphism $h^{(t)} : E \to E$ of E defined by translation. The skew projective line $\mathbb{P}(h^{(t)})$ associated to $h(t)$ is a quasi scheme over $\mathbb{C}$. We thus have a "one-parameter family" $\{\mathbb{P}(h^{(t)})\}_{t\in E(\mathbb{C})}$ of quasi schemes with $\mathbb{P}(h^{(0)}) = \mathrm{Qcoh}(\mathbb{P}^1 \times E)$.

References

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