A property for the formula of the sectional classes of classical scrolls [*†‡]

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Abstract

In this note, we investigate the formula of the sectional class of classical scrolls and we give an answer of a conjecture proposed in a previous paper.

1 Introduction

Let (X, L) be a polarized manifold of dimension n. Assume that L is very ample and let $\varphi : X \hookrightarrow \mathbb{P}^N$ be the morphism defined by $|L|$. Then φ is an embedding. In this situation, its dual variety $X^\vee \to (\mathbb{P}^N)^\vee$ is a hypersurface of N-dimensional projective space except some special types. Then the *class* $\mathrm{cl}(X, L)$ of (X, L) is defined by the following.

$$\mathrm{cl}(X,L) = \begin{cases} \deg(X^\vee), & \text{if } X^\vee \text{ is a hypersurface in } (\mathbb{P}^N)^\vee \\ 0, & \text{otherwise.} \end{cases}$$

As a generalization of this notion, in [3], we defined the *ith sectional class* $\mathrm{cl}_i(X, L)$ for any ample line bundle L and every integer i with $0 \leq i \leq n$ (see Definition 2.2).

Here we note the following fact: Assume that L is very ample. Then there exists a sequence of smooth subvarieties $X \supset X_1 \supset \cdots \supset X_{n-i}$ such that $X_j \in |L_{j-1}|$ and $\dim X_j = n - j$ for every integer j with $1 \leq j \leq n - i$, where $L_j = L|_{X_j}$ and $L_0 := L$. In particular, X_{n-i} is a smooth projective variety of dimension i and L_{n-i} is a very ample line bundle on X_{n-i}. Then $\mathrm{cl}_i(X, L)$ is equal to the class of (X_{n-i}, L_{n-i}). This is the reason why we call this invariant the ith sectional class.

In [4], we calculated the sectional class of special polarized manifolds. For example, we consider the case where (X, L) is a classical scroll over a smooth projective variety Y of dimension m such that $n := \dim X \geq 2m$. Namely, there exists an ample vector bundle $\mathcal{E}$ on Y of rank $r \geq m + 1$ such that $(X, L) \cong (\mathbb{P}_Y(\mathcal{E}), H(\mathcal{E}))$, where $H(\mathcal{E})$ is the tautological line bundle. Here we note that we need the assumption $n \geq 2m$ in order to define and compare $\mathrm{cl}_i(X, L)$ and $\mathrm{cl}_{2m-i}(X, L)$ for every integer i with $0 \leq i \leq m$. Then we get the following:

(i) If $m = 1$, then by [4, Example 2.1 (ix)] we have

$$\mathrm{cl}_i(X,L) = \begin{cases} s_1(\mathcal{E}), & \text{if } i = 0, \\ 2g(C) - 2 + 2c_1(\mathcal{E}), & \text{if } i = 1, \\ c_1(\mathcal{E}), & \text{if } i = 2, \\ 0, & \text{if } i \geq 3 \text{ and } n \geq 3. \end{cases} \tag{1}$$

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(ii) If $m = 2$, then by [4, Example 2.1 (x)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} s_2(\mathcal{E}), & \text{if } i = 0, \\ (s_1(\mathcal{E}) + K_S)s_1(\mathcal{E}) + 2s_2(\mathcal{E}), & \text{if } i = 1, \\ c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}), & \text{if } i = 2, \\ (c_1(\mathcal{E}) + K_S)c_1(\mathcal{E}) + 2c_2(\mathcal{E}), & \text{if } i = 3, \\ c_2(\mathcal{E}), & \text{if } i = 4, \\ 0, & \text{if } i \geq 5 \text{ and } n \geq 5. \end{cases} \tag{2}$$

(iii) If $m = 3$, then by [4, Example 2.1] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} s_3(\mathcal{E}), & \text{if } i = 0, \\ 3s_3(\mathcal{E}) + (s_1(\mathcal{E}) + K_Y)s_2(\mathcal{E}), & \text{if } i = 1, \\ 3s_3(\mathcal{E}) + 12(s_1(\mathcal{E}) + K_Y)s_2(\mathcal{E}) & \\ \quad +(s_1(\mathcal{E}) + K_Y)s_1(\mathcal{E})^2 + c_2(Y)s_1(\mathcal{E}), & \text{if } i = 2, \\ -c_3(Y) + 2c_3(\mathcal{E}) - 2c_1(\mathcal{E})c_2(\mathcal{E}) + 4c_1(\mathcal{E})^3 & \\ \quad +3K_Y c_1(\mathcal{E})^2 + 2c_2(Y)c_1(\mathcal{E}), & \text{if } i = 3, \\ 3c_3(\mathcal{E}) + 12(c_1(\mathcal{E}) + K_Y)c_2(\mathcal{E}) & \\ \quad +(c_1(\mathcal{E}) + K_Y)c_1(\mathcal{E})^2 + c_2(Y)c_1(\mathcal{E}), & \text{if } i = 4, \\ 3c_3(\mathcal{E}) + (c_1(\mathcal{E}) + K_Y)c_2(\mathcal{E}), & \text{if } i = 5, \\ c_3(\mathcal{E}), & \text{if } i = 6, \\ 0, & \text{if } i \geq 7 \text{ and } n \geq 7. \end{cases} \tag{3}$$

The above equations show that there exists a relation between $\mathrm{cl}_i(X, L)$ and $\mathrm{cl}_{2m-i}(X, L)$. Here we note that for every integer i with $0 \leq i \leq m$, $\mathrm{cl}_i(X, L)$ can be written by the Segre classes $s_1(\mathcal{E}), \ldots, s_m(\mathcal{E})$.

Definition 1.1 For every integer i with $0 \leq i \leq m$, we define the polynomial $F_i(t_1, \ldots, t_m) \in \mathbb{Z}[t_1, \ldots, t_m]$ such that the following equality holds.

$$F_i(s_1(\mathcal{E}), \ldots, s_m(\mathcal{E})) = \mathrm{cl}_i(X, L).$$

Then we see from the above that if $m = 1$, 2 and 3, then

$$\mathrm{cl}_j(X, L) = F_{2m-j}(c_1(\mathcal{E}), \ldots, c_m(\mathcal{E}))$$

for $m \leq j \leq 2m$. In general, we can prove the following theorem, which was proposed in [4] and is the main result of this paper.

Theorem 1.1 *Let a polarized manifold (X, L) be a classical scroll over a smooth projective variety Y with $\dim X = n$ and $\dim Y = m$. Let $\mathcal{E}$ be an ample vector bundle on Y such that $X \cong \mathbb{P}_Y(\mathcal{E})$ and $L = H(\mathcal{E})$. Let $F_i(t_1, \ldots, t_m)$ be the polynomial defined in Definition 1.1 for every integer i with $0 \leq i \leq m$. Assume that $n \geq 2m$. Then for any integer j with $m \leq j \leq 2m$ we have*

$$\mathrm{cl}_j(X, L) = F_{2m-j}(c_1(\mathcal{E}), \ldots, c_m(\mathcal{E})).$$

In particular

$$F_m(s_1(\mathcal{E}), \ldots, s_m(\mathcal{E})) = F_m(c_1(\mathcal{E}), \ldots, c_m(\mathcal{E})).$$

By Theorem 1.1 we can easily calculate $\mathrm{cl}_{2m-i}(X,L)$ (resp. $\mathrm{cl}_i(X,L)$) if we are able to calculate $\mathrm{cl}_i(X,L)$ (resp. $\mathrm{cl}_{2m-i}(X,L)$). By this relation we expect that we can get some useful information about $\mathrm{cl}_{2m-i}(X,L)$ (resp. $\mathrm{cl}_i(X,L)$) from several properties of $\mathrm{cl}_i(X,L)$ (resp. $\mathrm{cl}_{2m-i}(X,L)$). Moreover if $i=m$, then we have $\mathrm{cl}_m(X,L) = F_m(c_1(\mathcal{E}),\dots,c_m(\mathcal{E})) = F_m(s_1(\mathcal{E}),\dots,s_m(\mathcal{E}))$ by Theorem 1.1. So $\mathrm{cl}_m(X,L)$ may have special and interesting properties. We will study these on another occasion.

2 Preliminaries

Definition 2.1 (See [1, Definition 3.1].) Let (X,L) be a polarized manifold of dimension n, and i an integer with $0 \le i \le n$. Then the *ith sectional Euler number* $e_i(X,L)$ *of* (X,L) is defined by the following:

$$e_i(X,L) := \sum_{l=0}^{i}(-1)^l \binom{n-i+l-1}{l} c_{i-l}(X)L^{n-i+l}.$$

Definition 2.2 (See [3, Definitions 2.8 and 2.9]. See also [3, Remark 2.6].) Let (X,L) be a polarized manifold of dimension n and i an integer with $0 \le i \le n$. Then the *ith sectional class* of (X,L) is defined by the following.

$$\mathrm{cl}_i(X,L) = \begin{cases} e_0(X,L), & \text{if } i=0, \\ (-1)\{e_1(X,L)-2e_0(X,L)\}, & \text{if } i=1, \\ (-1)^i\{e_i(X,L)-2e_{i-1}(X,L)+e_{i-2}(X,L)\}, & \text{if } 2 \le i \le n. \end{cases}$$

Definition 2.3 Let Y be a smooth projective variety of dimension m and $\mathcal{E}$ a vector bundle of rank r on Y.

(i) The *Chern polynomial* $c_t(\mathcal{E})$ is defined by $c_t(\mathcal{E}) = \sum_{i\ge 0} c_i(\mathcal{E})t^i$.

(ii) For every integer j with $j \ge 0$, the *jth Segre class* $s_j(\mathcal{F})$ *of* $\mathcal{F}$ is defined by the following equation: $c_t(\mathcal{F}^\vee)s_t(\mathcal{F}) = 1$, where $c_t(\mathcal{F}^\vee)$ is the Chern polynomial of $\mathcal{F}^\vee$ and $s_t(\mathcal{F}) = \sum_{j\ge 0} s_j(\mathcal{F})t^j$.

Remark 2.1 (i) Let Y be a smooth projective variety and $\mathcal{F}$ a vector bundle on X. Let $\tilde{s}_j(\mathcal{F})$ be the Segre class which is defined in [5, Chapter 3]. Then $s_j(\mathcal{F}) = \tilde{s}_j(\mathcal{F}^\vee)$.

(ii) For every integer i with $1 \le i$, $s_i(\mathcal{F})$ can be written by using the Chern classes $c_j(\mathcal{F})$ with $1 \le j \le i$. (For example, $s_1(\mathcal{F}) = c_1(\mathcal{F})$, $s_2(\mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, and so on.)

Notation 2.1 Let (X,L) be an n-dimensional classical scroll over a smooth projective variety Y of dimension m. Let $\mathcal{E}$ be an ample vector bundle of rank r on Y such that $X = \mathbb{P}_Y(\mathcal{E})$ and $L = H(\mathcal{E})$. Let $p : X \to Y$ be the projection. Then $n = m + r - 1$. In this paper we assume that $r \ge m+1$, that is, $n \ge 2m$.

Proposition 2.1 *Let* (X,L) *be a classical scroll over a smooth projective variety* Y *of dimension* m*. We use notations in Notation 2.1. Then for every integer* i *with* $0 \le i \le n$ *the following holds.*

$$e_i(X,L) = \sum_{t=0}^{i}\sum_{k=0}^{i-t}(-1)^{i-t}\binom{m-t-2}{i-t-k} c_k(\mathcal{E})\, c_t(Y)\, s_{m-k-t}(\mathcal{E}).$$

Proof. See the first part of the proof in [2, Theorem 3.1]. □

3 Main result

Definition 3.1 Let Y be a smooth projective variety of dimension m and $\mathcal{E}$ a vector bundle on Y. Then for every integer i with $0 \leq i \leq m$ we define the polynomial $t_i(x_0, \ldots, x_i) \in \mathbb{Z}[x_0, \ldots, x_i]$ which satisfies the following.

$$c_i(\mathcal{E}) = t_i(s_0(\mathcal{E}), \ldots, s_i(\mathcal{E})). \tag{4}$$

For example, we see that $t_0(x_0) = 1$, $t_1(x_0, x_1) = x_1$, $t_2(x_0, x_1, x_2) = x_1^2 - x_2$ and so on.

Proposition 3.1 *Let Y be a smooth projective variety of dimension m and $\mathcal{E}$ a vector bundle over Y. For every integer i with $0 \leq i \leq m$, we have $s_i(\mathcal{E}) = t_i(c_0(\mathcal{E}), \ldots, c_i(\mathcal{E}))$.*

Proof. We prove this by induction.
(I) If $i = 0$, then this is true because $c_0(\mathcal{E}) = s_0(\mathcal{E}) = 1$.
(II) Assume that the assertion holds for every i with $i \leq k - 1$. So we consider the case $i = k$. Then by Definition 2.3 (ii)

$$\sum_{\substack{i+j=k \\ i\geq 0, j\geq 0}} (-1)^i c_i(\mathcal{E}) s_j(\mathcal{E}) = 0. \tag{5}$$

Hence by (5) we have

$$\begin{aligned} t_k(s_0(\mathcal{E}), \ldots, s_k(\mathcal{E})) &= c_k(\mathcal{E}) \\ &= (-1)^{k+1} \sum_{\substack{i+j=k \\ j\geq 1}} (-1)^i c_i(\mathcal{E}) s_j(\mathcal{E}) \\ &= (-1)^{k+1} \sum_{\substack{i+j=k \\ j\geq 1}} (-1)^i t_i(s_0(\mathcal{E}), \ldots, s_i(\mathcal{E})) s_j(\mathcal{E}). \end{aligned}$$

In particular, we have

$$t_k(x_0, \ldots, x_k) = (-1)^{k+1} \sum_{\substack{i+j=k \\ j\geq 1}} (-1)^i t_i(x_0, \ldots, x_i) x_j. \tag{6}$$

On the other hand, we see from the induction hypothesis and (6) that

$$\begin{aligned} s_k(\mathcal{E}) &= -\sum_{\substack{i+j=k \\ i\geq 1}} (-1)^i c_i(\mathcal{E}) s_j(\mathcal{E}) \\ &= (-1)^{k+1} \sum_{\substack{i+j=k \\ i\geq 1}} (-1)^j c_i(\mathcal{E}) t_j(c_0(\mathcal{E}), \ldots, c_j(\mathcal{E})) \\ &= t_k(c_0(\mathcal{E}), \ldots, c_k(\mathcal{E})). \end{aligned}$$

So we get the assertion. □

The following theorem which is Theorem 1.1 in Introduction is the main result of this note.

Theorem 3.1 *Let (X, L) be an n-dimensional classical scroll over a smooth projective variety Y of dimension m such that $n \geq 2m$. Let $F_i(t_1, \ldots, t_m)$ be the polynomial defined in Definition 1.1 for every integer i with $0 \leq i \leq m$. We use notations in Notation 2.1. Then for any integer j with $m \leq j \leq 2m$ we have*

$$\mathrm{cl}_j(X, L) = F_{2m-j}(c_1(\mathcal{E}), \ldots, c_m(\mathcal{E})).$$

In particular

$$F_m(s_1(\mathcal{E}), \ldots, s_m(\mathcal{E})) = F_m(c_1(\mathcal{E}), \ldots, c_m(\mathcal{E})).$$

Proof. First we prove the following.

Claim 3.1 *For any integer i with $0 \leq i \leq m$, we have*

$$\begin{aligned} e_{2m-2-i}(X,L) &= \sum_{t=0}^{i}\sum_{l=0}^{i-t}(-1)^{i-t}\binom{m-t-2}{i-t-l}c_{m-t-l}(\mathcal{E}^{\vee})c_t(Y)s_l(\mathcal{E}) + (m-i-1)c_m(Y), \\ e_{2m-1-i}(X,L) &= \sum_{t=0}^{i-1}\sum_{l=0}^{i-1-t}(-1)^{i-1-t}\binom{m-t-2}{i-1-t-l}c_{m-t-l}(\mathcal{E}^{\vee})c_t(Y)s_l(\mathcal{E}) + (m-i)c_m(Y), \\ e_{2m-i}(X,L) &= \sum_{t=0}^{i-2}\sum_{l=0}^{i-2-t}(-1)^{i-2-t}\binom{m-t-2}{i-2-t-l}c_{m-t-l}(\mathcal{E}^{\vee})c_t(Y)s_l(\mathcal{E}) + (m-i+1)c_m(Y). \end{aligned}$$

(We note that if $i=0$ (resp. $i=0,1$), then $\sum_{t=0}^{i-1}\sum_{l=0}^{i-1-t}(-1)^{i-1-t}\binom{m-t-2}{i-1-t-l}c_{m-t-l}(\mathcal{E}^{\vee})c_t(Y)s_l(\mathcal{E}) = 0$ *(resp.* $\sum_{t=0}^{i-2}\sum_{l=0}^{i-2-t}(-1)^{i-2-t}\binom{m-t-2}{i-2-t-l}c_{m-t-l}(\mathcal{E}^{\vee})c_t(Y)s_l(\mathcal{E}) = 0$*).)*

Proof. (A) First we treat $e_{2m-2-i}(X,L)$. Then by Proposition 2.1

$$e_{2m-2-i}(X,L) = \sum_{t=0}^{2m-2-i}\left(\sum_{k=0}^{2m-2-i-t}(-1)^{2m-2-i-t-k}\binom{m-t-2}{2m-2-i-t-k}c_k(\mathcal{E}^{\vee})c_t(Y)s_{m-k-t}(\mathcal{E})\right).$$

Here we note that

(7) $$2m-2-i-t \geq k.$$

We set

$$E(i,k,t) = (-1)^{2m-2-i-t-k}\binom{m-t-2}{2m-2-i-t-k}c_k(\mathcal{E}^{\vee})c_t(Y)s_{m-k-t}(\mathcal{E}).$$

If $E(i,k,t) \neq 0$, then the following two conditions hold by noting (7).

(8) $$0 \leq k \leq m.$$

(9) $$0 \leq t \leq m.$$

(10) $$k+t \leq \min\{m, 2m-2-i\}.$$

If $m-t-2>0$ and $m-t-2<2m-2-i-t-k$, then $\binom{m-t-2}{2m-2-i-t-k}=0$. Hence if $E(i,k,t)\neq 0$, then $m-t-2 \leq 0$ or $m-t-2 \geq 2m-2-i-t-k$, that is,

(11) $$t \geq m-2 \quad \text{or} \quad k \geq m-i.$$

(A.1) The case where $0 \leq i \leq m-3$.
We see from (8), (9), (10) and (11) that the possible cases of (k,t) are as follows.

$$\text{(A.1.1)}\begin{cases} k=0,1,2 \ \ t=m-2, \\ k=0,1, \ \ t=m-1, \\ k=0, \ \ t=m. \end{cases} \qquad \text{(A.1.2)}\begin{cases} k=m-i, \ \ t=i,i-1,\ldots,1,0 \\ k=m-i+1, \ \ t=i-1,\ldots,1,0 \\ \qquad \vdots \\ k=m, \ \ t=0. \end{cases}$$

In the case (A.1.1) we have

(12) $$\begin{aligned} \sum_{t=m-2}^{m}\sum_{k=0}^{m-t}E(i,k,t) &= c_{m-1}(Y)s_1(\mathcal{E}) + c_1(\mathcal{E}^{\vee})c_{m-1}(Y) + (m-i-1)c_m(Y) \\ &= (m-1-i)c_m(Y). \end{aligned}$$

On the other hand, in the case (A.1.2) we get

$$\sum_{k=m-i}^{m}\sum_{t=0}^{m-k}(-1)^{2m-2-i-t-k}\binom{m-t-2}{2m-2-i-t-k}c_k(\mathcal{E}^\vee)c_t(Y)s_{m-k-t}(\mathcal{E})$$
$$=\sum_{t=0}^{i}\sum_{k=m-i}^{m-t}(-1)^{2m-2-i-t-k}\binom{m-t-2}{2m-2-i-t-k}c_k(\mathcal{E}^\vee)c_t(Y)s_{m-k-t}(\mathcal{E}).$$

Here we put $j := k-(m-i)$. Then by $t \le i \le m-3$ we have

$$\sum_{t=0}^{i}\sum_{k=m-i}^{m-t}(-1)^{2m-2-i-t-k}\binom{m-t-2}{2m-2-i-t-k}c_k(\mathcal{E}^\vee)c_t(Y)s_{m-k-t}(\mathcal{E})$$
$$=\sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E})$$
$$=\sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E}).$$

Hence we have

$$\begin{aligned}(13)\qquad & e_{2m-2-i}(X,L)\\ &=\sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E})+(m-1-i)c_m(Y).\end{aligned}$$

(A.2) The case where $i = m-2$.
We see from (8), (9), (10) and (11) that the possible cases of (k,t) are as follows.

$$\text{(A.2.1)}\begin{cases} k=0,1,2 & t=m-2,\\ k=0,1, & t=m-1,\\ k=0, & t=m.\end{cases}\qquad \text{(A.2.2)}\begin{cases} k=2, & t=m-2,m-3,\dots,1,0\\ k=3, & t=m-3,\dots,1,0\\ & \vdots\\ k=m, & t=0.\end{cases}$$

Here we note that the case $(k,t)=(2,m-2)$ is contained in both the cases (A.2.1) and (A.2.2). So we count the case $(k,t)=(2,m-2)$ as the case (A.2.2).

In the case (A.2.1) we have

$$(14)\qquad \sum_{t=m-2}^{m}\sum_{k=0}^{m-t}E(i,k,t)=c_{m-1}(Y)s_1(\mathcal{E})+c_1(\mathcal{E}^\vee)c_{m-1}(Y)+c_m(Y)=c_m(Y).$$

On the other hand, in the case (A.2.2) by the same argument as above we get

$$\begin{aligned}(15)\qquad & \sum_{k=2}^{m}\sum_{t=0}^{m-k}E(i,k,t)\\ &=\sum_{t=0}^{m-2}\sum_{k=2}^{m-t}(-1)^{2m-2-i-t-k}\binom{m-t-2}{2m-2-i-t-k}c_k(\mathcal{E}^\vee)c_t(Y)s_{m-k-t}(\mathcal{E})\\ &=\sum_{t=0}^{m-2}\sum_{j=0}^{m-2-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j}c_{j+2}(\mathcal{E}^\vee)c_t(Y)s_{(m-2)-j-t}(\mathcal{E})\\ &=\sum_{t=0}^{m-2}\sum_{j=0}^{m-2-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+2}(\mathcal{E}^\vee)c_t(Y)s_{(m-2)-j-t}(\mathcal{E}).\end{aligned}$$

Hence we see from (14) and (15) that for $i = m-2$

$$(16) \qquad \begin{aligned} & e_{2m-2-i}(X,L) \\ & = \sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E}) + (m-1-i)c_m(Y). \end{aligned}$$

(A.3) The case where $i = m-1$.
We see from (8), (9), (10) and (11) that the possible cases of (k,t) are as follows.

$$\text{(A.3.1)} \left\{ \begin{array}{l} k=0,1 \quad t=m-2, \\ k=0 \quad t=m-1. \end{array} \right. \qquad \text{(A.3.2)} \left\{ \begin{array}{l} k=1, \quad t=m-2, m-3, \dots, 1, 0 \\ k=2, \quad t=m-3, \dots, 1, 0 \\ \qquad \vdots \\ k=m-1, \quad t=0. \end{array} \right.$$

Here we note that the case $(k,t) = (1, m-2)$ is contained in both the cases (A.3.1) and (A.3.2). So we count the case $(k,t) = (1, m-2)$ as the case (A.3.2).

In the case (A.3.1) we have

$$(17) \qquad \sum_{t=m-2}^{m-1}\sum_{k=0}^{m-1-t} E(i,k,t) = c_{m-1}(Y)s_1(\mathcal{E}).$$

On the other hand, in the case (A.3.2) by the same argument as above we get

$$(18) \qquad \begin{aligned} & \sum_{k=1}^{m-1}\sum_{t=0}^{m-1-k} E(i,k,t) \\ & = \sum_{t=0}^{m-2}\sum_{k=1}^{m-1-t}(-1)^{2m-2-(m-1)-t-k}\binom{m-t-2}{2m-2-(m-1)-t-k}c_k(\mathcal{E}^\vee)c_t(Y)s_{m-k-t}(\mathcal{E}) \\ & = \sum_{t=0}^{m-2}\sum_{j=0}^{m-2-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j}c_{j+1}(\mathcal{E}^\vee)c_t(Y)s_{(m-1)-j-t}(\mathcal{E}) \\ & = \sum_{t=0}^{m-1}\sum_{j=0}^{m-1-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j}c_{j+1}(\mathcal{E}^\vee)c_t(Y)s_{(m-1)-j-t}(\mathcal{E}). \end{aligned}$$

We note that in the final step of the above equalities we use $\binom{m-t-2}{m-2-t-j} = 0$ for $(t,j) = (0, m-1), (1, m-2), \dots, (m-1, 0)$. Moreover

$$(19) \qquad \begin{aligned} & \sum_{t=0}^{m-1}\sum_{j=0}^{m-1-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j}c_{j+1}(\mathcal{E}^\vee)c_t(Y)s_{(m-1)-j-t}(\mathcal{E}) \\ & = \sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E}) - c_1(\mathcal{E})c_{m-1}(Y). \end{aligned}$$

We also note that in the final step of the above equalities

$$\binom{m-t-2}{m-2-t-j} = \left\{ \begin{array}{ll} \binom{m-t-2}{j} & \text{if } t \leq m-2, \\ \binom{m-t-2}{j} - 1 & \text{if } t = m-1. \end{array} \right.$$

(Here we note that if $t = m-1$, then $j = 0$ in this case.)

Hence we see from (17), (18) and (19) that for $i = m-1$

$$\begin{aligned}(20)\qquad & e_{2m-2-i}(X,L)\\ &=\sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E})+(m-1-i)c_m(Y).\end{aligned}$$

(A.4) The case where $i = m$.
We see from (8), (9), (10) and (11) that the possible cases of (k,t) are as follows.

$$\text{(A.4.1) } k=0,\quad t=m-2. \qquad \text{(A.4.2) } \begin{cases} k=0, & t=m-2, m-3, \dots, 1, 0,\\ k=1, & t=m-3,\dots,1,0,\\ & \vdots \\ k=m-2, & t=0. \end{cases}$$

Here we note that the case $(k,t)=(0,m-2)$ is contained in both the cases (A.4.1) and (A.4.2). So we count the case $(k,t)=(0,m-2)$ as the case (A.4.2) and it suffices to calculate the case (A.4.2). By the same argument as above we get

$$\begin{aligned}(21)\qquad e_{2m-2-i}(X,L) &= \sum_{t=0}^{m-2}\sum_{j=0}^{m-2-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j}c_j(\mathcal{E}^\vee)c_t(Y)s_{m-j-t}(\mathcal{E})\\ &= \sum_{t=0}^{m}\sum_{j=0}^{m-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j}c_j(\mathcal{E}^\vee)c_t(Y)s_{m-j-t}(\mathcal{E}).\end{aligned}$$

We note that in the final step of the above equalities we use $\binom{m-t-2}{m-2-t-j}=0$ for $(t,j)=(0,m),(1,m-1),\dots,(m,0),(0,m-1),(1,m-2),\dots,(m-1,0)$.

On the other hand

$$(22)\qquad \binom{m-t-2}{m-2-t-j} = \begin{cases} \binom{m-t-2}{j} & \text{if } t\le m-2,\\ \binom{m-t-2}{j}-(-1)^j & \text{if } t=m-1, m.\end{cases}$$

(Here we note that if $t=m-1$ (resp. $t=m$), then $j=0,1$ (resp. $j=0$) in this case.)

Hence we see from (21) and (22) that for $i=m$

$$\begin{aligned}(23)\qquad & e_{2m-2-i}(X,L)\\ &=\sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{m-2-t-j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E})\\ &=\sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E})\\ &\quad +c_{m-1}(Y)s_1(\mathcal{E})+c_1(\mathcal{E}^\vee)c_{m-1}(Y)-c_m(Y)\\ &=\sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E})+(m-i-1)c_m(Y).\end{aligned}$$

By (13), (16), (20) and (23) for any i with $0\le i\le m$ we have

$$e_{2m-2-i}(X,L)=\sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^\vee)c_t(Y)s_{i-j-t}(\mathcal{E})+(m-i-1)c_m(Y).$$

Furthermore we set $l := i - t - j$. Then

$$\begin{aligned}
&\sum_{t=0}^{i}\sum_{j=0}^{i-t}(-1)^{m-2-t-j}\binom{m-t-2}{j}c_{j+m-i}(\mathcal{E}^{\vee})c_t(Y)s_{i-j-t}(\mathcal{E}) \\
&= \sum_{t=0}^{i}\sum_{l=0}^{i-t}(-1)^{m-2-i+l}\binom{m-t-2}{i-t-l}c_{m-t-l}(\mathcal{E}^{\vee})c_t(Y)s_l(\mathcal{E}) \\
&= \sum_{t=0}^{i}\sum_{l=0}^{i-t}(-1)^{i-t}\binom{m-t-2}{i-t-l}c_{m-t-l}(\mathcal{E})c_t(Y)s_l(\mathcal{E}).
\end{aligned}$$

Hence for every integer i with $0 \leq i \leq m$

$$e_{2m-2-i}(X,L) = \sum_{t=0}^{i}\sum_{l=0}^{i-t}(-1)^{i-t}\binom{m-t-2}{i-t-l}c_{m-t-l}(\mathcal{E})c_t(Y)s_l(\mathcal{E}) + (m-i-1)c_m(Y). \tag{24}$$

(B) Next we consider $e_{2m-1}(X,L)$ and $e_{2m}(X,L)$. Then by [2, Theorem 3.1 (3.1.1)] we have

$$e_{2m-1}(X,L) = mc_m(Y), \tag{25}$$
$$e_{2m}(X,L) = (m+1)c_m(Y). \tag{26}$$

(C) By (24), (25) and (26), we get the assertion of Claim 3.1. □

Here we set

$$E_i(c_0(\mathcal{E}),\dots,c_i(\mathcal{E});s_{m-i}(\mathcal{E}),\dots,s_m(\mathcal{E})) := \sum_{t=0}^{i}\sum_{k=0}^{i-t}(-1)^{i-t}\binom{m-t-2}{i-t-k}c_k(\mathcal{E})\,c_t(Y)\,s_{m-k-t}(\mathcal{E}).$$

Then by Proposition 2.1 we have

$$E_i(c_0(\mathcal{E}),\dots,c_i(\mathcal{E});s_{m-i}(\mathcal{E}),\dots,s_m(\mathcal{E})) = e_i(X,L). \tag{27}$$

Moreover by Claim 3.1 we have

$$\begin{aligned}
&(28)\quad e_{2m-2-i}(X,L) = E_i(s_0(\mathcal{E}),\dots,s_i(\mathcal{E});c_{m-i}(\mathcal{E}),\dots,c_m(\mathcal{E})) + (m-i-1)c_m(Y),\\
&(29)\quad e_{2m-1-i}(X,L) = E_{i-1}(s_0(\mathcal{E}),\dots,s_{i-1}(\mathcal{E});c_{m-i+1}(\mathcal{E}),\dots,c_m(\mathcal{E})) + (m-i)c_m(Y),\\
&(30)\quad e_{2m-i}(X,L) = E_{i-2}(s_0(\mathcal{E}),\dots,s_{i-2}(\mathcal{E});c_{m-i+2}(\mathcal{E}),\dots,c_m(\mathcal{E})) + (m-i+1)c_m(Y)
\end{aligned}$$

for every integer i with $0 \leq i \leq m$. By (27) and Definitions 1.1 and 2.2 we get

$$\begin{aligned}
(31)\qquad & F_i(s_1(\mathcal{E}),\dots,s_m(\mathcal{E})) \\
&= \mathrm{cl}_i(X,L) \\
&= (-1)^i\{E_i(c_0(\mathcal{E}),\dots,c_i(\mathcal{E});s_{m-i}(\mathcal{E}),\dots,s_m(\mathcal{E})) \\
&\quad -2E_{i-1}(c_0(\mathcal{E}),\dots,c_{i-1}(\mathcal{E});s_{m-i+1}(\mathcal{E}),\dots,s_m(\mathcal{E})) \\
&\quad +E_{i-2}(c_0(\mathcal{E}),\dots,c_{i-2}(\mathcal{E});s_{m-i+2}(\mathcal{E}),\dots,s_m(\mathcal{E}))\} \\
&= (-1)^i\{E_i(t_0(s_0(\mathcal{E})),\dots,t_i(s_0(\mathcal{E}),\dots,s_i(\mathcal{E}));s_{m-i}(\mathcal{E}),\dots,s_m(\mathcal{E})) \\
&\quad -2E_{i-1}(t_0(s_0(\mathcal{E})),\dots,t_{i-1}(s_0(\mathcal{E}),\dots,s_{i-1}(\mathcal{E}));s_{m-i+1}(\mathcal{E}),\dots,s_m(\mathcal{E})) \\
&\quad +E_{i-2}(t_0(s_0(\mathcal{E})),\dots,t_{i-2}(s_0(\mathcal{E}),\dots,s_{i-2}(\mathcal{E}));s_{m-i+2}(\mathcal{E}),\dots,s_m(\mathcal{E}))\}
\end{aligned}$$

for every integer i with $0 \leq i \leq m$. Here $t_i(x_0,\dots,x_i)$ denotes the polynomial which was defined in Definition 3.1.

On the other hand we see from (28), (29), (30) and (31) that for every integer i with $0 \leq i \leq m$

$$\begin{aligned}
\mathrm{cl}_{2m-i}(X,L) &= (-1)^{2m-i}\{e_{2m-i}(X,L) - 2e_{2m-i-1}(X,L) + e_{2m-i-2}(X,L)\} \\
&= (-1)^{i}\{E_{i-2}(s_0(\mathcal{E}),\ldots,s_{i-2}(\mathcal{E});c_{m-i+2}(\mathcal{E}),\ldots,c_m(\mathcal{E})) \\
&\quad -2E_{i-1}(s_0(\mathcal{E}),\ldots,s_{i-1}(\mathcal{E});c_{m-i+1}(\mathcal{E}),\ldots,c_m(\mathcal{E})) \\
&\quad +E_{i}(s_0(\mathcal{E}),\ldots,s_{i}(\mathcal{E});c_{m-i}(\mathcal{E}),\ldots,c_m(\mathcal{E})) \\
&\quad +(m-i+1)c_m(Y) - 2(m-i)c_m(Y) + (m-i-1)c_m(Y)\} \\
&= (-1)^{i}\{E_{i}(s_0(\mathcal{E}),\ldots,s_{i}(\mathcal{E});c_{m-i}(\mathcal{E}),\ldots,c_m(\mathcal{E})) \\
&\quad -2E_{i-1}(s_0(\mathcal{E}),\ldots,s_{i-1}(\mathcal{E});c_{m-i+1}(\mathcal{E}),\ldots,c_m(\mathcal{E})) \\
&\quad +E_{i-2}(s_0(\mathcal{E}),\ldots,s_{i-2}(\mathcal{E});c_{m-i+2}(\mathcal{E}),\ldots,c_m(\mathcal{E}))\} \\
&= (-1)^{i}\{E_{i}(t_0(c_0(\mathcal{E})),\ldots,t_i(c_0(\mathcal{E}),\ldots,c_i(\mathcal{E}));c_{m-i}(\mathcal{E}),\ldots,c_m(\mathcal{E})) \\
&\quad -2E_{i-1}(t_0(c_0(\mathcal{E})),\ldots,t_{i-1}(c_0(\mathcal{E}),\ldots,c_{i-1}(\mathcal{E}));c_{m-i+1}(\mathcal{E}),\ldots,c_m(\mathcal{E})) \\
&\quad +E_{i-2}(t_0(c_0(\mathcal{E})),\ldots,t_{i-2}(c_0(\mathcal{E}),\ldots,c_{i-2}(\mathcal{E}));c_{m-i+2}(\mathcal{E}),\ldots,c_m(\mathcal{E}))\} \\
&= F_i(c_1(\mathcal{E}),\ldots,c_m(\mathcal{E})).
\end{aligned}$$

Therefore we get the assertion of Theorem 3.1. $\square$

Finally we note the following.

Proposition 3.2 *Let (X,L) be an n-dimensional classical scroll over a smooth projective variety Y of dimension m such that $n \geq 2m+1$. Then $\mathrm{cl}_i(X,L) = 0$ for every integer i with $2m+1 \leq i \leq n$.*

Proof. By [2, Theorem 3.1 (3.1.1)], we see that $e_j(X,L) = (j-m+1)c_m(Y)$ for every integer j with $j \geq 2m-1$. Hence

$$\mathrm{cl}_i(X,L) = (-1)^i(e_i(X,L) - 2e_{i-1}(X,L) + e_{i-2}(X,L)) = 0.$$

This completes the proof. $\square$

References

[1] Y. Fukuma, *On the sectional invariants of polarized manifolds,* J. Pure Appl. Algebra **209** (2007), 99–117.

[2] Y. Fukuma, *Sectional invariants of scroll over a smooth projective variety,* Rend. Sem. Mat. Univ. Padova **121** (2009), 93–119.

[3] Y. Fukuma, *Sectional class of ample line bundles on smooth projective varieties,* Riv. Mat. Univ. Parma (N.S.) **6** (2015), 215–240.

[4] Y. Fukuma, *Calculations of sectional classes of special polarized manifolds,* preprint, http://www.math.kochi-u.ac.jp/fukuma/Cal-SC.html

[5] W. Fulton, *Intersection Theory,* Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 2. Springer-Verlag, Berlin, 1998.

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