Sectional class of ample line bundles on smooth projective varieties *†‡

YOSHIAKI FUKUMA

September 11, 2012

Abstract

Let X be an n-dimensional smooth projective variety defined over the field of complex numbers, let $L_1, \dots, L_{n-i}, A_1$ and A_2 be ample line bundles on X. In this paper, we will define the sectional class $\mathrm{cl}_i(X, L_1, \dots, L_{n-i}; A_1, A_2)$ for every integer i with $0 \leq i \leq n$, and we will investigate this invariant. In particular, for every integer i with $0 \leq i \leq n$, by setting $L_1 = \cdots = L_{n-i} = L$ and $A_1 = A_2 = L$, we give a classification of polarized manifolds (X, L) by the value of $\mathrm{cl}_i(X, L) := \mathrm{cl}_i(X, \underbrace{L, \cdots, L}_{n-i}; L, L)$.

1 Introduction

Let X be a smooth projective variety of dimension n defined over the field of complex numbers, and let L be an ample line bundle on X. Then (X, L) is called a *polarized manifold*. Assume that L is very ample and let $\varphi : X \hookrightarrow \mathbb{P}^N$ be the morphism defined by $|L|$. Then φ is an embedding. In this situation, its dual variety $X^\vee \to (\mathbb{P}^N)^\vee$ is a hypersurface of N-dimensional projective space except some special types. Then the *class* $\mathrm{cl}(X, L)$ of (X, L) is defined by the following.

$$\mathrm{cl}(X, L) = \begin{cases} \deg(X^\vee), & \text{if } X^\vee \text{ is a hypersurface in } (\mathbb{P}^N)^\vee \\ 0, & \text{otherwise.} \end{cases}$$

A lot of investigations by using $\mathrm{cl}(X, L)$ have been obtained (for example [21], [25], [30], [22], [26], [24], [1], [29] and so on). In this paper, we are going to define a generalization of this invariant. Let X be a smooth projective variety of dimension n and let $L_1, \dots, L_{n-i}, A_1$ and A_2 be ample (not necessarily very ample) line bundles on X. Then in Section 2 we will define the *sectional class* $\mathrm{cl}_i(X, L_1, \dots, L_{n-i}; A_1, A_2)$ for every integer i with $0 \leq i \leq n$ (see Definition 2.2), and we will study some fundamental properties concerning this invariant. In section 3, we consider the following special case: Let L be an ample (not necessarily very ample) line bundle on X and we set $L_1 = \cdots = L_{n-i} = L$ and $A_1 = A_2 = L$. Then we will define $\mathrm{cl}_i(X, L) := \mathrm{cl}_i(X, \underbrace{L, \cdots, L}_{n-i}; L, L)$. We will call this invariant the *ith sectional class of* (X, L). In section 3, we study this invariant $\mathrm{cl}_i(X, L)$ for the case where L is not necessarily very ample and will get some results about $\mathrm{cl}_i(X, L)$.

Here we note the following: Assume that L is very ample. Then there exists a member $X_j \in |L_{j-1}|$ such that each X_j is a smooth projective manifold of dimension $n - j$ and $L_j := L|_{X_j}$ for

*Key words and phrases. Ample vector bundle, (multi-)polarized manifold, class, sectional Euler number, sectional Betti number.

†2010 *Mathematics Subject Classification.* Primary 14C20; Secondary 14C17, 14J30, 14J35, 14J40, 14J60, 14M99, 14N15.

‡This research was partially supported by the Grant-in-Aid for Scientific Research (C) (No.20540045), Japan Society for the Promotion of Science, Japan.

every j with $1 \leq j \leq n-i$. In this case, we see that $\mathrm{cl}_i(X, L)$ is the class of the i dimensional polarized manifold (X_{n-i}, L_{n-i}). In particular, if $i = n$, then $\mathrm{cl}_n(X, L)$ is equal to the class $\mathrm{cl}(X, L)$ of (X, L) if L is very ample.

As we said above, there are a lot of works about the class $\mathrm{cl}(X, L)$ for very ample line bundles L, that is, the case where $i = n$ and L is very ample. Classifications of (X, L) concerning $\mathrm{cl}_i(X, L)$ are known for the following cases.

- The case where $i = n \leq 3$ and L is very ample (see [21], [25], [22]).
- The case where $i = 2$, $n \geq 2$ and L is very ample (see [30], [26], [24]).
- The case where $i = n = 2$ and L is ample (see [29]).

In this paper, we give classifications of (X, L) by the value of $\mathrm{cl}_i(X, L)$ for the following cases.

- The case where $i = 1$, $n \geq 3$, $\mathrm{cl}_1(X, L) \leq 4$ and L is ample.
- The case where $i = 2$, $n \geq 3$, $\mathrm{cl}_2(X, L) \leq 16$ and L is ample and spanned.
- The case where $i = 3$, $n \geq 3$, $\mathrm{cl}_3(X, L) \leq 8$ and L is ample and spanned.
- The case where $i = 4$, $n \geq 5$, $\mathrm{cl}_4(X, L) \leq 1$ (resp. $\mathrm{cl}_4(X, L) = 2$) and L is ample and spanned (resp. very ample).

In subsection 3.1, we calculate $\mathrm{cl}_i(X, L)$ for some special cases. The results in subsection 3.1 will be used in order to classify (X, L) by the value of $\mathrm{cl}_i(X, L)$. In subsections 3.2, 3.3, 3.4 and 3.5 we obtain the classification of (X, L) by the value of $\mathrm{cl}_1(X, L)$, $\mathrm{cl}_2(X, L)$, $\mathrm{cl}_3(X, L)$ and $\mathrm{cl}_4(X, L)$.

We see from the definition of the ith sectional class that it is somewhat hard to calculate this invariant in general (see also [17]). But we expect that the ith sectional class has properties similar to those of the class of i-dimensional projective manifolds, and we believe that this invariant is useful for investigating polarized manifolds. We also hope that we can give a characterization of special polarized manifolds by the value of sectional classes. This is the reason why we define this invariant.

In our paper for the future, we will define and study the sectional class for the case of ample vector bundles.

2 Definition and fundamental results

Definition 2.1 Let $L_1, \dots, L_m$ be ample line bundles on a smooth projective variety X. Then $(X, L_1, \dots, L_m)$ is called a *multi-polarized manifold of type m*.

Definition 2.2 Let X be a smooth projective variety of dimension $n \geq 1$, let i be an integer with $0 \leq i \leq n$ and let $L_1, \dots, L_{n-i}, A_1, A_2$ be ample line bundles on X.
Then the *ith sectional class* of $(X, L_1, \dots, L_{n-i}; A_1, A_2)$ is defined by the following.

$$
\begin{aligned}
&\mathrm{cl}_i(X, L_1, \dots, L_{n-i}; A_1, A_2) \\
&:= \begin{cases}
e_0(X, L_1, \dots, L_n), & \text{if } i = 0, \\
(-1)\{e_1(X, L_1, \dots, L_{n-1}) - e_0(X, L_1, \dots, L_{n-1}, A_1) \\
\qquad -e_0(X, L_1, \dots, L_{n-1}, A_2)\}, & \text{if } i = 1, \\
(-1)^i\{e_i(X, L_1, \dots, L_{n-i}) - e_{i-1}(X, L_1, \dots, L_{n-i}, A_1) \\
\qquad -e_{i-1}(X, L_1, \dots, L_{n-i}, A_2) + e_{i-2}(X, L_1, \dots, L_{n-i}, A_1, A_2)\}, & \text{if } 2 \leq i \leq n.
\end{cases}
\end{aligned}
$$

where $e_k(X, L_1, \dots, L_{n-k})$ is the kth sectional Euler number[1] of $(X, L_1, \dots, L_{n-k})$ [15, Definitions 3.1.1 and 5.1.1].

[1] For $k = n$ we set $e_k(X, L_1, \dots, L_{n-k}) := e(X) = \sum_{i=0}^{2n}(-1)^i h^i(X, \mathbb{C})$.

Remark 2.1 (1) If i is odd, then $e_i(X, L_1, \dots, L_{n-i})$ is even.

Proof. First we note that $1 \leq i$ because i is odd. Then by the definition of the ith sectional Betti number $b_i(X, L_1, \dots, L_{n-i})$ (see [15, Definitions 3.2.1 and 5.1.1]), we have

$$e_i(X, L_1, \dots, L_{n-i}) = 2\sum_{j=0}^{i-1}(-1)^j h^j(X, \mathbb{C}) + (-1)^i b_i(X, L_1, \dots, L_{n-i}). \tag{1}$$

On the other hand, since i is odd, $b_i(X, L_1, \dots, L_{n-i})$ is even by [15, Theorem 4.1 and Definition 5.1.1]. Hence $e_i(X, L_1, \dots, L_{n-i})$ is even. $\square$

So if i is odd and $A_1 = A_2 = A$, then we see that $\mathrm{cl}_i(X, L_1, \dots, L_{n-i}; A, A)$ is even.
(2) If $i = 0$, then $\mathrm{cl}_0(X, L_1, \dots, L_n; A_1, A_2) = L_1 \cdots L_n > 0$.

Definition 2.3 Let (X, L) be a polarized manifold of dimension n and let i be an integer with $0 \leq i \leq n$. Then the *ith sectional class* of (X, L) is defined by the following.

$$\mathrm{cl}_i(X, L) := \mathrm{cl}_i(X, \underbrace{L, \dots, L}_{n-i}; L, L).$$

Remark 2.2 Assume that L is very ample. Then there exists a sequence of smooth subvarieties $X \supset X_1 \supset \cdots \supset X_{n-i}$ such that $X_j \in |L_{j-1}|$ and $\dim X_j = n - j$ for every integer j with $1 \leq j \leq n - i$, where $L_j = L|_{X_j}$. In particular, X_{n-i} is a smooth projective variety of dimension i and L_{n-i} is a very ample line bundle on X_{n-i}. Then $\mathrm{cl}_i(X, L)$ is equal to the class of (X_{n-i}, L_{n-i}).

Remark 2.3 ([20, II-1]) Let X be an n-dimensional smooth projective variety and let L be a very ample line bundle on X. Let $X \hookrightarrow \mathbb{P}^N$ be the embedding defined by $|L|$. For every integer i with $0 \leq i \leq n$, Severi defined the notion of the ith *rank* $r_i(X)$ of X as follows.

$$r_i(X) = \int L^i (L^\vee)^{N-1-i}(CX).$$

Here CX denotes the conormal variety, $X^\vee$ denotes the dual variety of X and $L^\vee = \mathcal{O}_{X^\vee}(1)$. Then we see that $r_i(X) = \mathrm{cl}_{n-i}(X, L)$ (see [20, (6) Theorem in II]). We also note that if $i = 0$, then $r_0(X) = \mathrm{cl}_n(X, L)$ is called the *class* of X.

Remark 2.4 By Definitions 2.2 and 2.3 we see that

$$\mathrm{cl}_i(X, L) = \begin{cases} e_0(X, L), & \text{if } i = 0, \\ (-1)\{e_1(X, L) - 2e_0(X, L)\}, & \text{if } i = 1, \\ (-1)^i\{e_i(X, L) - 2e_{i-1}(X, L) + e_{i-2}(X, L)\}, & \text{if } 2 \leq i \leq n. \end{cases}$$

Here $e_i(X, L)$ is the ith sectional Euler number of (X, L) ([10, Definition 3.1]).

Proposition 2.1 *Let X be a smooth projective variety of dimension n and let i be an integer with $0 \leq i \leq n$. Let $L_1, \dots, L_{n-i}, A_1, A_2$ be ample line bundles on X. Then the following holds*[2]*.*

$$\begin{aligned} &\mathrm{cl}_i(X, L_1, \dots, L_{n-i}; A_1, A_2) \\ &= \begin{cases} b_0(X, L_1, \dots, L_n), & \textit{if } i = 0, \\ b_1(X, L_1, \dots, L_{n-1}) + b_0(X, L_1, \dots, L_{n-1}, A_1) - b_0(X) & \\ \quad + b_0(X, L_1, \dots, L_{n-1}, A_2) - b_0(X), & \textit{if } i = 1, \\ b_i(X, L_1, \dots, L_{n-i}) - b_{i-2}(X) + b_{i-1}(X, L_1, \dots, L_{n-i}, A_1) - b_{i-1}(X) & \\ \quad + b_{i-1}(X, L_1, \dots, L_{n-i}, A_2) - b_{i-1}(X) & \\ \quad + b_{i-2}(X, L_1, \dots, L_{n-i}, A_1, A_2) - b_{i-2}(X), & \textit{if } 2 \leq i \leq n. \end{cases} \end{aligned}$$

[2]Here $b_i(X)$ denotes the ith Betti number $h^i(X, \mathbb{C})$. For $i = n$, we set $b_i(X, L_1, \dots, L_{n-i}) := b_i(X)$.

Proof. Since

$$(-1)^i\left(2\sum_{j=0}^{i-1}(-1)^j b_j(X) - 4\sum_{j=0}^{i-2}(-1)^j b_j(X) + 2\sum_{j=0}^{i-3}(-1)^j b_j(X)\right) = -2b_{i-1}(X) - 2b_{i-2}(X),$$

the assertion holds by substituting the equality (1) in Remark 2.1 (1) for the formula in Definition 2.2. □

Corollary 2.1 *Let (X, L) be a polarized manifold of dimension n. For any integer i with $0 \leq i \leq n$, the following holds.*

$$\begin{aligned}&\mathrm{cl}_i(X, L)\\ &= \begin{cases} b_0(X, L), & \text{if } i = 0,\\ b_1(X, L) + 2b_0(X, L) - 2, & \text{if } i = 1,\\ b_i(X, L) - b_{i-2}(X) + 2b_{i-1}(X, L) - 2b_{i-1}(X) + b_{i-2}(X, L) - b_{i-2}(X), & \text{if } 2 \leq i \leq n.\end{cases}\end{aligned}$$

Next we study the non-negativity of the sectional class.

Theorem 2.1 *Let X be a smooth projective variety of dimension n and let i be an integer with $1 \leq i \leq n$. Let $L_1, \ldots, L_{n-i}, A_1, A_2$ be ample and spanned line bundles on X. Then*

$$\mathrm{cl}_i(X, L_1, \ldots, L_{n-i}; A_1, A_2) \geq 0.$$

Proof. (i) First we assume that $2 \leq i$. Then by Proposition 2.1, we get

$$\begin{aligned}&\mathrm{cl}_i(X, L_1, \ldots, L_{n-i}, A_1, A_2)\\ &= b_i(X, L_1, \ldots, L_{n-i}) - b_{i-2}(X) + b_{i-1}(X, L_1, \ldots, L_{n-i}, A_1) - b_{i-1}(X)\\ &\quad + b_{i-1}(X, L_1, \ldots, L_{n-i}, A_2) - b_{i-1}(X) + b_{i-2}(X, L_1, \ldots, L_{n-i}, A_1, A_2) - b_{i-2}(X).\end{aligned}$$

In general, for every ample and spanned line bundles $H_1, \ldots, H_{n-j}$, by [15, Proposition 4.1 and Definition 5.1.1] we have $b_j(X, H_1, \ldots, H_{n-j}) \geq b_j(X)$ for every integer j with $0 \leq j \leq n$. On the other hand, we obtain $b_i(X) \geq b_{i-2}(X)$ by the hard Lefschetz theorem [27, Corollary 3.1.40]. Therefore we get the assertion.
(ii) Next we assume that $i = 1$. Then by definition we have

$$\mathrm{cl}_1(X, L_1, \ldots, L_{n-1}, A_1, A_2) = 2g_1(X, L_1, \ldots, L_{n-1}) + L_1 \cdots L_{n-1}(A_1 + A_2) - 2,$$

where $g_1(X, L_1, \ldots, L_{n-1})$ is the first sectional geometric genus of $(X, L_1, \ldots, L_{n-1})$ (see [12, Definition 2.1 (2) and Remark 2.2 (2)]). We note that $g_1(X, L_1, \ldots, L_{n-1}) \geq 0$ by [14, Theorem 6.1.1], and $L_1 \cdots L_{n-1}A_k \geq 1$ for $k = 1, 2$. So we have $\mathrm{cl}_1(X, L_1, \ldots, L_{n-1}, A_1, A_2) \geq 0$. □

Remark 2.5 By (ii) in the proof of Theorem 2.1 $\mathrm{cl}_1(X, L_1, \ldots, L_{n-1}, A_1, A_2) \geq 0$ holds for any merely ample line bundles $L_1, \ldots, L_{n-i}, A_1, A_2$.

By Definition 2.3, Remark 2.1 (2) and Theorem 2.1 the following holds.

Corollary 2.2 *Let (X, L) be a polarized manifold of dimension n and let i be an integer with $0 \leq i \leq n$. Assume that L is base point free. Then $\mathrm{cl}_i(X, L) \geq 0$.*

Here we propose the following conjecture.

Conjecture 2.1 *Let X be a smooth projective variety of dimension n and let i be an integer with $0 \leq i \leq n$. Let $L_1, \ldots, L_{n-i}, A_1, A_2$ be ample line bundles on X. Then*

$$\mathrm{cl}_i(X, L_1, \ldots, L_{n-i}; A_1, A_2) \geq 0.$$

By Remark 2.1 (2) (resp. Remark 2.5), this conjecture is true for the case where $i = 0$ (resp. $i = 1$).

If $i = 2$ and $\kappa(X) \geq 0$, then we can get the following lower bound.

Theorem 2.2 *Let X be a smooth projective variety of dimension n with $\kappa(X) \geq 0$ and let $L_1, \dots, L_{n-2}, A_1, A_2$ be ample line bundles on X. Then the following inequality holds.*

$$\begin{aligned}\mathrm{cl}_2(X, L_1, \dots, L_{n-2}; A_1, A_2) \quad &\geq \quad \frac{1}{2n}\left(\sum_{j=1}^{n-2} L_j\right)^2 L_1 \cdots L_{n-2} + \frac{1}{2}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2} \\ &+ \sum_{j=1}^{2}(L_1 + \cdots + L_{n-2} + A_j)L_1 \cdots L_{n-2}A_j + L_1 \cdots L_{n-2}A_1A_2.\end{aligned}$$

Proof. First we note that

$$\begin{aligned}\mathrm{cl}_2(X, L_1, \dots, L_{n-2}; A_1, A_2) \quad &= \quad e_2(X, L_1, \dots, L_{n-2}) + 2g_1(X, L_1, \dots, L_{n-2}, A_1) - 2 \\ &+2g_1(X, L_1, \dots, L_{n-2}, A_2) - 2 + L_1 \dots L_{n-2}A_1A_2.\end{aligned}$$

From [15, Theorem 5.3.1], we have

$$e_2(X, L_1, \dots, L_{n-2}) \geq \frac{1}{2n}\left(\sum_{j=1}^{n-2} L_j\right)^2 L_1 \cdots L_{n-2} + \frac{1}{2}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2}.$$

Moreover since $\kappa(X) \geq 0$ we have

$$2g_1(X, L_1, \dots, L_{n-2}, A_k) - 2 \geq (L_1 + \cdots + L_{n-2} + A_k)L_1 \cdots L_{n-2}A_k$$

for $k = 1, 2$. So we get the assertion. $\square$

Next we consider the value of the sectional class of a reduction of multi-polarized manifolds.

Proposition 2.2 *Let $(X, L_1, \cdots, L_{n-i}, A_1, A_2)$ be a multi-polarized manifold of type $n-i+2$ with $\dim X = n$, where i is an integer with $0 \leq i \leq n$. Let $(Y, H_1, \cdots, H_{n-i}, B_1, B_2)$ be a multi-polarized manifold of type $n-i+2$ such that $(X, L_1, \cdots, L_{n-i}, A_1, A_2)$ is a composite of simple blowing ups of $(Y, H_1, \cdots, H_{n-i}, B_1, B_2)$ and let γ be the number of its simple blowing ups[3]. Then*

$$\begin{aligned}&\mathrm{cl}_i(X, L_1, \dots, L_{n-i}; A_1, A_2) \\ &:= \begin{cases} \mathrm{cl}_0(Y, H_1, \dots, H_n; B_1, B_2) - \gamma, & \text{if } i = 0, \\ \mathrm{cl}_1(Y, H_1, \dots, H_{n-1}; B_1, B_2) - 2\gamma, & \text{if } i = 1, \\ \mathrm{cl}_i(Y, H_1, \dots, H_{n-i}; B_1, B_2), & \text{if } 2 \leq i \leq n-1 \text{ or } i = n \geq 2.\end{cases}\end{aligned}$$

Proof. By Definition 2.2, Remark 2.1 and [15, Proposition 5.3.1] and its proof, we get the assertion. $\square$

Corollary 2.3 *Let (X, L) be a polarized manifold of dimension $n \geq 2$ and let (Y, H) be a polarized manifold such that (X, L) is a composite of simple blowing ups of (Y, H) and let γ be the number of its simple blowing ups. Then for every integer i with $0 \leq i \leq n$, we have*

$$\mathrm{cl}_i(X, L) \quad := \quad \begin{cases} \mathrm{cl}_0(Y, H) - \gamma, & \text{if } i = 0, \\ \mathrm{cl}_1(Y, H) - 2\gamma, & \text{if } i = 1, \\ \mathrm{cl}_i(Y, H), & \text{if } 2 \leq i \leq n-1 \text{ or } i = n \geq 2.\end{cases}$$

Proof. By putting $L_1 := L, \cdots, L_{n-i} := L$, $A_1 := L$, $A_2 := L$, $H_1 := H, \cdots, H_{n-i} := H$, $B_1 := H$ and $B_2 := H$, we get the assertion by Proposition 2.2. $\square$

[3]For the definition of a simple blowing up, see [12, Definition 1.5].

3 On classification of polarized manifolds (X, L) by the sectional class

In this section, we study a classification of polarized manifolds (X, L) by the ith sectional class $\mathrm{cl}_i(X, L)$.

Notation 3.1 (1) Let Y be a projective variety and let $\mathcal{E}$ be a vector bundle on Y. Then $\mathbb{P}_Y(\mathcal{E})$ denotes the projective bundle over Y associated with $\mathcal{E}$ and $H(\mathcal{E})$ denotes the tautological line bundle.

(2) Let (X, L) be a hyperquadric fibration over a smooth curve C. We put $\mathcal{E} := f_*(L)$. Then $\mathcal{E}$ is a locally free sheaf of rank $n+1$ on C. Let $\pi : \mathbb{P}_C(\mathcal{E}) \to C$ be the projective bundle. Then $X \in |2H(\mathcal{E}) + \pi^*(B)|$ for some $B \in \mathrm{Pic}(C)$ and $L = H(\mathcal{E})|_X$, where $H(\mathcal{E})$ is the tautological line bundle of $\mathbb{P}_C(\mathcal{E})$. We put $e := \deg \mathcal{E}$ and $b := \deg B$.

Definition 3.1 Let $\mathcal{F}$ be a vector bundle on a smooth projective variety X. Then for every integer j with $j \geq 0$, the *jth Segre class $s_j(\mathcal{F})$ of $\mathcal{F}$* is defined by the following equation: $c_t(\mathcal{F}^\vee)s_t(\mathcal{F}) = 1$, where $\mathcal{F}^\vee := Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, $c_t(\mathcal{F}^\vee)$ is the Chern polynomial of $\mathcal{F}^\vee$ and $s_t(\mathcal{F}) = \sum_{j\geq 0} s_j(\mathcal{F})t^j$.

Remark 3.1 (a) Let $\mathcal{F}$ be a vector bundle on a smooth projective variety X. Let $\tilde{s}_j(\mathcal{F})$ be the jth Segre class which is defined in [19, Chapter 3]. Then $s_j(\mathcal{F}) = \tilde{s}_j(\mathcal{F}^\vee)$.
(b) For every integer i with $1 \leq i$, $s_i(\mathcal{F})$ can be written by using the Chern classes $c_j(\mathcal{F})$ with $1 \leq j \leq i$. (For example, $s_1(\mathcal{F}) = c_1(\mathcal{F})$, $s_2(\mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, and so on.)

3.1 Calculations on the sectional class of some special polarized manifolds

Here we calculate the sectional class of some special polarized manifolds which will be used in the following subsection. See also [17].

Example 3.1.1 Let (X, L) be a polarized manifold of dimension $n \geq 3$ and let $g(X, L)$ be the sectional genus. Assume that L is spanned and $g(X, L) \leq q(X) + 2$. Then (X, L) is one of the following types (see [6], [7] and [8]).

(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(b) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(c) A scroll over a smooth curve.

(d) A Del Pezzo manifold[4] with $L^n \geq 2$.

(e) X is a double covering of $\mathbb{P}^n$ branched along a smooth hypersurface of degree 6, and L is the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$.

(f) A scroll over a smooth surface S and (X, L) satisfies one of the types (2-1), (2-2) and (2-3) in [8, Theorem 3.3].

(g) A hyperquadric fibration over a smooth curve C and (X, L) satisfies one of the types (3-1) and (3-2) in [8, Theorem 3.3].

Here we calculate the ith sectional class of the above (e), (f) and (g).
(I) If (X, L) is the type (e), then by [17, Proposition 2.2 in Example 2.1 (vii.7)], we have

[4]Here we assume that L is spanned. So we see that $L^n \geq 2$

i	0	$1 \leq i$
$\mathrm{cl}_i(X, L)$	2	$6 \cdot 5^{i-1}$

(II) Next we consider the case (f). Here we use the same notation as in [8, Theorem 3.3].
(II.1) First we assume that (X, L) is the type (2-1) in [8, Theorem 3.3]. Then we have $K_S = -2H_\alpha - 2H_\beta$, $c_1(\mathcal{E}) = 2H_\alpha + 3H_\beta$ and $c_2(\mathcal{E}) = (H_\alpha + 2H_\beta)(H_\alpha + H_\beta) = 3$. Hence $K_S^2 = 8$, $K_S c_1(\mathcal{E}) = -10$, $c_1(\mathcal{E})^2 = 12$ and $L^n = s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 9$. On the other hand since $c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2 = 4$, by [13, Corollary 3.1 (3.1.2)] we have

i	0	1	2	3
$\mathrm{e}_i(X, L)$	9	-2	7	8

Therefore

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	9	20	20	8

(II.2) Next we consider the type (2-2) in [8, Theorem 3.3]. Then $K_S = -3H + E$ and $\mathcal{E} = (2H - E)^{\oplus 2}$. Hence $K_S^2 = 8$, $c_1(\mathcal{E})^2 = (4H - 2E)^2 = 12$, $c_2(\mathcal{E}) = (2H - E)^2 = 3$, $K_S c_1(\mathcal{E}) = -10$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 9$. We also note that $c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2 = 4$. Hence we have

i	0	1	2	3
$\mathrm{e}_i(X, L)$	9	-2	7	8

Therefore

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	9	20	20	8

(II.3) Next we consider the type (2-3) in [8, Theorem 3.3]. Then $K_S = -2H(\mathcal{F}) + c_1(\mathcal{F})F = -2H(\mathcal{F}) + F$, $\mathcal{E} = H(\mathcal{F}) \otimes p^*\mathcal{G}$, $\deg \mathcal{F} = 1$ and $\deg \mathcal{G} = 1$. Hence $K_S^2 = 4H(\mathcal{F})^2 - 4 = 0$, $c_1(\mathcal{E})^2 = (2H(\mathcal{F}) + F)^2 = 8$, $c_2(\mathcal{E}) = c_2(p^*\mathcal{G}) + H(\mathcal{G})c_1(p^*\mathcal{G}) + H(\mathcal{G})^2 = 2$, $K_S c_1(\mathcal{E}) = -4H(\mathcal{G})^2 = -4$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 6$. We also note that $c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2 = 0$. Hence by [13, Corollary 3.1 (3.1.2)] we have

i	0	1	2	3
$\mathrm{e}_i(X, L)$	6	-4	2	0

Therefore

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	6	16	16	8

(III) Finally we consider the case (g).
(III.1) First we assume that (X, L) is the type in the type (3-1) in [8, Theorem 3.3]. Then by [17, Example 2.1 (viii)] we have

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	6	16	16	8

(III.2) Next we consier the type (3-2) in [8, Theorem 3.3]. Then $e = d - 3$ and $b = 6 - d$. So by [17, Example 2.1 (viii)] we have

i	0	1	$2 \leq i \leq n$
$\mathrm{cl}_i(X,L)$	d	$2d+2$	$4(6-d)(i-1)+4(d-1)$

Here we note that $3 \leq d \leq 9$ holds in this case, and if $d=8$ (resp. $d \neq 8$), then $3 \leq n \leq 4$ (resp. $n=3$).

Example 3.1.2 Let (X,L) be a polarized manifold of dimension $n \geq 3$. Assume that $q(X)=0$, L is spanned and $g(X,L)=3$. Then (X,L) is one of (I-2), (III), (IV), (IV′) and (V) in [18, Theorem 2.1]. Here we calculate the second sectional class of (X,L), which will be used in Theorem 3.3.2.

(A) First we consider the case (I-2) in [18, Theorem 2.1]. Then by [17, Example 2.1 (viii)] we have $\mathrm{cl}_2(X,L) = 8e + 8b + 4(g(C)-1) = 8e + 8b - 4 = 28$.

(B) Next we consider the case (III) in [18, Theorem 2.1].
(B.1a) If (X,L) is the type (III-1a), then $n=5$ and $\mathrm{cl}_2(X,L) = c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}) = 27$ by [17, Example 2.1 (x)].
(B.1b) If (X,L) is the type (III-1b), then $n=4$. If $(S,\mathcal{E}) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^n}(2))$, then by [17, Example 2.1 (x)] we have $\mathrm{cl}_2(X,L) = c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}) = 27$.
If $(S,\mathcal{E}) = (\mathbb{P}^2, T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^n}(1))$, then by [17, Example 2.1 (x)] we have $\mathrm{cl}_2(X,L) = c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}) = 27$.
(B.1c) If (X,L) is the type (III-1c), then $S \cong \mathbb{P}^2$, $\mathrm{rank}(\mathcal{E}) = 2$ and $c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^2}(4)$. Hence $\mathrm{cl}_2(X,L) = c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}) = 27$.
(B.2) If (X,L) is the type (III-2), then S is a Del Pezzo surface with $K_S^2 = 2$ and $\mathcal{E}$ is an ample vector bundle of rank two on S with $c_1(\mathcal{E})^2 = 8$ and $K_S c_1(\mathcal{E}) = -4$. Hence $\mathrm{cl}_2(X,L) = c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}) = 26$.

(C) Next we consider the case (IV) in [18, Theorem 2.1]. By [17, Proposition 2.1 in Example 2.1 (vii.6)] we have $\mathrm{cl}_2(X,L) = 4 \cdot 3^2 = 36$.

(D) Next we consider the case (IV′) in [18, Theorem 2.1]. Since $\mathrm{cl}_2(X,L)$ and $\mathrm{cl}_3(X,L)$ are invariant under simple blowing ups by Corollary 2.3, we have $\mathrm{cl}_2(X,L) = 4 \cdot 3^2 = 36$.

(E) Next we consider the case (V) in [18, Theorem 2.1].
(E.1) If (X,L) is the type (V-1), then by [17, Proposition 2.2 in Example 2.1 (vii.7)] we have $\mathrm{cl}_2(X,L) = 8 \cdot 7^1 = 56$.
(E.2) If (X,L) is the type (V-2), then (X,L) is a Mukai manifold, that is, $\mathcal{O}_X(K_X + (n-2)L) = \mathcal{O}_X$. Hence by [9, Example 2.10 (7)] we have $g_2(X,L) = 1$ and $\chi_2^H(X,L) = 1 - h^1(\mathcal{O}_X) + g_2(X,L) = 2$, where $\chi_2^H(X,L)$ is the second sectional H-arithmetic genus of (X,L) (see [10, Definition 2.2 and Remark 2.1 (5)]). Furthermore by [11, Proposition 3.1] we have

$$h_2^{1,1}(X,L) = 10\chi_2^H(X,L) - (K_X + (n-2)L)^2 L^{n-2} + 2h^1(\mathcal{O}_X) = 20.$$

Here $h_2^{1,1}(X,L)$ denotes the second sectional Hodge number of type $(1,1)$ (see [10, Definition 3.1 (3)]). Hence by [10, Theorem 3.1 (3.1.1), (3.1.3) and (3.1.4)] we get $b_2(X,L) = 2g_2(X,L) + h_2^{1,1}(X,L) = 22$. Since $b_1(X,L) = 2g(X,L) = 6$ (see [10, Remark 3.1 (2)]) and $b_0(X,L) = L^n$, we have

$$\begin{aligned} e_2(X,L) &= 2b_0(X) - 2b_1(X) + b_2(X,L) = 2 - 2 \cdot 0 + 22 = 24, \\ e_1(X,L) &= 2b_0(X) - b_1(X,L) = 2 - 6 = -4, \\ e_0(X,L) &= b_0(X,L) = 4. \end{aligned}$$

Therefore we get $\mathrm{cl}_2(X,L) = 24 - 2(-4) + 4 = 36$.

Example 3.1.3 Let (X,L) be a polarized manifold of dimension $n \geq 3$ such that $h^0(L) \geq n+1$ and $L^n \leq 2$. Then we see that $\Delta(X,L) \leq 1$ and (X,L) is one of the following types[5].

(i) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(ii) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(iii) X is a double covering of $\mathbb{P}^n$ whose branch locus is of degree $2g(X,L)+2$ and L is the pull back of $\mathcal{O}_{\mathbb{P}^n}(1)$. In this case we see that $g(X,L) \geq 1$, and if $g(X,L) = 1$, then (X,L) is a Del Pezzo manifold.

If (X,L) is the type (iii), then by [17, Proposition 2.2 in Example 2.1 (vii.7)] we have $\mathrm{cl}_i(X,L) = (2g(X,L)+2)(2g(X,L)+1)^{i-1}$ for $i \geq 1$ and $\mathrm{cl}_0(X,L) = 2$.

Example 3.1.4 Let (X,L) be a polarized manifold of dimension $n \geq 3$ such that $b_2(X,L) = h^2(X,\mathbb{C})+1$. Here we calculate $\mathrm{cl}_i(X,L)$ if (X,L) is the type (e) in [16, Theorem 3.1].
(i) If $(S,\mathcal{E})$ is the type (e.1), then $c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^2}(3)$, $c_1(\mathcal{E})^2 = 9$, $c_2(S) = 3$, $c_2(\mathcal{E}) = 2$, $K_S^2 = 9$, $K_S c_1(\mathcal{E}) = -9$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 7$. Hence by [17, Example 2.1 (x)]

i	0	1	2	3
$\mathrm{cl}_i(X,L)$	7	14	12	4

In this case, $(X,L) = (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ is a Del Pezzo 3-fold with $L^3 = 7$.

(ii) If $(S,\mathcal{E})$ is the type (e.2), then $c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{Q}^2}(2)$, $c_1(\mathcal{E})^2 = 8$, $c_2(S) = 4$, $c_2(\mathcal{E}) = 2$, $K_S^2 = 8$, $K_S c_1(\mathcal{E}) = -8$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 6$. Hence by [17, Example 2.1 (x)]

i	0	1	2	3
$\mathrm{cl}_i(X,L)$	6	12	12	4

In this case, $(X,L) = (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ is a Del Pezzo 3-fold with $L^3 = 6$.

(iii) If $(S,\mathcal{E})$ is the type (e.3), then $c_1(\mathcal{E}) = 2H(\mathcal{F}) + \pi^* c_1(\mathcal{G})$, $c_1(\mathcal{E})^2 = 8$, $c_2(S) = 0$, $c_2(\mathcal{E}) = 2$, $K_S^2 = 0$, $K_S c_1(\mathcal{E}) = -4$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 6$. Hence by [17, Example 2.1 (x)]

i	0	1	2	3
$\mathrm{cl}_i(X,L)$	6	16	16	8

(iv) If $(S,\mathcal{E})$ is the type (e.4), then there exists a line bundle $\mathcal{O}_{\mathbb{P}^2}(2b)$ such that the branch locus $C \in |\mathcal{O}_{\mathbb{P}^2}(2b)|$. In this case $c_1(\mathcal{E}) = f^*\mathcal{O}_{\mathbb{P}^2}(2)$, $c_1(\mathcal{E})^2 = 8$, $c_2(S) = 2c_2(\mathbb{P}^2) + 2g(C) - 2 = 4b^2 - 6b + 6$, $c_2(\mathcal{E}) = 2$, $K_S^2 = 2(b-3)^2$, $K_S c_1(\mathcal{E}) = 4(b-3)$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 6$. Hence by [17, Example 2.1 (x)]

i	0	1	2	3
$\mathrm{cl}_i(X,L)$	6	$4b+8$	$4b^2+2b+6$	$4b$

If $b = 1$, then $(X,L) = (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ is a Del Pezzo 3-fold with $L^3 = 6$.

3.2 The case where $i = 1$

In this subsection, we consider the case where $i = 1$. Here we assume that $n \geq 3$. In this case by [10, Remark 3.1 (2)] we have

$$\mathrm{cl}_1(X,L) = -e_1(X,L) + 2e_0(X,L) = 2g(X,L) - 2 + 2L^n. \tag{2}$$

[5] $\Delta(X,L)$ denotes the Δ-genus of (X,L) (see [5, (2.2)]).

Since $g(X,L) \geq 0$ and $L^n \geq 1$, we see that $\mathrm{cl}_1(X,L) \geq 0$. We also note that $c_1(X,L)$ is even.

Next we consider a classification of (X,L) with small $\mathrm{cl}_1(X,L)$.

(I) First we consider the case where $\mathrm{cl}_1(X,L) = 0$.

Proposition 3.2.1 Let (X,L) be a polarized manifold of dimension $n \geq 3$. If $\mathrm{cl}_1(X,L) = 0$, then (X,L) is isomorphic to $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

Proof. If $\mathrm{cl}_1(X,L) = 0$, then we have $g(X,L) = 0$ and $L^n = 1$ from the equality (2). Therefore we see from [5, (12.1) Theorem and (5.10) Theorem] that (X,L) is isomorphic to $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. $\square$

(II) Next we consider the case where $\mathrm{cl}_1(X,L) = 2$.

Proposition 3.2.2 Let (X,L) be a polarized manifold of dimension $n \geq 3$. If $\mathrm{cl}_1(X,L) = 2$, then (X,L) is one of the following types.

(a) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(b) A Del Pezzo manifold and $L^n = 1$. In this case, X is a weighted hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3,2,1,\ldots,1)$.

(c) A scroll over an elliptic curve B and $L^n = 1$. In this case, $(X,L) = (\mathbb{P}_B(\mathcal{E}), H(\mathcal{E}))$, where $\mathcal{E}$ is an ample vector bundle of rank n on B with $c_1(\mathcal{E}) = 1$.

Proof. Then by the equality (2) we have $(g(X,L), L^n) = (0,2)$ or $(1,1)$. If (X,L) is the first type, then by [5, (12.1) Theorem and (5.10) Theorem] (X,L) is the type (a) above. If (X,L) is the last type, then we see from [5, (12.3) Theorem] that (X,L) is either the type (b) or the type (c) above. $\square$

(III) Next we consider the case where $\mathrm{cl}_1(X,L) = 4$.

Proposition 3.2.3 Let (X,L) be a polarized manifold of dimension $n \geq 3$. If $\mathrm{cl}_1(X,L) = 4$, then (X,L) is one of the following types.

(a) $(\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}), H(\mathcal{E}))$, where $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

(b) A Del Pezzo manifold and $L^n = 2$. In this case, X is a double covering of $\mathbb{P}^n$ branched along a smooth hypersurface of degree 4 and L is the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$.

(c) A scroll over an elliptic curve B and $L^n = 2$. In this case, $(X,L) = (\mathbb{P}_B(\mathcal{E}), H(\mathcal{E}))$, where $\mathcal{E}$ is an ample vector bundle of rank n on B with $c_1(\mathcal{E}) = 2$.

(d) $K_X = (3-n)L$ and $L^n = 1$ hold[6].

(e) (X,L) is a simple blowing up of (M,A), where M is a double covering of $\mathbb{P}^n$ with branch locus being a smooth hypersurface of degree 6 and $A = \pi^*(\mathcal{O}_{\mathbb{P}^n}(1))$, where $\pi : M \to \mathbb{P}^n$ is its double covering.

(f) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where $(S,\mathcal{E})$ is one of the types 1), 2-i) and 4-b) in [4, (2.25) Theorem].

(g) A hyperquadric fibration over a smooth curve C. In this case C is one of the following types[7].

[6] For some examples of this type, see [3, §2]

[7] We use Notation 3.1 (2).

(g.1) C is an elliptic curve, $b = 1$ and $e = 0$.

(g.2) $C \cong \mathbb{P}^1$, $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ and $b = 5$.

(h) $(\mathbb{P}_C(\mathcal{E}), H(\mathcal{E}))$, where C is a smooth curve of genus two and $\mathcal{E}$ is an ample vector bundle of rank n on C with $c_1(\mathcal{E}) = 1$.

Proof. By the equality (2) we have $(g(X,L), L^n) = (0,3)$, $(1,2)$ or $(2,1)$. If $(g(X,L), L^n) = (0,3)$, then by [5, (12.1) Theorem and (5.10) Theorem] (X,L) is the type (a) above. If $(g(X,L), L^n) = (1,2)$, then we see from [5, (12.3) Theorem] that (X,L) is either the type (b) or (c) above. If $(g(X,L), L^n) = (2,1)$, then by using a list of a classification of polarized manifolds with $g(X,L) = 2$ and $L^n = 1$ (see [3, (1.10) Theorem, (3.7) and (3.30) Theorem]) we see that (X,L) is one of the types from (c) to (h) above. $\square$

3.3 The case where $i = 2$

Here we classify polarized manifolds (X,L) such that L is spanned and $\mathrm{cl}_2(X,L) \leq 15$.

Theorem 3.3.1 Let (X,L) be a polarized manifold (X,L) with $\dim X = n \geq 3$. Assume that L is spanned and $\mathrm{cl}_2(X,L) \leq 15$. Then (X,L) is one of the following.

(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. In this case $\mathrm{cl}_2(X,L) = 0$.

(b) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$. In this case $\mathrm{cl}_2(X,L) = 2$.

(c) A scroll over a smooth curve. In this case $3 \leq \mathrm{cl}_2(X,L) \leq 15$.

(d) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where $(S,\mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. In this case $\mathrm{cl}_2(X,L) = 3$.

(e) A Del Pezzo manifold (X,L) with $L^n \geq 2$. In this case $\mathrm{cl}_2(X,L) = 12$.

Proof. We note that

$$\begin{aligned}\mathrm{cl}_2(X,L) &= b_2(X,L) - b_0(X) + 2(b_1(X,L) - b_1(X)) + b_0(X,L) - b_0(X)\\ &= (b_2(X,L) - b_2(X)) + (b_2(X) - b_0(X)) + 4(g(X,L) - h^1(\mathcal{O}_X)) + (b_0(X,L) - b_0(X)).\end{aligned}$$

We also note that $b_0(X,L) \geq 1 = b_0(X)$ and $b_2(X) \geq b_0(X)$. Since L is spanned, we have $b_2(X,L) \geq b_2(X)$ and $g(X,L) \geq h^1(\mathcal{O}_X)$ by [10, Proposition 3.3 (2)] and [2, Theorem 7.2.10]. Hence we get the following.

- If $0 \leq \mathrm{cl}_2(X,L) \leq 3$, then $g(X,L) = h^1(\mathcal{O}_X)$ holds.
- If $4 \leq \mathrm{cl}_2(X,L) \leq 7$, then $g(X,L) \leq h^1(\mathcal{O}_X) + 1$ holds.
- If $8 \leq \mathrm{cl}_2(X,L) \leq 11$, then $g(X,L) \leq h^1(\mathcal{O}_X) + 2$ holds.
- If $\mathrm{cl}_2(X,L) = 12$, then $g(X,L) \leq h^1(\mathcal{O}_X) + 2$ or $L^n = 1$ holds.
- If $\mathrm{cl}_2(X,L) = 13$, then $g(X,L) \leq h^1(\mathcal{O}_X) + 2$ or $L^n \leq 2$ holds.
- If $\mathrm{cl}_2(X,L) = 14$, then $g(X,L) \leq h^1(\mathcal{O}_X) + 2$ or $L^n \leq 2$ or $b_2(X,L) = b_2(X)$ holds.
- If $\mathrm{cl}_2(X,L) = 15$, then $g(X,L) \leq h^1(\mathcal{O}_X) + 2$ or $L^n \leq 2$ or $b_2(X,L) \leq b_2(X) + 1$ holds.

Hence by [11, Theorem 4.1], [16, Theorem 3.1], Exampless 3.1.1, 3.1.3 and 3.1.4 and [17, Example 2.1], we get the assertion. $\square$

Next we consider the case where $\mathrm{cl}_2(X,L) = 16$.

Theorem 3.3.2 Let (X, L) be a polarized manifold (X, L) with $\dim X = n \geq 3$. Assume that L is spanned and $\mathrm{cl}_2(X, L) = 16$. Then (X, L) is one of the following.

(a) $(\mathbb{P}_C(\mathcal{E}), H(\mathcal{E}))$, where C is a smooth projective curve and $\mathcal{E}$ is an ample vector bundle of rank n on C with $c_1(\mathcal{E}) = 16$.

(b) A hyperquadric fibration over an elliptic curve with $e = 4$, $b = -2$ and $\mathcal{E}$ is ample[8].

(c) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ and $(S, \mathcal{E}) \cong (\mathbb{P}_C(\mathcal{F}), \pi^*(\mathcal{G}) \otimes H(\mathcal{F}))$, where C is an elliptic curve, $\mathcal{F}$ and $\mathcal{G}$ are indecomposable vector bundles of rank two on C with $\deg \mathcal{F} = 1$ and $\deg \mathcal{G} = 1$, and $\pi : \mathbb{P}_C(\mathcal{F}) \to C$ is the projection map.

Proof. By the same argument as the proof of Theorem 3.3.1, either of the following 4 types occurs.

$$\text{(i) } g(X, L) \leq h^1(\mathcal{O}_X) + 2. \quad \text{(ii) } L^n \leq 2. \quad \text{(iii) } b_2(X, L) \leq b_2(X) + 1.$$
$$\text{(iv) } g(X, L) = h^1(\mathcal{O}_X) + 3,\ L^n = 3 \text{ and } b_2(X, L) = b_2(X) + 2.$$

If (X, L) satisfies one of the cases (i), (ii), or (iii), then we see from [11, Theorem 4.1], [16, Theorem 3.1], Examples 3.1.1, 3.1.3 and 3.1.4, and [17, Example 2.1] that (X, L) is one of the types (a), (b) and (c) in Theorem 3.3.2. So we may assume that the case (iv) occurs. Then $\Delta(X, L) = n + L^n - h^0(L) \leq n + 3 - (n + 1) \leq 2$.

Claim 3.1 *$h^1(\mathcal{O}_X) = 0$ holds.*

Proof. If $\Delta(X, L) \leq 1$, then by [5, (5.10) Theorem and (6.7) Corollary] we get the assertion. So we may assume that $\Delta(X, L) = 2$. Then since L is spanned, $h^0(L) = n + 1$ and $L^n = 3$, the morphism $X \to \mathbb{P}^n$ defined by $|L|$ is a triple covering. So by [27, Theorem 7.1.15], we get the assertion. □

Hence $g(X, L) = 3$. Since $\mathrm{Bs}|L| = \emptyset$, we see from Example 3.1.2 that this case (iv) cannot occur. □

3.4 The case where $i = 3$

Here we consider a classification of (X, L) such that L is spanned and $\mathrm{cl}_3(X, L) \leq 8$.

Theorem 3.4.1 Let (X, L) be a polarized manifold with $\dim X = n \geq 3$. Assume that L is spanned and $\mathrm{cl}_3(X, L) \leq 8$. Then (X, L) is one of the following.

(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. In this case $\mathrm{cl}_3(X, L) = 0$.

(b) A scroll over a smooth curve. In this case $\mathrm{cl}_3(X, L) = 0$.

(c) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$. In this case $\mathrm{cl}_3(X, L) = 2$.

(d) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$. In this case $\mathrm{cl}_3(X, L) = 4$.

(e) A simple blowing up of $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$. In this case $\mathrm{cl}_3(X, L) = 4$.

(f) $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^* \mathcal{O}_{\mathbb{P}^1}(1))$. In this case $\mathrm{cl}_3(X, L) = 4$.

(g) $(\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^2 p_i^* \mathcal{O}_{\mathbb{P}^2}(1))$. In this case $\mathrm{cl}_3(X, L) = 6$.

(h) A hyperquadric fibration over a smooth curve C, and one of the following holds[9].

[8] We use Notation 3.1 (2).
[9] We use Notation 3.1 (2).

(h.1) $g(C)=1$, $n=3$, $L^3=6$, $e=4$, $b=-2$, and $\mathcal{E}$ is ample. In this case $\mathrm{cl}_3(X,L)=8$.

(h.2) $g(C)=0$, $n=3$, $L^3=9$, $e=6$, $b=-3$ and $\mathcal{E}\cong\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}(2)\oplus\mathcal{O}_{\mathbb{P}^1}(2)$ (see [3, (3.30) Theorem 9)]). In this case $\mathrm{cl}_3(X,L)=8$.

(i) $(\mathbb{P}_S(\mathcal{E}),H(\mathcal{E}))$ and $(S,\mathcal{E})$ is one of the following.

(i.1) $(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}(1))$. In this case $\mathrm{cl}_3(X,L)=0$.

(i.2) $(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}(2))$. In this case $\mathrm{cl}_3(X,L)=4$.

(i.3) $(\mathbb{Q}^2,\mathcal{O}_{\mathbb{Q}^2}(1)\oplus\mathcal{O}_{\mathbb{Q}^2}(1))$. In this case $\mathrm{cl}_3(X,L)=4$.

(i.4) S is a double covering $f:S\to\mathbb{P}^2$ branched along a smooth hypersurface of degree 2, and $\mathcal{E}=f^*(\mathcal{O}_{\mathbb{P}^n}(1))\oplus f^*(\mathcal{O}_{\mathbb{P}^n}(1))$. In this case $\mathrm{cl}_3(X,L)=4$.

(i.5) $(\mathbb{P}^2,T_{\mathbb{P}^2})$. In this case $\mathrm{cl}_3(X,L)=6$.

(i.6) $(\mathbb{P}_C(\mathcal{F}),\pi^*(\mathcal{G})\otimes H(\mathcal{F}))$, where C is an elliptic curve, $\mathcal{F}$ and $\mathcal{G}$ are indecomposable vector bundles of rank two on C with $\deg\mathcal{F}=1$ and $\deg\mathcal{G}=1$, and $\pi:\mathbb{P}_C(\mathcal{F})\to C$ is the projection map. In this case $\mathrm{cl}_3(X,L)=8$.

(i.7) S is a double covering $f:S\to\mathbb{P}^2$ branched along a smooth hypersurface of degree 4, and $\mathcal{E}\cong f^*(\mathcal{O}_{\mathbb{P}^2}(1))\oplus f^*(\mathcal{O}_{\mathbb{P}^2}(1))$. In this case $\mathrm{cl}_3(X,L)=8$.

(i.8) $(\mathbb{P}^1_\alpha\times\mathbb{P}^1_\beta,[H_\alpha+2H_\beta]\oplus[H_\alpha+H_\beta])$ and H_α (resp. H_β) is the ample generator of $\mathrm{Pic}(\mathbb{P}_\alpha)$ (resp. $\mathrm{Pic}(\mathbb{P}_\beta)$). In this case $\mathrm{cl}_3(X,L)=8$.

(i.9) S is the blowing up of $\mathbb{P}^2$ at a point and $\mathcal{E}\cong[2H-E]^{\oplus 2}$, where H is the pull-back of $\mathcal{O}_{\mathbb{P}^2}(1)$ and E is the exceptional divisor. In this case $\mathrm{cl}_3(X,L)=8$.

Proof. We note that

$$\begin{aligned}\mathrm{cl}_3(X,L) &= b_3(X,L)-b_1(X)+2(b_2(X,L)-b_2(X))+b_1(X,L)-b_1(X)\\ &= (b_3(X,L)-b_3(X))+(b_3(X)-b_1(X))+2(b_2(X,L)-b_2(X))+2(g(X,L)-h^1(\mathcal{O}_X)).\end{aligned}$$

We also note that $\mathrm{cl}_3(X,L)$ is even and $b_3(X)\geq b_1(X)$. Since L is spanned, we have $b_3(X,L)\geq b_3(X)$, $b_2(X,L)\geq b_2(X)$ and $g(X,L)\geq h^1(\mathcal{O}_X)$ by [10, Proposition 3.3 (2)] and [2, Theorem 7.2.10]. Hence we get the following.

- If $0\leq\mathrm{cl}_3(X,L)\leq 2$, then $b_2(X,L)\leq b_2(X)+1$ holds.
- If $\mathrm{cl}_3(X,L)=4$, then $b_2(X,L)\leq b_2(X)+1$ or $g(X,L)=h^1(\mathcal{O}_X)$ holds.
- If $\mathrm{cl}_3(X,L)=6$, then $b_2(X,L)\leq b_2(X)+1$ or $g(X,L)\leq h^1(\mathcal{O}_X)+1$ holds.
- If $\mathrm{cl}_3(X,L)=8$, then $b_2(X,L)\leq b_2(X)+1$ or $g(X,L)\leq h^1(\mathcal{O}_X)+2$ holds.

By [11, Theorem 4.1], [16, Theorem 3.1], Examples 3.1.1 and 3.1.4, and [17, Example 2.1] we get the assertion[10]. □

3.5 The case where $i=4$

Here we consider a classification of (X,L) such that L is spanned (resp. very ample) and $\mathrm{cl}_4(X,L)\leq 1$ (resp. $\mathrm{cl}_4(X,L)=2$).

Theorem 3.5.1 Let (X,L) be a polarized manifold (X,L) with $\dim X=n\geq 4$. Assume that L is spanned and $\mathrm{cl}_4(X,L)\leq 1$. Then (X,L) is one of the following.

[10] We note that the type (2-1) (resp. (2-2), (2-3), (3-1) and (3-2)) in [8, Theorem 3.3] corresponds to (i.8) (resp. (i.9), (i.6), (h.1) and (h.2)).

(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. In this case $\mathrm{cl}_4(X,L)=0$.

(b) A scroll over a smooth curve. In this case $\mathrm{cl}_4(X,L)=0$.

Proof. We note that the following equality holds.

$$\begin{aligned}\mathrm{cl}_4(X,L) &= b_4(X,L)-b_2(X)+2(b_3(X,L)-b_3(X))+b_2(X,L)-b_2(X)\\ &= b_4(X,L)-b_4(X)+(b_4(X)-b_2(X))+2(b_3(X,L)-b_3(X))+b_2(X,L)-b_2(X).\end{aligned}$$

Since L is spanned, we see from [10, Proposition 3.3 (2)] that $b_4(X,L)\geq b_4(X)$, $b_3(X,L)\geq b_3(X)$ and $b_2(X,L)\geq b_2(X)$ hold. Furthermore by the strong Lefschetz theorem, we have $b_4(X)\geq b_2(X)$. Hence if $\mathrm{cl}_4(X,L)\leq 1$, then $b_2(X,L)\leq b_2(X)+1$. Since $n\geq 4$, we can easily check that (X,L) is one of the above types by [11, Theorem 4.1], [16, Theorem 3.1] and [17, Example 2.1]. □

Remark 3.5.1 If L is spanned, then there does not exist (X,L) with $\mathrm{cl}_4(X,L)=1$.

Theorem 3.5.2 Let (X,L) be a polarized manifold (X,L) with $\dim X = n\geq 5$. Assume that L is very ample and $\mathrm{cl}_4(X,L)=2$. Then (X,L) is $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

Proof. By the same argument as above, (X,L) with $\mathrm{cl}_4(X,L)=2$ satisfies one of the following.

$$\text{(I) } b_2(X,L)\leq b_2(X)+1. \qquad \text{(II) } b_4(X,L)=b_4(X).$$

(I) If $b_2(X,L)\leq b_2(X)+1$ holds, then by [11, Theorem 4.1], [16, Theorem 3.1] and [17, Example 2.1] we see that (X,L) with $\mathrm{cl}_4(X,L)=2$ is $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.
(II) Next we assume that $b_4(X,L)=b_4(X)$ holds. Then by [11, Theorem 4.2], we see that (X,L) is one of the following types since we assume that $n\geq 5$.

(II.1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(II.2) A scroll over a smooth projective curve.

(II.3) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is an ample vector bundle of rank $n-1$ on S.

(II.4) X is the Plücker embedding of $G(2,5)$ and $L=\mathcal{O}_X(1)$. In this case $n=6$.

(II.5) X is a nonsingular hyperplane section of the Plücker embedding of $G(2,5)$ in $\mathbb{P}^9$ and $L=\mathcal{O}_X(1)$. In this case $n=5$.

Then by calculating $\mathrm{cl}_4(X,L)$, we see from [17, Example 2.1] that $\mathrm{cl}_4(X,L)=0$ (resp. 0, $c_2(\mathcal{E})$, 5 and 5) if (X,L) is the type (II.1) (resp. (II.2), (II.3), (II.4) and (II.5)). Hence we find that the type (II.3) is possible and in this case $c_2(\mathcal{E})=2$. But by [28, Theorem 6.1] and [23], the rank of $\mathcal{E}$ is two and so we have $n=3$. This contradicts the assumption that $n\geq 5$. □

References

[1] E. Ballico, M. Bertolini and C. Turrini, *On the class of some projective varieties,* Collect. Math. 48 (1997), 281–287.

[2] M. C. Beltrametti and A. J. Sommese, *The adjunction theory of complex projective varieties,* de Gruyter Expositions in Math. 16, Walter de Gruyter, Berlin, NewYork, (1995).

[3] T. Fujita, *Classification of polarized manifolds of sectional genus two,* the Proceedings of "Algebraic Geometry and Commutative Algebra" in Honor of Masayoshi Nagata (1987), 73–98.

[4] T. Fujita, *Ample vector bundles of small c_1-sectional genera,* J. Math. Kyoto Univ. 29 (1989), 1-16.

[5] T. Fujita, *Classification Theories of Polarized Varieties,* London Math. Soc. Lecture Note Ser. 155, Cambridge University Press, (1990).

[6] Y. Fukuma, *On polarized 3-folds (X, L) with $g(L) = q(X) + 1$ and $h^0(L) \geq 4$,* Ark. Mat. 35 (1997), 299–311.

[7] Y. Fukuma, *On sectional genus of quasi-polarized 3-folds,* Trans. Amer. Math. Soc. 351 (1999), 363–377.

[8] Y. Fukuma, *On complex manifolds polarized by an ample line bundle of sectional genus $q(X) + 2$,* Math. Z. 234 (2000), 573–604.

[9] Y. Fukuma, *On the sectional geometric genus of quasi-polarized varieties, I,* Comm. Algebra 32 (2004), 1069–1100.

[10] Y. Fukuma, *On the sectional invariants of polarized manifolds,* J. Pure Appl. Algebra 209 (2007), 99–117.

[11] Y. Fukuma, *A classification of polarized manifolds by the sectional Betti number and the sectional Hodge number,* Adv. Geom. 8 (2008), 591–614.

[12] Y. Fukuma, *Invariants of ample line bundles on projective varieties and their applications, I,* Kodai Math. J. 31 (2008), 219–256.

[13] Y. Fukuma, *Sectional invariants of scroll over a smooth projective variety,* Rend. Sem. Mat. Univ. Padova 121 (2009), 93–119.

[14] Y. Fukuma, *Invariants of ample line bundles on projective varieties and their applications, II,* Kodai Math. J. 33 (2010), 416–445.

[15] Y. Fukuma, *Invariants of ample vector bundles on smooth projective varieties,* Riv. Mat. Univ. Parma (8), 2 (2011), 273–297.

[16] Y. Fukuma, *Classification of polarized manifolds by the second sectional Betti numbers,* preprint, http://www.math.kochi-u.ac.jp/fukuma/SSBN+1.html, to appear in Hokkaido Math. J.

[17] Y. Fukuma, *Calculations of sectional classes of special polarized manifolds,* preprint, http://www.math.kochi-u.ac.jp/fukuma/Cal-SC.html

[18] Y. Fukuma and H. Ishihara *Complex manifolds polarized by an ample and spanned line bundle of sectional genus three,* Arch. Math. (Basel) 71 (1998), 159-168.

[19] W. Fulton, *Intersection Theory,* Ergebnisse der Mathematik und ihrer Grenzgebiete 2 (1984), Springer-Verlag.

[20] S. L. Kleiman, *Tangency and Duality,* in Vanquver Conference in Algebraic Geometry, Canad. Math. Soc. Conf. Proc., 6 (1986), 163–225.

[21] A. Lanteri, *On the class of a projective algebraic surface,* Arch. Math. (Basel) 45 (1985), 79–85.

[22] A. Lanteri, *On the class of an elliptic projective surface,* Arch. Math. (Basel) 64 (1995), 359–368.

[23] A. Lanteri and F. Russo, *A footnote to a paper by Noma,* Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl. 4, No.2, (1993), 131-132.

[24] A. Lanteri and F. Tonoli, *Ruled surfaces with small class,* Comm. Algebra 24 (1996), 3501–3512.

[25] A. Lanteri and C. Turrini, *Projective threefolds of small class,* Abh. Math. Sem. Univ. Hamburg 57 (1987), 103–117.

[26] A. Lanteri and C. Turrini, *Projective surfaces with class less than or equal to twice the degree,* Math. Nachr. 175 (1995), 199–207.

[27] R. Lazarsfeld, *Positivity in Algebraic Geometry I, II,* Ergebnisse der Mathematik, Springer-Verlag, 2004.

[28] A. Noma, *Classification of rank-*2 *ample and spanned vector bundles on surfaces whose zero loci consist of general points,* Trans. Amer. Math. Soc. 342 (1994), 867–894.

[29] M. Palleschi and C. Turrini, *On polarized surfaces with a small generalized class,* Extracta Math. 13 (1998), 371–381.

[30] C. Turrini and E. Verderio, E, *Projective surfaces of small class,* Geom. Dedicata 47 (1993), 1–14.

Department of Mathematics
Faculty of Science
Kochi University
Akebono-cho, Kochi 780-8520
Japan
E-mail: fukuma@kochi-u.ac.jp