

# Calculations of sectional Euler numbers and sectional Betti numbers of special polarized manifolds

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## 1 Introduction

In this note, we will calculate the  $i$ th sectional Euler number  $e_i(X, L)$  and the  $i$ th sectional Betti number  $b_i(X, L)$  of some special polarized manifolds  $(X, L)$ . We also note that results in this note are useful for classifications of polarized manifolds (for example see [5]). At any time, we will update this note if we complete calculations of sectional Euler numbers and sectional Betti numbers of new example<sup>1</sup>.

## 2 Preliminaries

**Notation 2.1** Let  $(X, L)$  be a polarized manifold of dimension  $n$ . For every integers  $i$  and  $j$  with  $0 \leq i \leq n$  and  $0 \leq j \leq i$ , we put

$$C_j^i(X, L) := \sum_{l=0}^j (-1)^l \binom{n-i+l-1}{l} c_{j-l}(X) L^l,$$

**Definition 2.1** ([3]) Let  $(X, L)$  be a polarized manifold of dimension  $n$ , and let  $i$  and  $j$  be integers with  $0 \leq j \leq i \leq n$ .

(i) The  $i$ -th sectional Euler number  $e_i(X, L)$  of  $(X, L)$  is defined by the following:

$$e_i(X, L) := C_i^i(X, L) L^{n-i}.$$

(ii) The  $i$ -th sectional Betti number  $b_i(X, L)$  of  $(X, L)$  is defined by the following:

$$b_i(X, L) := \begin{cases} e_0(X, L) & \text{if } i = 0, \\ (-1)^i \left( e_i(X, L) - \sum_{j=0}^{i-1} 2(-1)^j h^j(X, \mathbb{C}) \right) & \text{if } 1 \leq i \leq n. \end{cases}$$

**Remark 2.1** (i) For every integers  $i$  and  $j$  with  $0 \leq j \leq i \leq n$ ,  $e_i(X, L)$ ,  $b_i(X, L)$  and  $w_i^j(X, L)$  are integer (see [3]).

(ii) If  $i = 0$ , then  $e_0(X, L) = b_0(X, L) = L^n$ . If  $i = n$ , then  $e_n(X, L) = e(X)$  and  $b_n(X, L) = h^n(X, \mathbb{C})$ .

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<sup>1</sup>If you find a mistake in this note, please let me know.

### 3 Calculations

**Example 3.1** The case where  $(X, L)$  is  $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ .

Then

$$e_i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = e(\mathbb{P}^i) = i + 1$$

and

$$b_i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = b(\mathbb{P}^i) = \begin{cases} 1, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$$

**Example 3.2** The case where  $(X, L)$  is  $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$ .

Then

$$b_n(\mathbb{Q}^n) = \begin{cases} 2, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

$$b_{n-1}(\mathbb{Q}^n) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd,} \end{cases}$$

$$b_i(\mathbb{Q}^n) = \begin{cases} 1, & \text{if } i \text{ is even with } i \leq n-2, \\ 0, & \text{if } i \text{ is odd with } i \leq n-2, \end{cases}$$

Hence

$$e_i(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)) = e_i(\mathbb{Q}^i) = \begin{cases} i+2, & \text{if } i \text{ is even,} \\ i+1, & \text{if } i \text{ is odd,} \end{cases}$$

and

$$b_i(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)) = (-1)^i \left( e_i(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)) - 2 \sum_{j=0}^{i-1} b_j(\mathbb{Q}^n) \right) = \begin{cases} 2, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$$

**Example 3.3** The case where  $(X, L)$  is  $(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$ .

Set  $H = \mathcal{O}_{\mathbb{P}^4}(1)$ . Then  $c_1(\mathbb{P}^4) = 5H$ ,  $c_2(\mathbb{P}^4) = 10H^2$ ,  $c_3(\mathbb{P}^4) = 10H^3$ ,  $c_4(\mathbb{P}^4) = 5H^4 = 5$ .

Hence

$$\begin{aligned} e_0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= (2H)^4 = 16, \\ e_1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= \sum_{l=0}^1 (-1)^l \binom{2+l}{l} c_{1-l}(X) (2H)^{3+l} = -8, \\ e_2(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= \sum_{l=0}^2 (-1)^l \binom{1+l}{l} c_{2-l}(X) (2H)^{2+l} = 8, \\ e_3(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= \sum_{l=0}^3 (-1)^l \binom{l}{l} c_{3-l}(X) (2H)^{1+l} = 4, \\ e_4(\mathbb{P}^4) &= e(\mathbb{P}^4) = 5. \end{aligned}$$

On the other hand, since

$$b_i(\mathbb{P}^4) = \begin{cases} 1, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd,} \end{cases}$$

we have

$$\begin{aligned} b_0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= 16, \\ b_1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= 10, \\ b_2(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= 6, \\ b_3(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= 0, \\ b_4(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) &= 1. \end{aligned}$$

**Example 3.4** The case where  $(X, L)$  is  $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$ .  
Set  $H = \mathcal{O}_{\mathbb{Q}^3}(1)$ . Then  $c_1(\mathbb{Q}^3) = 3H$ ,  $c_2(\mathbb{Q}^3) = 10H^2$ ,  $c_3(\mathbb{Q}^3) = 2H^3 = 4$ .

Hence

$$\begin{aligned} e_0(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= (2H)^3 = 16, \\ e_1(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= \sum_{l=0}^1 (-1)^l \binom{1+l}{l} c_{1-l}(X) (2H)^{2+l} = -8, \\ e_2(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= \sum_{l=0}^2 (-1)^l \binom{l}{l} c_{2-l}(X) (2H)^{1+l} = 8, \\ e_3(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= e(\mathbb{Q}^3) = 4. \end{aligned}$$

On the other hand, since

$$b_i(\mathbb{Q}^3) = \begin{cases} 1, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd,} \end{cases}$$

we have

$$\begin{aligned} b_0(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= 16, \\ b_1(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= 10, \\ b_2(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= 6, \\ b_3(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2)) &= 0. \end{aligned}$$

**Example 3.5** The case where  $(X, L)$  is  $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$ .  
Set  $H = \mathcal{O}_{\mathbb{P}^3}(1)$ . Then  $c_1(\mathbb{P}^3) = 4H$ ,  $c_2(\mathbb{P}^3) = 6H^2$ ,  $c_3(\mathbb{P}^3) = 4H^3$ .

Hence

$$\begin{aligned} e_0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= (3H)^3 = 27, \\ e_1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= \sum_{l=0}^1 (-1)^l \binom{1+l}{l} c_{1-l}(X) (3H)^{2+l} = -18, \\ e_2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= \sum_{l=0}^2 (-1)^l c_{2-l}(X) (3H)^{1+l} = 9, \\ e_3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= e(\mathbb{P}^3) = 4. \end{aligned}$$

On the other hand, since

$$b_i(\mathbb{P}^3) = \begin{cases} 1, & \text{if } i \text{ is even,} \\ 0, & \text{if } i \text{ is odd,} \end{cases}$$

we have

$$\begin{aligned} b_0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= 16, \\ b_1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= 10, \\ b_2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= 6, \\ b_3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) &= 0. \end{aligned}$$

**Example 3.6** The case where  $(X, L)$  is a Veronese fibration over a smooth curve  $C$  (see [2, (13.10)]).

Then there exists a vector bundle  $\mathcal{E}$  of rank three on  $C$  such that  $X = \mathbb{P}_C(\mathcal{E})$  and  $L = 2H(\mathcal{E}) + f^*(B)$ , where  $f: X \rightarrow C$  is its fibration and  $B \in \text{Pic}(C)$ . Set  $e := \deg \mathcal{E}$  and  $b := \deg B$ . First we

calculate  $e_i(X, L)$ . Here we note that  $2g(C) - 2 + e + 2b = 0$ ,  $L^3 = 8e + 12b$  and  $g_1(X, L) = 1 + 2e + 2b$ . Then

$$e_0(X, L) = L^3 = 8e + 12b, e_1(X, L) = 2 - 2g_1(X, L) = -4e - 4b.$$

Next we calculate  $e_2(X, L)$ . Since

$$\begin{aligned} c_2(X) &= \sum_{j=0}^2 \sum_{k=0}^j \binom{3-k}{j-k} c_k(f^*(\mathcal{E}^\vee)) H(\mathcal{E})^{j-k} c_{j-k}(f^*(\mathcal{T}_C)) \\ &= 3c_1(f^*(\mathcal{T}_C))H(\mathcal{E}) + 3H(\mathcal{E})^2 + 2c_1(f^*(\mathcal{E}^\vee))H(\mathcal{E}), \end{aligned}$$

we have

$$\begin{aligned} e_2(X, L) &= \sum_{l=0}^2 (-1)^l \binom{2+l}{l} c_{2-l}(X) (2H + f^*(B))^{1+l} \\ &= 20e + 27b. \end{aligned}$$

Next we calculate  $e_3(X, L)$ . We note that  $e_3(X, L) = e(X)$ . Since

$$\begin{aligned} b_0(X) &= 1, \\ b_1(X) &= 2g(C), \\ b_2(X) &= 2, \\ b_3(X) &= 2g(C), \end{aligned}$$

we have  $e_3(X, L) = e(X) = 6 - 6g(C) = 3e + 6b$ .

Furthermore we calculate  $b_i(X, L)$ . Then

$$\begin{aligned} b_0(X, L) &= 8e + 12e, \\ b_1(X, L) &= 2(1 + 2e + 2b), \\ b_2(X, L) &= 19e + 25b, \\ b_3(X, L) &= 2 - e - 2b. \end{aligned}$$

**Example 3.7** The case where  $(X, L)$  is a Del Pezzo manifold.

Here we note that by [2, (8.11) Theorem], we have  $L^n \leq 8$  and  $(X, L)$  is one of the following:

(3.7.1)  $(X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ .

First we calculate  $e_i(X, L)$ . Since

$$\begin{aligned} e_i(X, L) &= \sum_{l=0}^i (-1)^l \binom{n-i+l-1}{l} c_{i-l}(X) L^{n-i+l} \\ &= \sum_{l=0}^i (-1)^l \binom{n-i+l-1}{l} \binom{n+1}{i-l} 2^{n-i+l}, \end{aligned}$$

we have

$$\begin{aligned} e_0(X, L) &= \left( (-1)^0 \binom{2}{0} \binom{4}{0} 2^0 \right) 2^3 = 8, \\ e_1(X, L) &= \left( (-1)^0 \binom{1}{0} \binom{4}{1} 2^0 + (-1)^1 \binom{2}{1} \binom{4}{0} 2^1 \right) 2^2 = 0, \\ e_2(X, L) &= \left( (-1)^0 \binom{0}{0} \binom{4}{2} 2^0 + (-1)^1 \binom{1}{1} \binom{4}{1} 2^1 + (-1)^2 \binom{2}{2} \binom{4}{0} 2^2 \right) 2 = 4, \\ e_3(X, L) &= \left( (-1)^0 \binom{-1}{0} \binom{4}{3} 2^0 + (-1)^1 \binom{0}{1} \binom{4}{2} 2^1 + (-1)^2 \binom{1}{2} \binom{4}{1} 2^2 + (-1)^3 \binom{2}{3} \binom{4}{0} 2^3 \right) 2^0 = 4. \end{aligned}$$

Next we calculate  $b_i(X, L)$ . Since

$$b_j(X, \mathbb{C}) = \begin{cases} 1, & \text{if } j = 0, 2, \\ 0, & \text{if } j = 1, 3, \end{cases}$$

we have

$$\begin{aligned} b_0(X, L) &= e_0(X, L) = 8, \\ b_1(X, L) &= -e_1(X, L) + 2b_0(X) = 2, \\ b_2(X, L) &= e_2(X, L) - 2(b_0(X) - b_1(X)) = 2, \\ b_3(X, L) &= -e_3(X, L) + 2(b_0(X) - b_1(X) + b_2(X)) = 0. \end{aligned}$$

(3.7.2)  $X$  is the blowing up of  $\mathbb{P}^3$  at a point and  $L = \pi^*(\mathcal{O}_{\mathbb{P}^3}(2)) - E$ , where  $\pi : X \rightarrow \mathbb{P}^3$  is its birational morphism and  $E$  is the exceptional divisor. Then by [3, Theorem 3.2] and (3.7.1) above, we have

$$\begin{aligned} e_0(X, L) &= 7, \\ e_1(X, L) &= 0, \\ e_2(X, L) &= 5, \\ e_3(X, L) &= 6. \end{aligned}$$

and

$$\begin{aligned} b_0(X, L) &= 7, \\ b_1(X, L) &= 2, \\ b_2(X, L) &= 3, \\ b_3(X, L) &= 0. \end{aligned}$$

(3.7.3)  $(X, L)$  is either

$$(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^* \mathcal{O}_{\mathbb{P}^1}(1)), (\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^2 p_i^* \mathcal{O}_{\mathbb{P}^2}(1)) \text{ or } (\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}), H(T_{\mathbb{P}^2}))$$

where  $p_i$  is the  $i$ th projection and  $T_{\mathbb{P}^2}$  is the tangent bundle of  $\mathbb{P}^2$ .

(3.7.3.1) The case where  $(X, L) \cong (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^* \mathcal{O}_{\mathbb{P}^1}(1))$ .

Since  $\mathcal{T}_X \cong \oplus_{j=1}^3 p_j^* \mathcal{T}_{\mathbb{P}^1}$ , we have

$$\begin{aligned} c_1(\mathcal{T}_X) &= \sum_{j=1}^3 p_j^* c_1(\mathcal{T}_{\mathbb{P}^1}) \\ &= \sum_{j=1}^3 p_j^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)), \\ c_2(\mathcal{T}_X) &= p_1^* c_1(\mathcal{T}_{\mathbb{P}^1}) p_2^* c_1(\mathcal{T}_{\mathbb{P}^1}) + p_1^* c_1(\mathcal{T}_{\mathbb{P}^1}) p_3^* c_1(\mathcal{T}_{\mathbb{P}^1}) + p_2^* c_1(\mathcal{T}_{\mathbb{P}^1}) p_3^* c_1(\mathcal{T}_{\mathbb{P}^1}) \\ &= p_1^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)) p_2^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)) + p_1^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)) p_3^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)) + p_2^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)) p_3^* c_1(\mathcal{O}_{\mathbb{P}^1}(2)) \\ c_3(X) &= e(X). \end{aligned}$$

On the other hand

$$\begin{aligned} b_0(X) &= 1, \\ b_1(X) &= 2q(X) = 0, \\ b_2(X) &= 3b_2(\mathbb{P}^1) + b_1(\mathbb{P}^1)b_1(\mathbb{P}^1) + 2b_0(\mathbb{P}^1)b_1(\mathbb{P}^1)^2 = 3, \\ b_3(X) &= 3b_2(\mathbb{P}^1) + b_1(\mathbb{P}^1)b_1(\mathbb{P}^1) + 2b_0(\mathbb{P}^1)b_1(\mathbb{P}^1)^2 = 3. \end{aligned}$$

Therefore

$$\begin{aligned}
e_0(X, L) &= L^3 = 6, \\
e_1(X, L) &= \sum_{l=0}^1 (-1)^l \binom{1+l}{l} c_{1-l}(X) L^{2+l} = 0, \\
e_2(X, L) &= \sum_{l=0}^2 (-1)^l \binom{l}{l} c_{2-l}(X) L^{1+l} = 6, \\
e_3(X, L) &= e(X) = 5,
\end{aligned}$$

and

$$\begin{aligned}
b_0(X, L) &= e_0(X, L) = 6, \\
b_1(X, L) &= -e_1(X, L) + 2b_0(X) = 2, \\
b_2(X, L) &= e_2(X, L) - 2(b_0(X) - b_1(X)) = 4, \\
b_3(X, L) &= -e_3(X, L) + 2(b_0(X) - b_1(X) + b_2(X)) = 3.
\end{aligned}$$

(3.7.3.2) The case where  $(X, L) \cong (\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^2 p_i^* \mathcal{O}_{\mathbb{P}^2}(1))$ .

Since  $\mathcal{T}_X \cong \oplus_{j=1}^2 p_j^*(\mathcal{T}_{\mathbb{P}^2})$ , we have

$$\begin{aligned}
c_1(\mathcal{T}_X) &= \sum_{j=1}^2 p_j^* c_1(\mathcal{T}_{\mathbb{P}^2}) \\
&= \sum_{j=1}^2 p_j^* c_1(\mathcal{O}_{\mathbb{P}^2}(3)), \\
c_2(\mathcal{T}_X) &= p_1^* c_2(\mathcal{T}_{\mathbb{P}^2}) + p_1^* c_1(\mathcal{T}_{\mathbb{P}^2}) p_2^* c_1(\mathcal{T}_{\mathbb{P}^2}) + p_2^* c_2(\mathcal{T}_{\mathbb{P}^2}) \\
&= 3p_1^* \mathcal{O}_{\mathbb{P}^2}(1)^2 + 9p_1^* \mathcal{O}_{\mathbb{P}^2}(1) p_2^* \mathcal{O}_{\mathbb{P}^2}(1) + 3p_2^* \mathcal{O}_{\mathbb{P}^2}(1)^2, \\
c_3(\mathcal{T}_X) &= p_1^* c_2(\mathcal{T}_{\mathbb{P}^2}) p_2^* c_1(\mathcal{T}_{\mathbb{P}^2}) + p_1^* c_1(\mathcal{T}_{\mathbb{P}^2}) p_2^* c_2(\mathcal{T}_{\mathbb{P}^2}) \\
&= 9p_1^* \mathcal{O}_{\mathbb{P}^2}(1)^2 p_2^* \mathcal{O}_{\mathbb{P}^2}(1) + 9p_1^* \mathcal{O}_{\mathbb{P}^2}(1) p_2^* \mathcal{O}_{\mathbb{P}^2}(1)^2, \\
c_4(X) &= e(X).
\end{aligned}$$

On the other hand

$$\begin{aligned}
b_0(X) &= 1, \\
b_1(X) &= 2q(X) = 0, \\
b_2(X) &= b_2(\mathbb{P}^2) + b_1(\mathbb{P}^2)b_1(\mathbb{P}^2) + b_2(\mathbb{P}^2) = 2, \\
b_3(X) &= 2b_3(\mathbb{P}^2) + 2b_2(\mathbb{P}^2)b_1(\mathbb{P}^2) = 0, \\
b_4(X) &= 2b_4(\mathbb{P}^2) + 2b_3(\mathbb{P}^2)b_1(\mathbb{P}^2) + b_2(\mathbb{P}^2)b_2(\mathbb{P}^2) = 3.
\end{aligned}$$

Therefore

$$\begin{aligned}
e_0(X, L) &= L^4 = 6, \\
e_1(X, L) &= \sum_{l=0}^1 (-1)^l \binom{2+l}{l} c_{1-l}(X) L^{3+l} = 0, \\
e_2(X, L) &= \sum_{l=0}^2 (-1)^l \binom{1+l}{l} c_{2-l}(X) L^{2+l} = 6, \\
e_3(X, L) &= \sum_{l=0}^3 (-1)^l \binom{l}{l} c_{3-l}(X) L^{1+l} = 6, \\
e_4(X, L) &= e(X) = 9,
\end{aligned}$$

and

$$\begin{aligned}
b_0(X, L) &= e_0(X, L) = 6, \\
b_1(X, L) &= -e_1(X, L) + 2b_0(X) = 2, \\
b_2(X, L) &= e_2(X, L) - 2(b_0(X) - b_1(X)) = 4, \\
b_3(X, L) &= -e_3(X, L) + 2(b_0(X) - b_1(X) + b_2(X)) = 0, \\
b_4(X, L) &= e_4(X, L) - 2(b_0(X) - b_1(X) + b_2(X) - b_3(X)) = 3.
\end{aligned}$$

(3.7.3.3) The case where  $(X, L) \cong (\mathbb{P}_{\mathbb{P}^2}(\mathcal{T}_{\mathbb{P}^2}), H(\mathcal{T}_{\mathbb{P}^2}))$ .

Then by [4, Corollary 3.1 (3.1.2) and Corollary 3.3 (3.3.2)] we have

$$\begin{aligned}
e_0(X, L) &= s_2(\mathcal{T}_{\mathbb{P}^2}) = K_{\mathbb{P}^2}^2 - c_2(\mathbb{P}^2) = 6, \\
e_1(X, L) &= -(c_1(\mathcal{T}_{\mathbb{P}^2}) + K_{\mathbb{P}^2})c_1(\mathcal{T}_{\mathbb{P}^2}) = 0, \\
e_2(X, L) &= c_2(\mathbb{P}^2) + c_2(\mathcal{T}_{\mathbb{P}^2}) = 6, \\
e_3(X, L) &= 2e(\mathbb{P}^2) = 6,
\end{aligned}$$

and

$$\begin{aligned}
b_0(X, L) &= e_0(X, L) = 6, \\
b_1(X, L) &= (c_1(\mathcal{T}_{\mathbb{P}^2}) + K_{\mathbb{P}^2})c_1(\mathcal{T}_{\mathbb{P}^2}) + 2 = 2, \\
b_2(X, L) &= b_2(X) + c_2(\mathbb{P}^2) - 1 = 4, \\
b_3(X, L) &= b_3(X) = 0.
\end{aligned}$$

(3.7.4) The case where  $(X, L)$  is a linear section of the Grassmann variety  $\text{Gr}(5, 2)$  parametrizing lines in  $\mathbb{P}^4$ , embedded in  $\mathbb{P}^9$  via the Plücker embedding. Then  $L^n = 5$ .

First we review the Chern class of  $\text{Gr}(p, q)$  parametrizing  $\mathbb{P}^{q-1}$  in  $\mathbb{P}^{p-1}$ . Let  $S$  (resp.  $Q$ ) be the universal subbundle (resp. the universal quotient bundle) of  $\text{Gr}(p, q)$ . Then

$$c(\text{Gr}(p, q)) = c(S^\vee \otimes Q). \quad (1)$$

We note that  $\text{rank} S = q$  and  $\text{rank} Q = p - q$ . From (1),

$$ch(\mathcal{T}_{\text{Gr}(p, q)}) = ch(S^\vee)ch(Q) \quad (2)$$

holds. Since  $ch(Q) + ch(S) = p$ , we have

$$ch(S) = q - \sum_{k \geq 1} ch_k(Q).$$

On the other hand

$$ch_k(S^\vee) = q - \sum_{k \geq 1} (-1)^k ch_k(Q). \quad (3)$$

Next we explain the Schubert calculus. For  $\lambda = (\lambda_0, \dots, \lambda_d)$  with  $p - q \geq \lambda_0 \geq \dots \geq \lambda_d \geq 0$ , we set

$$\{\lambda_0, \dots, \lambda_d\} = \det(c_{\lambda_i + j - 1}(\text{Gr}(p, q)))_{0 \leq i, j \leq p}.$$

Then  $c_m(\text{Gr}(p, q)) = \{m, 0, \dots, 0\}$ . We note that the following equality holds.

$$\{\lambda\} \cdot c_m(\text{Gr}(p, q)) = \sum \{\mu\}, \quad (4)$$

where the sum over  $\mu$  with  $p - q \geq \mu_0 \geq \lambda_0 \geq \dots \geq \mu_p \geq \lambda_p$  and  $\sum_{i=0}^p \lambda_i = m + \sum_{i=0}^p \mu_i$ .

Moreover we have

$$\int c_1(\mathrm{Gr}(p, q))^s \{\lambda_0, \dots, \lambda_d\} = \frac{k!}{a_0! \cdots a_d!} \prod_{i < j} (a_j - a_i). \quad (5)$$

Here  $a_i = p - q + i - \lambda_i$ ,  $k = \sum_{i=0}^p a_i - \frac{p(p+1)}{2}$ .

Here we consider the case where  $p = 5$  and  $q = 2$ . Then first we calculate  $c_j(\mathrm{Gr}(5, 2))$  for  $1 \leq j \leq 5$ . From (2) and (3), we have

$$\begin{aligned} ch(\mathrm{Gr}(5, 2)) &= ch(S^\vee)ch(Q) \\ &= \left( 2 - \sum_{k \geq 1} (-1)^k ch_k(Q) \right) \left( 3 + \sum_{k \geq 1} ch_k(Q) \right). \end{aligned}$$

Using this, we get the following. (Here we note that  $c_j(Q) = 0$  for  $j \geq 4$  because  $\mathrm{rank} Q = 3$ .)

$$\begin{aligned} c_1(\mathrm{Gr}(5, 2)) &= 5c_1(Q) \\ c_2(\mathrm{Gr}(5, 2)) &= 12c_1(Q)^2 - c_2(Q) \\ c_3(\mathrm{Gr}(5, 2)) &= 20c_1(Q)^3 - 10c_1(Q)c_2(Q) + 5c_3(Q) \\ c_4(\mathrm{Gr}(5, 2)) &= 28c_1(Q)^4 - 38c_1(Q)^2c_2(Q) + 20c_1(Q)c_3(Q) + 7c_2(Q)^2 \\ c_5(\mathrm{Gr}(5, 2)) &= 36c_1(Q)^5 - 90c_1(Q)^3c_2(Q) + 40c_1(Q)^2c_3(Q) + 45c_1(Q)c_2(Q)^2 - 10c_2(Q)c_3(Q). \end{aligned}$$

Next we use the Schubert calculus. First from (5) we get the following.

$$\begin{aligned} c_1(Q)^6 &= 5, \\ c_1(Q)^4c_2(Q) &= 3, \\ c_1(Q)^3c_3(Q) &= 1. \end{aligned}$$

Next we calculate  $c_2(Q)^2c_1(Q)^2$ . Since  $\{2, 0\} \cdot \{2, 0\} = \{3, 1\} + \{2, 2\}$ , we have

$$\begin{aligned} c_2(Q)^2c_1(Q)^2 &= \int c_1(\mathrm{Gr}(5, 2))^2 \{3, 1\} + \int c_1(\mathrm{Gr}(5, 2))^2 \{2, 2\} \\ &= 2. \end{aligned}$$

Next we calculate  $c_2(Q)c_3(Q)c_1(Q)$ . Since  $\{2, 0\} \cdot \{3, 0\} = \{3, 2\}$ , we have

$$\begin{aligned} c_2(Q)c_3(Q)c_1(Q) &= \int c_1(\mathrm{Gr}(5, 2))^2 \{3, 2\} \\ &= 1. \end{aligned}$$

Hence

$$\begin{aligned} c_1(\mathrm{Gr}(5, 2))L^5 &= 5c_1(Q)^6 = 25 \\ c_2(\mathrm{Gr}(5, 2))L^4 &= 12c_1(Q)^6 - c_1(Q)^4c_2(Q) = 57 \\ c_3(\mathrm{Gr}(5, 2))L^3 &= 20c_1(Q)^6 - 10c_1(Q)^4c_2(Q) + 5c_1(Q)^3c_3(Q) = 75 \\ c_4(\mathrm{Gr}(5, 2))L^2 &= 28c_1(Q)^6 - 38c_1(Q)^4c_2(Q) + 20c_1(Q)^3c_3(Q) + 7c_1(Q)^2c_2(Q)^2 = 60 \\ c_5(\mathrm{Gr}(5, 2))L &= 36c_1(Q)^6 - 90c_1(Q)^4c_2(Q) + 40c_1(Q)^3c_3(Q) \\ &\quad + 45c_1(Q)^2c_2(Q)^2 - 10c_1(Q)c_2(Q)c_3(Q) \\ &= 30. \end{aligned}$$



Therefore

$$\begin{aligned}
e_1(X, L) &= c_1(X)L^5 - 5L^6 = 0, \\
e_2(X, L) &= c_2(X)L^4 - 4c_1(X)L^5 + 10L^6 = 7, \\
e_3(X, L) &= c_3(X)L^3 - 3c_2(X)L^4 + 6c_1(X)L^5 - 10L^6 = 4, \\
e_4(X, L) &= c_4(X)L^2 - 2c_3(X)L^3 + 3c_2(X)L^4 - 4c_1(X)L^5 + 5L^6 = 6, \\
e_5(X, L) &= c_5(X)L - c_4(X)L^2 + c_3(X)L^3 - c_2(X)L^4 + c_1(X)L^5 - L^6 = 8.
\end{aligned}$$

Next we calculate  $b_i(X, L)$ . Since  $b_0(X) = b_2(X) = 1$ ,  $b_4(X) = b_6(X) = b_8(X) = 2$ ,  $b_{10}(X) = b_{12}(X) = 1$  and  $b_j(X) = 0$  for every positive odd integer  $j$ , we have

$$\begin{aligned}
b_0(X, L) &= 5, \\
b_1(X, L) &= 2, \\
b_2(X, L) &= 5, \\
b_3(X, L) &= 0, \\
b_4(X, L) &= 2, \\
b_5(X, L) &= 4, \\
b_6(X, L) &= 2.
\end{aligned}$$

(3.7.5) The case where  $(X, L)$  is a complete intersection of two hyperquadrics in  $\mathbb{P}^{n+2}$ . Then  $L^n = 4$ . First we calculate  $e(X)$  in this case. In general we can prove the following.

**Lemma 3.1** *Let  $(X, L)$  be a complete intersection of two hypersurfaces of degree  $s$  and  $t$  in  $\mathbb{P}^{n+2}$ . Then*

$$e(X) = -\frac{s}{t^2}(1-t)^{n+3} \sum_{k=0}^{n-1} \left(\frac{s}{t}\right)^k + \frac{s}{t^2} \sum_{j=0}^{n-1} \left(\frac{s}{t}\right)^j \sum_{k=0}^{2+j} (-t)^k \binom{n+3}{n+3-k} + (-s)^{n+1}(-t).$$

*Proof.* Let  $c_j := c_j(X)$  and  $H := \mathcal{O}_X(1)$ . Then the following holds (see [7, Example 3.2.12]).

$$(1+H)^{n+3} = C(X)(1+sH)(1+tH).$$

Here  $C(X) = (1+c_1+\dots+c_n)$ . Hence

$$\begin{aligned}
(c_n + sc_{n-1}H) + t(c_{n-1}H + sc_{n-2}H^2) &= \binom{n+3}{n} H^n \\
(c_{n-1} + sc_{n-2}H) + t(c_{n-2}H + sc_{n-3}H^2) &= \binom{n+3}{n-1} H^{n-1} \\
&\vdots \\
(c_2 + sc_1H) + t(c_1H + sc_0H^2) &= \binom{n+3}{2} H^2
\end{aligned}$$

Hence

$$\begin{aligned}
&c_n + sc_{n-1}H + (-t)^{n-2} \cdot t(c_1H^{n-1} + sc_0H^n) \\
&= \left( \binom{n+3}{n} + (-t) \binom{n+3}{n-1} + \dots + (-t)^{n-3} \binom{n+3}{3} + (-t)^{n-2} \binom{n+3}{2} \right) H^n
\end{aligned}$$

Moreover since  $c_1 H^{n-1} = \mathcal{O}(n-s-t+3)H^{n-1}$ , we have  $c_1 H^{n-1} + s c_0 H^n = (n-t+3)H^n$ .  
Therefore

$$\begin{aligned}
& c_n + s c_{n-1} H \\
&= \left( \binom{n+3}{n} + (-t) \binom{n+3}{n-1} + \cdots \right. \\
&\quad \left. \cdots + (-t)^{n-3} \binom{n+3}{3} + (-t)^{n-2} \binom{n+3}{2} + (-t)^{n-1} \binom{n+3}{1} + (-t)^n \right) H^n \\
&= -\frac{s}{t^2} \left( (-t)^3 \binom{n+3}{n} + (-t)^4 \binom{n+3}{n-1} + \cdots \right. \\
&\quad \left. \cdots + (-t)^n \binom{n+3}{3} + (-t)^{n+1} \binom{n+3}{2} + (-t)^{n+2} \binom{n+3}{1} + (-t)^{n+3} \right) \\
&= -\frac{s}{t^2} \left( (1-t)^{n+3} - 1 - (-t)^1 \binom{n+3}{n+2} - (-t)^2 \binom{n+3}{n+1} \right) \\
&= -\frac{s}{t^2} \left( (1-t)^{n+3} - 1 + t \binom{n+3}{n+2} - t^2 \binom{n+3}{n+1} \right).
\end{aligned}$$

By the same argument as above for every  $j$  with  $1 \leq j \leq n-1$  we have

$$\begin{aligned}
& c_j H^{n-j} + s c_{j-1} H^{n-j+1} \\
&= \frac{s}{(-t)^{n-j+2}} \left( (1-t)^{n+3} - \sum_{k=0}^{n+2-j} (-t)^k \binom{n+3}{n+3-k} \right).
\end{aligned}$$

Hence

$$c_n = -\frac{s}{t^2} (1-t)^{n+3} \sum_{k=0}^{n-1} \left(\frac{s}{t}\right)^k + \frac{s}{t^2} \sum_{j=0}^{n-1} \left(\frac{s}{t}\right)^j \sum_{k=0}^{2+j} (-t)^k \binom{n+3}{n+3-k} + (-s)^{n+1} (-t). \quad (6)$$

□

**Lemma 3.2** *Let  $(X, L)$  be a complete intersection of two hyperquadrics in  $\mathbb{P}^{n+2}$ . Then*

$$e(X) = \begin{cases} 2n+4, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

*Proof.* By Lemma 3.1 we have

$$c_n = (-2)^{n+2} + \frac{1}{2} \left( n(-1)^n + \sum_{j=1}^n \sum_{k=0}^{n+2-j} (-2)^k \binom{n+3}{n+3-k} \right).$$

Next we prove the following.

**Claim 3.1**

$$\begin{aligned}
& (-2)^{n+2} + \frac{1}{2} \left( n(-1)^n + \sum_{j=1}^n \sum_{k=0}^{n+2-j} (-2)^k \binom{n+3}{n+3-k} \right) \\
&= \begin{cases} 0, & n \text{ is odd,} \\ 2n+4, & n \text{ is even.} \end{cases} \quad (7)
\end{aligned}$$

*Proof.* First we note the following.

$$\begin{aligned}
& \sum_{j=1}^n \sum_{k=0}^{n+2-j} (-2)^k \binom{n+2}{n+2-k} \tag{8} \\
&= \sum_{j=1}^n \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} + \sum_{j=1}^n (-2)^{n+2-j} \binom{n+2}{j} \\
&= \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} + \sum_{k=0}^1 (-2)^k \binom{n+2}{n+2-k} + \sum_{j=1}^n (-2)^{n+2-j} \binom{n+2}{j},
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^n \sum_{k=0}^{n+2-j} (-2)^k \binom{n+2}{n+3-k} &= \sum_{j=1}^n \sum_{k=1}^{n+2-j} (-2)^k \binom{n+2}{n+3-k} \tag{9} \\
&= \sum_{j=1}^n \sum_{k=0}^{n+1-j} (-2)^{k+1} \binom{n+2}{n+2-k} \\
&= \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^{k+1} \binom{n+2}{n+2-k} \\
&\quad + \sum_{k=0}^1 (-2)^{k+1} \binom{n+2}{n+2-k} \\
&= -2 \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} \\
&\quad + \sum_{k=0}^1 (-2)^{k+1} \binom{n+2}{n+2-k}.
\end{aligned}$$

Then from (8) and (9) we have

$$\begin{aligned}
& \sum_{j=1}^n \sum_{k=0}^{n+2-j} (-2)^k \left( \binom{n+2}{n+2-k} + \binom{n+2}{n+2-k} \right) \\
&= - \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} - \sum_{k=0}^1 (-2)^k \binom{n+2}{n+2-k} + \sum_{j=1}^n (-2)^{n+2-j} \binom{n+2}{j} \\
&= - \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} + 4n + 6 + (-1)^{n+2} - (-2)^{n+2}. \tag{10}
\end{aligned}$$

Here we prove (7) by induction on  $n$ .

If  $n = 1$  and  $2$ , then (7) holds.

Next we assume that (7) holds for  $n - 1$  is odd. Then by assumption we have the following equality.

$$(-2)^{n+1} + \frac{1}{2} \left( (n-1)(-1)^{n-1} + \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} \right) = 0. \tag{11}$$

Then by using (11), we have

$$\begin{aligned}
& (-2)^{n+2} + \frac{1}{2} \left( n(-1)^n + \sum_{j=1}^n \sum_{k=0}^{n+2-j} (-2)^k \binom{n+3}{n+3-k} \right) \\
&= (-2)^{n+2} + \frac{1}{2} \left( n(-1)^n - \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} + 4n+6 \right. \\
&\quad \left. + (-1)^{n+2} - (-2)^{n+2} \right) \\
&= (-2)^{n+2} + \frac{1}{2} \left( n - \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} + 4n+6 + (-1)^{n+2} - (-2)^{n+2} \right) \\
&= (-2)^{n+2} + \frac{1}{2} \left( n - \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} + 4n+6+1 - (-2)^{n+2} \right) \\
&= (-2)^{n+2} + \frac{1}{2} (5n+7+2(-2)^{n+1}(n-1)(-1)^{n-1}) \\
&= 2n+4.
\end{aligned}$$

Next we assume that (7) holds for  $n-1$  is even. Then by assumption we have the following equality.

$$(-2)^{n+1} + \frac{1}{2} \left( (n-1)(-1)^{n-1} + \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} \right) = 2n+2. \quad (12)$$

Then by using (12), we have

$$\begin{aligned}
& (-2)^{n+2} + \frac{1}{2} \left( n(-1)^n + \sum_{j=1}^n \sum_{k=0}^{n+2-j} (-2)^k \binom{n+3}{n+3-k} \right) \\
&= (-2)^{n+2} + \frac{1}{2} \left( -n - \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} + 4n+6 + (-1)^{n+2} - (-2)^{n+2} \right) \\
&= (-2)^{n+2} + \frac{1}{2} \left( 3n+5 - \sum_{j=1}^{n-1} \sum_{k=0}^{n+1-j} (-2)^k \binom{n+2}{n+2-k} - (-2)^{n+2} \right) \\
&= (-2)^{n+2} + \frac{1}{2} (3n+5+2(-2)^{n+1} + (n-1)(-1)^{n-1} - 2(2n+2) - (-2)^{n+2}) \\
&= 0.
\end{aligned}$$

This completes the proof of Claim 3.1. □

From Claim 3.1 we get Lemma 3.2. □

**Remark 3.1** Let  $(X, L)$  be a complete intersection of two hypersurfaces of degree  $s$  and  $t$  in  $\mathbb{P}^{n+2}$ . Then from (6) we can write  $e(X)$  as follows.

$$e(X) = (-1)^n st \left( \sum_{k=0}^n (-1)^k \binom{n+3}{k} \left( \sum_{j=0}^{n-k} s^{n-k-j} t^j \right) \right).$$

*Proof.*

$$\begin{aligned}
c_n &= -\frac{s}{t^2}(1-t)^{n+3} \left( 1 + \frac{s}{t} + \cdots + \left(\frac{s}{t}\right)^{n-1} \right) + \frac{s}{t^2} \left\{ \left( 1 + (-t) \binom{n+3}{n+2} + (-t)^2 \binom{n+3}{n+1} \right) \right. \\
&\quad \left. + \frac{s}{t} \left( 1 + (-t) \binom{n+3}{n+2} + (-t)^2 \binom{n+3}{n+1} + (-t)^3 \binom{n+3}{n} \right) \right. \\
&\quad \left. + \cdots + \left(\frac{s}{t}\right)^{n-1} \left( 1 + (-t) \binom{n+3}{n+2} + \cdots + (-t)^{n+1} \binom{n+3}{2} \right) \right\} + (-s)^{n+1}(-t) \\
&= -\frac{s}{t^2} \left( (-t)^3 \binom{n+3}{n} + \cdots + (-t)^{n+3} \right) \left( 1 + \frac{s}{t} + \cdots + \left(\frac{s}{t}\right)^{n-1} \right) \\
&\quad + \frac{s}{t^2} \left( \sum_{j=1}^{n-1} \left(\frac{s}{t}\right)^j \sum_{k=P_1}^j (-t)^{2+k} \binom{n+3}{n+1-k} \right) + (-s)^{n+1}(-t) \\
&= -\frac{s}{t^2} \left( (-t)^3 \binom{n+3}{n} + \cdots + (-t)^{n+3} \right) - \frac{s^2}{t^3} \left( (-t)^4 \binom{n+3}{n-1} + \cdots + (-t)^{n+3} \right) \\
&\quad - \frac{s^3}{t^4} \left( (-t)^5 \binom{n+3}{n-2} + \cdots + (-t)^{n+3} \right) \cdots - \frac{s^n}{t^{n+1}} \left( (-t)^{n+2} \binom{n+3}{1} + (-t)^{n+3} \right) \\
&\quad + (-s)^{n+1}(-t) \\
&= (-s) \left( (-t) \binom{n+3}{n} + \cdots + (-t)^{n+1} \right) + s^2 \left( (-t) \binom{n+3}{n-1} + \cdots + (-t)^n \right) \\
&\quad - s^3 \left( (-t) \binom{n+3}{n-2} + \cdots + (-t)^{n-1} \right) \cdots + (-s)^n \left( (-t) \binom{n+3}{1} + (-t)^2 \right) + (-s)^{n+1}(-t) \\
&= \sum_{j=1}^{n+1} (-s)(-t)^j \binom{n+3}{n+1-j} + \sum_{j=1}^n (-s)^2(-t)^j \binom{n+3}{n-j} + \sum_{j=1}^{n-1} (-s)^3(-t)^j \binom{n+3}{n-1-j} \\
&\quad + \cdots + \sum_{j=1}^2 (-s)^n(-t)^j \binom{n+3}{2-j} + \sum_{j=1}^1 (-s)^{n+1}(-t)^j \binom{n+3}{1-j} \\
&= st \left( \sum_{j=1}^{n+1} (-t)^{j-1} \binom{n+3}{n+1-j} + \sum_{j=1}^n (-s)(-t)^{j-1} \binom{n+3}{n-j} + \cdots + \sum_{j=1}^1 (-s)^n(-t)^{j-1} \binom{n+3}{1-j} \right) \\
&= st \left( \sum_{j=0}^n (-t)^j \binom{n+3}{n-j} + \sum_{j=0}^{n-1} (-s)(-t)^j \binom{n+3}{n-1-j} + \cdots + \sum_{j=0}^0 (-s)^n(-t)^j \binom{n+3}{-j} \right) \\
&= st \left( \sum_{k=0}^n (-1)^{n-k} \binom{n+3}{k} \left( \sum_{j=0}^{n-k} s^{n-k-j} t^j \right) \right) \\
&= (-1)^n st \left( \sum_{k=0}^n (-1)^k \binom{n+3}{k} \left( \sum_{j=0}^{n-k} s^{n-k-j} t^j \right) \right).
\end{aligned}$$

So we get the assertion.  $\square$

Here we go back to the case (3.7.5). In this case, there exists a smooth ladder  $X \supset X_1 \supset \cdots \supset X_{n-1}$  of  $L$  such that  $(X_j, L_j)$  is complete intersection of two hyperquadrics in  $\mathbb{P}^{n-j+2}$ . Since  $e_i(X, L) = e(X_{n-i})$ , we see that

$$e_i(X, L) = \begin{cases} 2i + 4, & \text{if } i \text{ is even with } i \geq 2, \\ 0, & \text{if } i \text{ is odd with } i \geq 3. \end{cases}$$

We also note that

$$e_i(X, L) = \begin{cases} 4, & \text{if } i = 0, \\ 0, & \text{if } i = 1. \end{cases}$$

Next we calculate  $b_i(X, L)$ . Then

$$b_i(X, L) = \begin{cases} 2i + 4 - 2\frac{i}{2} = i + 4, & \text{if } i \text{ is even with } i \geq 2, \\ 0 + 2\frac{i+1}{2} = i + 1, & \text{if } i \text{ is odd with } i \geq 3. \end{cases}$$

We also note that

$$b_i(X, L) = \begin{cases} 4, & \text{if } i = 0, \\ 2, & \text{if } i = 1. \end{cases}$$

(3.7.6) The case where  $X$  is a hypercubic in  $\mathbb{P}^{n+1}$  and  $L = \mathcal{O}_X(1)$ .

Here we consider more general case than this. In general we can prove the following claim.

**Lemma 3.3** *Let  $(X, L)$  be a polarized manifold of dimension  $n$  such that  $X$  is a hypersurface of degree  $m$  and  $L = \mathcal{O}_X(1)$ . Then*

$$\begin{aligned} e_i(X, L) &= \frac{1}{m} ((1-m)^{i+2} - 1 + m(i+2)), \\ b_i(X, L) &= \begin{cases} \frac{1}{m} ((1-m)^{i+2} - 1 + m(i+2)) - i, & \text{if } i \text{ is even with } i \leq n-1, \\ -\frac{1}{m} ((1-m)^{i+2} - 1 + m(i+2)) + i + 1, & \text{if } i \text{ is odd with } i \leq n-1. \end{cases} \end{aligned}$$

*Proof.* First we calculate  $e_n(X, L)$ . Let  $c_j := c_j(X)$  and  $H := \mathcal{O}_X(1)$ . Then the following holds (see [7, Example 3.2.12]).

$$(1+H)^{n+2} = (1+c_1+\cdots+c_n)(1+mH).$$

Hence

$$\begin{aligned} c_n + mc_{n-1}H &= \binom{n+2}{n} H^n \\ c_{n-1} + mc_{n-2}H &= \binom{n+2}{n-1} H^{n-1} \\ &\vdots \\ c_1 + mH &= \binom{n+2}{1} H \end{aligned}$$

So we have

$$\begin{aligned} c_n &= (-m)^n H^n + \frac{1}{m^2} \left( (-m)^2 \binom{n+2}{2} + (-m)^3 \binom{n+2}{3} + \cdots + (-m)^{n+1} \binom{n+2}{n+1} \right) H^n \\ &= m(-m)^n + \frac{1}{m^2} \left( 1-m \right)^{n+2} - 1 - (-m) \binom{n+2}{1} - (-m)^{n+2} \Big) m \\ &= \frac{1}{m} ((1-m)^{n+2} - 1 + m(n+2)). \end{aligned}$$

On the other hand, in this case, there exists a smooth ladder  $X \supset X_1 \supset \cdots \supset X_{n-1}$  of  $L$  such that  $(X_j, L_j)$  is a hypersurface of degree  $m$  in  $\mathbb{P}^{n-j+1}$ . Since  $e_i(X, L) = e(X_{n-i})$ , by the above argument we see that

$$e_i(X, L) = \frac{1}{m} ((1-m)^{i+2} - 1 + m(i+2)).$$

Next we calculate  $b_i(X, L)$ . Since

$$b_i(X) = \begin{cases} 1, & \text{if } i \text{ is even with } i \leq n-2, \\ 0, & \text{if } i \text{ is odd with } i \leq n-2, \end{cases}$$

we have

$$\begin{aligned} b_i(X, L) &= \begin{cases} \frac{1}{m}((1-m)^{i+2} - 1 + m(i+2)) - 2 \cdot \frac{i}{2}, & \text{if } i \text{ is even with } i \leq n-1, \\ -\frac{1}{m}((1-m)^{i+2} - 1 + m(i+2)) + 2 \cdot \frac{i+1}{2}, & \text{if } i \text{ is odd with } i \leq n-1. \end{cases} \\ &= \begin{cases} \frac{1}{m}((1-m)^{i+2} - 1 + m(i+2)) - i, & \text{if } i \text{ is even with } i \leq n-1, \\ -\frac{1}{m}((1-m)^{i+2} - 1 + m(i+2)) + i + 1, & \text{if } i \text{ is odd with } i \leq n-1. \end{cases} \end{aligned}$$

This completes the proof of Lemma 3.3.  $\square$

(3.7.7) The case where  $X$  is a double covering of  $\mathbb{P}^n$  branched along a smooth hypersurface of degree 4, and  $L$  is the pull-back of  $\mathcal{O}_{\mathbb{P}^n}(1)$ . Here we consider more general case than this.

**Lemma 3.4** *Let  $X$  be a double covering of  $\mathbb{P}^n$  branched along a smooth hypersurface of degree  $m$  with even  $m \geq 4$ , and  $L$  is the pull-back of  $\mathcal{O}_{\mathbb{P}^n}(1)$ . Then*

$$\begin{aligned} e_i(X, L) &= i + 2 - \frac{1}{m}(m-1 + (1-m)^{i+1}), \\ b_i(X, L) &= \left( i + 2 - \frac{1}{m}(m-1 + (1-m)^{i+1}) \right) + (-1)^{i+1} \begin{cases} i & \text{if } i \text{ is even,} \\ i + 1 & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

*Proof.* First we calculate  $e_n(X, L)$ . Let  $B$  be the branch locus. Then

$$e(X) = 2e(\mathbb{P}^n) - e(B).$$

Hence by Lemma 3.3

$$\begin{aligned} e_n(X, L) &= e(X) \\ &= 2e(\mathbb{P}^n) - e(B) \\ &= 2n + 2 - \frac{1}{m}((1-m)^{n+1} + m(n+1) - 1) \\ &= n + 2 - \frac{1}{m}(m-1 + (1-m)^{n+1}). \end{aligned}$$

Next we consider  $e_i(X, L)$ . First we note that  $\Delta(X, L) = 1$  in this case. Since  $\text{Bs}|L| = \emptyset$ , there exists a smooth ladder  $X \supset X_1 \supset \cdots \supset X_{n-1}$  of  $L$ . Then we see that  $\Delta(X_j, L_j) = 1$  and  $L_j^{n-j} = 2$ , where  $L_j := L|_{X_j}$  because  $g(X, L) = m/2 - 1 \geq 1 = \Delta(X, L)$  and  $L^n = 2 = 2\Delta(X, L)$ . Hence  $X_j$  is a double covering of  $\mathbb{P}^{n-j}$  branched along a smooth hypersurface of degree 4, and  $L_j$  is the pull-back of  $\mathcal{O}_{\mathbb{P}^{n-j}}(1)$ . Since  $e_i(X, L) = e(X_{n-i})$ , by the above argument we see that for every integer  $i$  with  $i \geq 1$ , we have

$$e_i(X, L) = i + 2 - \frac{1}{m}(m-1 + (1-m)^{i+1}). \quad (13)$$

Here we note that  $e_0(X, L) = L^n = 2 = 0 + 2 - \frac{1}{4}(3 + (-3)^{0+1})$ . Hence (13) also holds for  $i = 0$ .

Next we calculate  $b_i(X, L)$ . By the Barth-type theorem (see e.g. [8, Theorem 7.1.15]), we have

$$b_i(X) = \begin{cases} 1, & \text{if } i \text{ is even with } i \neq n, \\ 0, & \text{if } i \text{ is odd with } i \neq n. \end{cases}$$

Hence we have

$$\begin{aligned}
b_i(X, L) &= (-1)^i \left( e_i(X, L) - 2 \sum_{j=0}^{i-1} (-1)^j b_j(X) \right) \\
&= (-1)^i \left( i + 2 - \frac{1}{m} (m-1 + (1-m)^{i+1}) \right) - 2(-1)^i \cdot \begin{cases} \frac{i-1+1}{2} & \text{if } i \text{ is even,} \\ \frac{i-1}{2} + 1 & \text{if } i \text{ is odd,} \end{cases} \\
&= \left( i + 2 - \frac{1}{m} (m-1 + (1-m)^{i+1}) \right) + (-1)^{i+1} \begin{cases} i & \text{if } i \text{ is even,} \\ i+1 & \text{if } i \text{ is odd.} \end{cases}
\end{aligned}$$

We get the assertion of Lemma 3.4.  $\square$

(3.7.8) The case where  $(X, L)$  is a weighted hypersurface of degree 6 in the weighted projective space  $\mathbb{P}(3, 2, 1, \dots, 1)$ . Then  $L^n = 1$  and  $\text{Bs}|L| = \{p\}$  (see [1, (16.7) Theorem and Appendix 1]).

In this case, there exists a smooth ladder  $X \supset X_1 \supset \dots \supset X_{n-1}$  of  $L$  such that  $(X_j, L_j)$  is a weighted hypersurface of degree 6 in the weighted projective space  $\mathbb{P}(3, 2, 1, \dots, 1)$ . Since  $e_i(X, L) = e(X_{n-i})$ , in order to calculate  $e_i(X, L)$  for  $i \geq 1$ , it suffices to calculate  $e(X)$ .

Let  $\pi : X^* \rightarrow X$  be the blowing up at  $p \in X$ . Then  $\pi^*(L) - E$  is base point free and let  $f : X^* \rightarrow \mathbb{P}^{n-1}$  be the morphism defined by  $|\pi^*(L) - E|$ . In this case, there exists a projective bundle  $p : V \rightarrow \mathbb{P}^{n-1}$  and a double covering  $\rho : X^* \rightarrow V$  such that  $f = p \circ \rho$ . Here we note that  $V = \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}})$ . Let  $H_V$  be the tautological line bundle of  $V$  and let  $B$  be the branch locus of  $\rho$ . Then there exist  $B_1 \in |H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1)|$  and  $B_2 \in |3H_V|$  such that  $B_1 \cong \mathbb{P}^{n-1}$  and  $B = B_1 + B_2$ . Here we note that the following equality holds.

$$e(X) = e(X^*) - e(E) + 1, \quad (14)$$

$$e(X^*) = 2e(V) - e(B), \quad (15)$$

$$e(B) = e(B_1) + e(B_2). \quad (16)$$

Therefore in order to calculate  $e(X)$ , we need the value of  $e(E)$ ,  $e(B_1)$ ,  $e(B_2)$ , and  $e(V)$ .

First we note that

$$e(E) = e(\mathbb{P}^{n-1}) = n \quad (17)$$

and

$$e(B_1) = e(\mathbb{P}^{n-1}) = n. \quad (18)$$

Next we calculate  $e(V)$ . By [1, Proof of Lemma in Appendix 2], we see that there exist the following three exact sequence:

$$0 \rightarrow 2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1) \rightarrow \mathcal{T}_V \rightarrow \mathcal{T}_{\mathbb{P}^{n-1}|V} \rightarrow 0, \quad (19)$$

$$0 \rightarrow \mathcal{O}_V \rightarrow H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))^\vee \otimes \pi^*(\mathcal{O}_{\mathbb{P}^{n-1}}(1)) \rightarrow \mathcal{T}_{\mathbb{P}^{n-1}|V} \rightarrow 0, \quad (20)$$

$$0 \rightarrow \mathcal{T}_{B_2} \rightarrow \mathcal{T}_V|_{B_2} \rightarrow (3H_V)|_{B_2} \rightarrow 0. \quad (21)$$

From (19), we have

$$c(\mathcal{T}_V) = c(2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))c(\mathcal{T}_{\mathbb{P}^{n-1}|V}). \quad (22)$$

Hence

$$\begin{aligned}
c_n(V) &= (2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))c_{n-1}(\mathcal{T}_{\mathbb{P}^{n-1}|V}) \\
&= (2H_V - 2\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))(n(\pi^*\mathcal{O}_{\mathbb{P}^{n-1}}(1))^{n-1}).
\end{aligned}$$



By (20), we get

$$c((\pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1))^{\oplus n}) = c(\mathcal{O}_V)c(\pi^* \mathcal{T}_{\mathbb{P}^{n-1}}). \quad (23)$$

Hence

$$c_{n-1}(p^* \mathcal{T}_{\mathbb{P}^{n-1}}) = \binom{n}{n-1} \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{n-1}.$$

Therefore

$$\begin{aligned} c_n(V) &= (2H_V - 2\pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1))(n(\pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1))^{n-1}) \\ &= 2nH_V \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{n-1} \\ &= 2n. \end{aligned} \quad (24)$$

Next we calculate  $e(B_2)$ . Before this, we note the following. Let  $\mathcal{E} := \mathcal{O}_{\mathbb{P}^{n-1}}(2) \oplus \mathcal{O}_{\mathbb{P}^{n-1}}$  and let  $H(\mathcal{E})$  be the tautological line bundle of  $\mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{E})$ . Then  $V = \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{E})$  and  $H(\mathcal{E}) = H_V$ . In this case, since  $c_j(\mathcal{E}) = 0$  for any  $j \geq 2$ , we have  $s_j(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^{n-1}}(2)^j$ . Therefore

$$H_V^j \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{n-j} = \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{n-j} s_{j-1}(\mathcal{E}) = 2^{j-1}. \quad (25)$$

From (21), we have

$$c(\mathcal{T}|_{B_2}) = c(\mathcal{T}_{B_2})c(3H_V|_{B_2}). \quad (26)$$

From (26) we obtain the following:

$$\begin{aligned} c_{n-1}(B_2) + c_{n-2}(B_2)(3H_V|_{B_2}) &= c_{n-1}(V)B_2 \\ c_{n-2}(B_2) + c_{n-3}(B_2)(3H_V|_{B_2}) &= c_{n-2}(V)B_2 \\ &\vdots \\ c_1(B_2) + 3H_V|_{B_2} &= c_1(V)B_2 \end{aligned}$$

Therefore

$$\begin{aligned} c_{n-1}(B_2) &= 3(c_{n-1}(V)H_V + (-3)c_{n-2}(V)H_V^2 \\ &\quad + \cdots + (-3)^{n-2}c_1(V)H^{n-1} + (-3)^{n-1}H^n). \end{aligned} \quad (27)$$

On the other hand, by (22) we have

$$\begin{aligned} c_j(V) &= (2H_V - 2\pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1))c_{j-1}(\mathcal{T}_{\mathbb{P}^{n-1}}|_V) + c_j(\mathcal{T}_{\mathbb{P}^{n-1}}|_V) \\ &= 2 \binom{n}{j-1} H_V \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{j-1} + \left( \binom{n}{j} - 2 \binom{n}{j-1} \right) \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)^j. \end{aligned}$$

Hence by using (25) we get

$$\begin{aligned} c_j(V)H^{n-j} &= 2 \binom{n}{j-1} H_V^{n-j+1} \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{j-1} + \left( \binom{n}{j} - 2 \binom{n}{j-1} \right) H_V^{n-j} \pi^* \mathcal{O}_{\mathbb{P}^{n-1}}(1)^j \\ &= 2^{n-j+1} \binom{n}{j-1} + 2^{n-j-1} \left( \binom{n}{j} - 2 \binom{n}{j-1} \right) \\ &= 2^{n-j} \binom{n}{j-1} + 2^{n-j-1} \binom{n}{j}. \end{aligned}$$

Therefore

$$\sum_{j=1}^{n-1} (-3)^{n-j-1} c_j(V)H^{n-j} = 2 \sum_{j=1}^{n-1} (-6)^{n-j-1} \binom{n}{j-1} + \sum_{j=1}^{n-1} (-6)^{n-j-1} \binom{n}{j}. \quad (28)$$

On the other hand

$$\begin{aligned}
2 \sum_{j=1}^{n-1} (-6)^{n-j-1} \binom{n}{j-1} &= \frac{1}{18} \sum_{j=1}^{n-1} (-6)^{n-j+1} \binom{n}{j-1} \\
&= \frac{1}{18} ((1 + (-6))^n - (-6)n - 1) \\
&= \frac{1}{18} ((-5)^n + 6n - 1),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1}^{n-1} (-6)^{n-j-1} \binom{n}{j} &= -\frac{1}{6} \sum_{j=1}^{n-1} (-6)^{n-j} \binom{n}{j} \\
&= -\frac{1}{6} ((1 + (-6))^n - (-6)^n - 1) \\
&= \frac{1}{6} ((-6)^n - (-5)^n + 1).
\end{aligned}$$

Since  $H_V^n = 2^{n-1}$ , from (27) we get

$$\begin{aligned}
c_{n-1}(B_2) & \tag{29} \\
&= 3 \left( \frac{1}{18} ((-5)^n + 6n - 1) + \frac{1}{6} ((-6)^n - (-5)^n + 1) + (-3)^{n-1} 2^{n-1} \right) \\
&= -\frac{1}{3} (-5)^n + \frac{3n+1}{3}.
\end{aligned}$$

From (18) and (29), we have

$$e(B) = e(B_1) + e(B_2) = 2n + \frac{1 - (-5)^n}{3}. \tag{30}$$

By (24) and (30) we get

$$e(X^*) = 2e(V) - e(B) = 2n + \frac{(-5)^n - 1}{3}. \tag{31}$$

Therefore by (17) and (31)

$$e(X) = e(X^*) - e(E) + 1 = n + \frac{(-5)^n + 2}{3}. \tag{32}$$

So we see that

$$e_i(X, L) = i + \frac{(-5)^i + 2}{3} \tag{33}$$

for every integer  $i$  with  $1 \leq i \leq n$ . Here we note that this equality holds for the case where  $i = 0$ .

Next we calculate  $b_i(X, L)$ . Since

$$b_j(X) = \begin{cases} 1, & \text{if } j \text{ is even with } j \neq n-1, \\ 0, & \text{if } j \text{ is odd with } j \neq n-1, \end{cases}$$

we have

$$\begin{aligned}
b_i(X, L) &= (-1)^i \left( e_i(X, L) - 2 \sum_{j=0}^{i-1} (-1)^j b_j(X) \right) \\
&= (-1)^i \left( i + \frac{(-5)^i + 2}{3} \right) - 2(-1)^i \cdot \begin{cases} \frac{i}{2}, & \text{if } i \text{ is even,} \\ \frac{i+1}{2} + 1, & \text{if } i \text{ is odd,} \end{cases} \\
&= \begin{cases} (-1)^i \frac{(-5)^i + 2}{3}, & \text{if } i \text{ is even,} \\ (-1)^i \frac{(-5)^i - 1}{3}, & \text{if } i \text{ is odd.} \end{cases}
\end{aligned}$$

**Example 3.8** The case where  $(X, L)$  is a hyperquadric fibration over a smooth curve  $C$ . Let  $f : X \rightarrow C$  be its morphism. We put  $\mathcal{E} := f_*(L)$ . Then  $\mathcal{E}$  is a locally free sheaf of rank  $n + 1$  on  $C$ . Let  $\pi : \mathbb{P}_C(\mathcal{E}) \rightarrow C$  be the projection. Then there exists an embedding  $i : X \hookrightarrow \mathbb{P}_C(\mathcal{E})$  such that  $f = \pi \circ i$ ,  $X \in |2H(\mathcal{E}) + \pi^*(B)|$  for some  $B \in \text{Pic}(C)$  and  $L = H(\mathcal{E})|_X$ . Let  $e := \deg \mathcal{E}$  and  $b := \deg B$ . Then by [6, Theorem 3.1], we see that the following holds. Let  $(X, L)$  be a hyperquadric fibration over a smooth curve  $C$  with  $\dim X = n \geq 3$ , and let  $i$  be an integer with  $0 \leq i \leq n$ . Then

$$e_i(X, L) = (-1)^i (2e + (i + 1)b) + \begin{cases} 2(i + 1)(1 - g(C)) & \text{if } i \text{ is odd,} \\ 2i(1 - g(C)) & \text{if } i \text{ is even.} \end{cases}$$

**Example 3.9** The case where  $(X, L)$  is a scroll over a smooth curve  $C$ . Then by [4, Corollary 3.1 (3.1.1) and Corollary 3.3 (3.3.1)], we see that the following holds. Let  $\mathcal{E}$  be an ample vector bundle of rank  $n$  on  $C$  such that  $X = \mathbb{P}_C(\mathcal{E})$  and  $L = H(\mathcal{E})$ .

$$e_i(X, L) = \begin{cases} i(2 - 2g(C)) & \text{if } i \geq 1, \\ \deg \mathcal{E} & \text{if } i = 0. \end{cases}$$

$$b_i(X, L) = \begin{cases} h^i(X, \mathbb{C}) & \text{if } i \geq 1, \\ \deg \mathcal{E} & \text{if } i = 0. \end{cases}$$

**Example 3.10** The case where  $(X, L)$  is a scroll over a smooth surface  $S$ . Let  $\mathcal{E}$  be an ample vector bundle of rank  $n - 1$  on  $S$  such that  $X = \mathbb{P}_S(\mathcal{E})$  and  $L = H(\mathcal{E})$ . Then by [4, Corollary 3.1 (3.1.2) and Corollary 3.3 (3.3.2)], we see that the following holds.

$$e_i(X, L) = \begin{cases} (i - 1)c_2(S) & \text{if } i \geq 3, \\ c_2(S) + c_2(\mathcal{E}) & \text{if } i = 2, \\ -(c_1(\mathcal{E}) + K_S)c_1(\mathcal{E}) & \text{if } i = 1, \\ s_2(\mathcal{E}) & \text{if } i = 0. \end{cases}$$

$$b_i(X, L) = \begin{cases} h^i(X, \mathbb{C}) & \text{if } m \geq i \geq 3, \\ h^2(X, \mathbb{C}) + c_2(\mathcal{E}) - 1 & \text{if } i = 2, \\ c_1(\mathcal{E})(c_1(\mathcal{E}) + K_S) + 2 & \text{if } i = 1, \\ s_2(\mathcal{E}) & \text{if } i = 0. \end{cases}$$

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