Sectional class of ample vector bundles on smooth projective varieties, I: The case of ample line bundles *†‡

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Abstract

Let X be an n-dimensional smooth projective variety defined over the field of complex numbers, let $\mathcal{E}$ $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$ be ample vector bundles with $\mathrm{rank}(\mathcal{E}) = r \leq n$, $\mathrm{rank}(\mathcal{F}_1) = \mathrm{rank}(\mathcal{F}_2) = r + 1$ and $\mathrm{rank}(\mathcal{G}) = r + 2$. In this paper, we will define the generalized sectional class $\mathrm{cl}_{n,r}(X, \mathcal{E}; \mathcal{F}_1, \mathcal{F}_2; \mathcal{G})$, and we will investigate this invariant for some special cases. In particular, for every integer i with $0 \leq i \leq n-1$, by setting $\mathcal{E} := L^{\oplus n-i}$, $\mathcal{F}_1 := L^{\oplus n-i+1}$, $\mathcal{F}_2 := L^{\oplus n-i+1}$ and $\mathcal{G} := L^{\oplus n-i+2}$, we give a classification of polarized manifolds (X, L) by the value of $\mathrm{cl}_i(X, L) := \mathrm{cl}_{n,n-i}(X, \mathcal{E}; \mathcal{F}_1, \mathcal{F}_2; \mathcal{G})$.

1 Introduction

Let X be a smooth projective variety of dimension n defined over the field of complex numbers, and let L be an ample line bundle on X. Then (X, L) is called a *polarized manifold*. Assume that L is very ample and let $\varphi : X \hookrightarrow \mathbb{P}^N$ be the morphism defined by $|L|$. Then φ is an embedding. In this situation, its dual variety $X^\vee \to (\mathbb{P}^N)^\vee$ is a hypersurface of N-dimensional projective space except some special types. Then the *class* $\mathrm{cl}(X, L)$ of (X, L) is defined by the following.

$$\mathrm{cl}(X, L) = \begin{cases} \deg(X^\vee), & \text{if } X^\vee \text{ is a hypersurface in } (\mathbb{P}^N)^\vee \\ 0, & \text{otherwise.} \end{cases}$$

A lot of investigations by using $\mathrm{cl}(X, L)$ have been obtained (for example [22], [26], [34], [23], [27], [25], [1], [31] and so on). In this paper, we are going to define a generalization of this invariant. Let X be a smooth projective variety of dimension n and let $\mathcal{E}$, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$ be ample (not necessarily very ample) vector bundles on X with $\mathrm{rank}(\mathcal{E}) = r$, $\mathrm{rank}(\mathcal{F}_1) = \mathrm{rank}(\mathcal{F}_2) = r + 1$ and $\mathrm{rank}(\mathcal{G}) = r + 2$ such that $r \leq n$. Then in Section 3 we will define the *generalized sectional class* $\mathrm{cl}_{n,r}(X, \mathcal{E}; \mathcal{F}_1, \mathcal{F}_2; \mathcal{G})$ of $(X, \mathcal{E}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{G})$ (see Definition 3.4), and for future works we will study some fundamental properties concerning this invariants. Our main purpose is to study this invariant in general. But in this paper, as the first step, we consider the following case: Let L be an ample (not necessarily very ample) line bundle on X and we set $\mathcal{E} := L^{\oplus n-i}$, $\mathcal{F}_1 := L^{\oplus n-i+1}$, $\mathcal{F}_2 := L^{\oplus n-i+1}$ and $\mathcal{G} := L^{\oplus n-i+2}$, where i is an integer with $0 \leq i \leq n$. We note that $\mathrm{rank}(\mathcal{E}) = n - i$. Then we will define

$$\mathrm{cl}_i(X, L) := \mathrm{cl}_{n,n-i}\left(X, \mathcal{E}; \mathcal{F}_1, \mathcal{F}_2; \mathcal{G}\right)$$

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We will call this invariant the *ith sectional class of* (X, L). In this paper, we mainly study this invariant $\mathrm{cl}_i(X, L)$ for the case where L is not necessarily very ample and will get some results about $\mathrm{cl}_i(X, L)$.

Here we note the following: Assume that L is very ample. Then there exists a member $X_j \in |L_{j-1}|$ such that each X_j is a smooth projective manifold of dimension $n-j$ and $L_j := L_{j-1}|_{X_j}$ for every j with $1 \leq j \leq n-i$. In this case, we see that $\mathrm{cl}_i(X, L)$ is the class of the i dimensional polarized manifold (X_{n-i}, L_{n-i}). In particular, if $i = n$, then $\mathrm{cl}_n(X, L)$ is equal to the class $\mathrm{cl}(X, L)$ of (X, L) if L is very ample.

As we said above, there are a lot of works about the class $\mathrm{cl}(X, L)$ for very ample line bundles L, that is, the case where $i = n$ and L is very ample.

Classifications of (X, L) concerning $\mathrm{cl}_i(X, L)$ are known for the following cases.

- The case where $i = n \leq 3$ and L is very ample (see [22], [26], [23]).
- The case where $i = 2$, $n \geq 2$ and L is very ample (see [34], [27], [25]).
- The case where $i = n = 2$ and L is ample (see [31]).

In this paper, we give classifications of (X, L) by the value of $\mathrm{cl}_i(X, L)$ for the following cases.

- The case where $i = 1$, $n \geq 3$, $\mathrm{cl}_1(X, L) \leq 4$ and L is ample.
- The case where $i = 2$, $n \geq 3$, $\mathrm{cl}_2(X, L) \leq 16$ and L is ample and spanned.
- The case where $i = 3$, $n \geq 3$, $\mathrm{cl}_3(X, L) \leq 8$ and L is ample and spanned.
- The case where $i = 4$, $n \geq 5$, $\mathrm{cl}_4(X, L) \leq 1$ (resp. $\mathrm{cl}_4(X, L) = 2$) and L is ample and spanned (resp. very ample).

Moreover in this paper we also give some interesting problems (see Problems 3.1 and 4.1, and Conjecture 4.1).

In Section 4, we calculate $\mathrm{cl}_i(X, L)$ for some special cases. The results in Section 4 will be used in order to classify (X, L) by the value of $\mathrm{cl}_i(X, L)$. In Sections 5, 6, 7 and 8 we obtain the classification of (X, L) by the value of $\mathrm{cl}_1(X, L)$, $\mathrm{cl}_2(X, L)$, $\mathrm{cl}_3(X, L)$ and $\mathrm{cl}_4(X, L)$.

We are planning of studying a classification of $(X, \mathcal{E}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{G})$ by the value of $\mathrm{cl}_{n,r}(X, \mathcal{E}; \mathcal{F}_1, \mathcal{F}_2; \mathcal{G})$ in a future paper.

The content of this paper includes the content of the paper entitled "Sectional class of ample line bundles on smooth projective varieties", which was cited in [16].

2 Preliminaries

Definition 2.1 Let (X, L) be a polarized manifold of dimension n.

(1) We say that (X, L) is a *scroll* (resp. *quadric fibration*) *over a normal projective variety* Y *of dimension* m with $1 \leq m < n$ if there exists a surjective morphism with connected fibers $f : X \to Y$ such that $K_X + (n-m+1)L = f^*A$ (resp. $K_X + (n-m)L = f^*A$) for some ample line bundle A on Y.

(2) (X, L) is called a *hyperquadric fibration over a smooth curve* C if (X, L) is a quadric fibration over C and the morphism $f : X \to C$ is the contraction morphism of an extremal ray. In this case, $h^2(X, \mathbb{C}) = 2$, $(F, L_F) \cong (\mathbb{Q}^{n-1}, \mathcal{O}_{\mathbb{Q}^{n-1}}(1))$ for any general fiber F of f, and any fiber of f is irreducible and reduced.

Notation 2.1 (1) Let (X,L) be a hyperquadric fibration over a smooth curve C. We put $\mathcal{E} := f_*(L)$. Then $\mathcal{E}$ is a locally free sheaf of rank $n+1$ on C. Let $\pi : \mathbb{P}_C(\mathcal{E}) \to C$ be the projective bundle. Then $X \in |2H(\mathcal{E}) + \pi^*(B)|$ for some $B \in \mathrm{Pic}(C)$ and $L = H(\mathcal{E})|_X$, where $H(\mathcal{E})$ is the tautological line bundle of $\mathbb{P}_C(\mathcal{E})$. We put $e := \deg \mathcal{E}$ and $b := \deg B$.

(2) (See [7, (13.10) Chapter II].) Let (M,A) be a $\mathbb{P}^2$-bundle over a smooth curve C and $A|_F = \mathcal{O}_{\mathbb{P}^2}(2)$ for any fiber F of it. Let $f : M \to C$ be the fibration and $\mathcal{E} := f_*(K_M + 2A)$. Then $\mathcal{E}$ is a locally free sheaf of rank 3 on C, and $M \cong \mathbb{P}_C(\mathcal{E})$ such that $H(\mathcal{E}) = K_M + 2A$. In this case, $A = 2H(\mathcal{E}) + f^*(B)$ for a line bundle B on C, and by the canonical bundle formula $K_M = -3H(\mathcal{E}) + f^*(K_C + \det\mathcal{E})$. Here we set $e := \deg \mathcal{E}$ and $b := \deg B$.

Definition 2.2 Let $\mathcal{F}$ be a vector bundle on a smooth projective variety X. Then for every integer j with $j \geq 0$, the *jth Segre class $s_j(\mathcal{F})$ of $\mathcal{F}$* is defined by the following equation: $c_t(\mathcal{F}^\vee)s_t(\mathcal{F}) = 1$, where $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, $c_t(\mathcal{F}^\vee)$ is the Chern polynomial of $\mathcal{F}^\vee$ and $s_t(\mathcal{F}) = \sum_{j\geq 0} s_j(\mathcal{F})t^j$.

Remark 2.1 (a) Let $\mathcal{F}$ be a vector bundle on a smooth projective variety X. Let $\tilde{s}_j(\mathcal{F})$ be the jth Segre class which is defined in [20, Chapter 3]. Then $s_j(\mathcal{F}) = \tilde{s}_j(\mathcal{F}^\vee)$.
(b) For every integer i with $1 \leq i$, $s_i(\mathcal{F})$ can be written by using the Chern classes $c_j(\mathcal{F})$ with $1 \leq j \leq i$. (For example, $s_1(\mathcal{F}) = c_1(\mathcal{F})$, $s_2(\mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, and so on.)

Definition 2.3 Let $L_1, \ldots, L_m$ be ample line bundles on a smooth projective variety X. Then $(X, L_1, \ldots, L_m)$ is called a *multi-polarized manifold of type m*.

Theorem 2.1 *Let (X,L) be a polarized manifold of dimension $n \geq 3$. Assume that L is spanned. If $b_2(X,L) = h^2(X,\mathbb{C}) + 1$, then (X,L) is one of the following types.*

(a) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(b) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$.

(c) *A simple blowing up of (X,L) of type* (b).

(d) $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^*\mathcal{O}_{\mathbb{P}^1}(1))$, *where p_i is the ith projection.*

(e) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, *where S is a smooth projective surface and $\mathcal{E}$ is an ample vector bundle of rank two on S with $c_2(\mathcal{E}) = 2$. In particular $(S, \mathcal{E})$ is one of the following.*

(e.1) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$.

(e.2) $(\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1) \oplus \mathcal{O}_{\mathbb{Q}^2}(1))$.

(e.3) $(\mathbb{P}_C(\mathcal{F}), \pi^*(\mathcal{G}) \otimes H(\mathcal{F}))$, *where C is an elliptic curve, $\mathcal{F}$ and $\mathcal{G}$ are indecomposable vector bundles of rank two on C with $\deg \mathcal{F} = 1$ and $\deg \mathcal{G} = 1$, and $\pi : \mathbb{P}_C(\mathcal{F}) \to C$ is the projection map.*

(e.4) *S is a double covering $f : S \to \mathbb{P}^2$ of $\mathbb{P}^2$ and $\mathcal{E} \cong f^*(\mathcal{O}_{\mathbb{P}^2}(1)) \oplus f^*(\mathcal{O}_{\mathbb{P}^2}(1))$.*

Proof. See [18, Theorem 3.1]. □

3 Definition and fundamental results

In this section, we will give the definition of the *generalized sectional class* $\mathrm{cl}_{n,r}(X, \mathcal{E}; \mathcal{F}_1, \mathcal{F}_2; \mathcal{G})$ for ample vector bundles $\mathcal{E}$, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$ on X with rank $\mathcal{E} = r \leq \dim X$, rank $\mathcal{F}_1 = r+1$, rank $\mathcal{F}_2 = r+1$ and rank $\mathcal{G} = r+2$. Moreover we will give some fundamental results.

Definition 3.1 (See also [16, Definition 2.1.3].) Let X be a smooth projective variety of dimension n and let $\mathcal{E}$ be a vector bundle on X. Let r be the rank of $\mathcal{E}$. Assume that $r \leq n$. For every integer j with $0 \leq j \leq n-r$ we set

$$C_j^{n,r}(X,\mathcal{E}) \quad := \quad \sum_{k=0}^{j} c_k(X)s_{j-k}(\mathcal{E}^\vee).$$

Definition 3.2 (See also [16, Definitions 3.1.1 and 3.2.1].) Let $(X,\mathcal{E})$ be a generalized polarized manifold of dimension n with $1 \leq \operatorname{rank} \mathcal{E} = r \leq n$. (Here we use notation in Definition 3.1.) Then the *ith c_r-sectional Euler number $e_{n,r}(X,\mathcal{E})$ of $(X,\mathcal{E})$* and the *c_r-sectional Betti number $b_{n,r}(X,\mathcal{E})$ of $(X,\mathcal{E})$* are defined by the following.

$$\begin{aligned} e_{n,r}(X,\mathcal{E}) &:= C_{n-r}^{n,r}(X,\mathcal{E})c_r(\mathcal{E}) \\ b_{n,r}(X,\mathcal{E}) &:= \begin{cases} (-1)^{n-r}\left(e_{n,r}(X,\mathcal{E}) - \sum_{j=0}^{n-r-1} 2(-1)^j h^j(X,\mathbb{C})\right), & \text{if } r < n, \\ e_{n,n}(X,\mathcal{E}), & \text{if } r = n. \end{cases} \end{aligned}$$

Remark 3.1 If $n-r$ is odd, then $e_{n,r}(X,\mathcal{E})$ is even.

Proof. First we note that $r < n$ because $n-r$ is odd. Then by the definition of $b_{n,r}(X,\mathcal{E})$, we have

$$e_{n,r}(X,\mathcal{E}) = 2\sum_{j=0}^{n-r-1}(-1)^j h^j(X,\mathbb{C}) + (-1)^{n-r}b_{n,r}(X,\mathcal{E}).$$

On the other hand, since $n-r$ is odd, $b_{n-r}(X,\mathcal{E})$ is even by [16, Theorem 4.1]. Hence $e_{n,r}(X,\mathcal{E})$ is even. □

Definition 3.3 ([16, Definition 5.1.1]) Let $(X, L_1,\dots,L_{n-i})$ be a multi-polarized manifold of dimension n, where i is an integer with $0 \leq i \leq n-1$. Then we define the *ith sectional Euler number* $e_i(X,L_1,\dots,L_{n-i})$ and the *ith sectional Betti number* $b_i(X,L_1,\dots,L_{n-i})$ are defined as follows.

$$\begin{aligned} e_i(X,L_1,\dots,L_{n-i}) &:= e_{n,n-i}(X, L_1\oplus\cdots\oplus L_{n-i}), \\ b_i(X,L_1,\dots,L_{n-i}) &:= b_{n,n-i}(X, L_1\oplus\cdots\oplus L_{n-i}). \end{aligned}$$

Proposition 3.1 *Let i be an integer with $0 \leq i \leq n-1$ and let $(X, L_1,\dots,L_{n-i})$ be a multi-polarized manifold of type $n-i$. Assume that a line bundle L is ample and $L_k = L$ for every integer k with $1 \leq k \leq n-i$. Then we have*

$$e_i(X,L_1,\dots,L_{n-i}) = e_i(X,L), \quad b_i(X,L_1,\dots,L_{n-i}) = b_i(X,L).$$

Here $e_i(X,L)$ (resp. $b_i(X,L)$) is the ith sectional Euler number (resp. the ith sectional Betti number) which was defined in [13, Definition 3.1 (1) and (2)].

Proof. See [16, Proposition 5.2.1]. □

Definition 3.4 Let X be a smooth projective variety of dimension n. Let $\mathcal{E}$, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$ be ample vector bundles on X with $\operatorname{rank} \mathcal{E} = r$, $\operatorname{rank} \mathcal{F}_1 = r+1$, $\operatorname{rank} \mathcal{F}_2 = r+1$ and $\operatorname{rank} \mathcal{G} = r+2$. Assume that $r \leq n$. Then the *generalized sectional class* $\mathrm{cl}_{n,r}(X,\mathcal{E};\mathcal{F}_1,\mathcal{F}_2;\mathcal{G})$ is defined by the following.

$$\begin{aligned} &\mathrm{cl}_{n,r}(X,\mathcal{E};\mathcal{F}_1,\mathcal{F}_2;\mathcal{G}) \\ &:= \begin{cases} (-1)^{n-r}\{e_{n,r}(X,\mathcal{E}) - e_{n,r+1}(X,\mathcal{F}_1) - e_{n,r+1}(X,\mathcal{F}_2) + e_{n,r+2}(X,\mathcal{G})\}, & \text{if } r \leq n-2. \\ -e_{n,n-1}(X,\mathcal{E}) + e_{n,n}(X,\mathcal{F}_1) + e_{n,n}(X,\mathcal{F}_2), & \text{if } r = n-1. \\ e_{n,n}(X,\mathcal{E}), & \text{if } r = n. \end{cases} \end{aligned}$$

Remark 3.2 If $n-r$ is odd and $\mathcal{F}_1=\mathcal{F}_2=\mathcal{F}$, then by Remark 3.1 we see that $\mathrm{cl}_{n,r}(X,\mathcal{E};\mathcal{F},\mathcal{F};\mathcal{G})$ is even.

Here we will consider a special case.

Definition 3.5 Let X be a smooth projective variety of dimension $n\geq 1$. Let i be an integer with $0\leq i\leq n$. Let $L_1,\dots,L_{n-i},A_1,A_2$ be ample line bundles on X.
Then the *ith sectional class* of $(X,L_1,\dots,L_{n-i};A_1,A_2)$ is defined by the following:

$$\begin{aligned}&\mathrm{cl}_i(X,L_1,\dots,L_{n-i};A_1,A_2)\\&:=\begin{cases}\mathrm{cl}_{n,n-i}(X,L_1\oplus\cdots\oplus L_{n-i};L_1\oplus\cdots\oplus L_{n-i}\oplus A_1,\\ \qquad\qquad L_1\oplus\cdots\oplus L_{n-i}\oplus A_2;L_1\oplus\cdots\oplus L_{n-i}\oplus A_1\oplus A_2), & \text{if } 0\leq i\leq n-1.\\ (-1)^n\left\{e(X)-e_{n-1}(X,A_1)-e_{n-1}(X,A_2)+e_{n-2}(X,A_1,A_2)\right\}, & \text{if } i=n\geq 2.\\ -e(X)+\deg A_1+\deg A_2, & \text{if } i=n=1.\end{cases}\end{aligned}$$

Remark 3.3 (1) Assume that $0\leq i\leq n-1$. By Definition 3.4 and [16, Definition 5.1.1] we have

$$\begin{aligned}&\mathrm{cl}_i(X,L_1,\dots,L_{n-i};A_1,A_2)\\&:=\begin{cases}e_0(X,L_1,\dots,L_n), & \text{if } i=0,\\ (-1)\{e_1(X,L_1,\dots,L_{n-1})-e_0(X,L_1,\dots,L_{n-i},A_1)\\ \qquad -e_0(X,L_1,\dots,L_{n-i},A_2)\}, & \text{if } i=1,\\ (-1)^i\{e_i(X,L_1,\dots,L_{n-i})-e_{i-1}(X,L_1,\dots,L_{n-i},A_1)\\ \qquad -e_{i-1}(X,L_1,\dots,L_{n-i},A_2)+e_{i-2}(X,L_1,\dots,L_{n-i},A_1,A_2)\}, & \text{if } 2\leq i\leq n-1.\end{cases}\end{aligned}$$

(2) If i is odd and $A_1=A_2=A$, then by Remark 3.2 we see that $\mathrm{cl}_i(X,L_1,\dots,L_{n-i};A,A)$ is even.
(3) If $i=0$, then $\mathrm{cl}_0(X,L_1,\dots,L_n;A_1,A_2)=L_1\cdots L_n$.

Definition 3.6 Let (X,L) be a polarized manifold of dimension n and let i be an integer with $0\leq i\leq n$. Then the *ith sectional class* of (X,L) is defined by the following:

$$\mathrm{cl}_i(X,L):=\mathrm{cl}_i(X,\underbrace{L,\dots,L}_{n-i};L,L).$$

Proposition 3.2 *Let (X,L) be a polarized manifold of dimension n. For any integer i with $0\leq i\leq n$, the following holds.*

$$\mathrm{cl}_i(X,L)=\sum_{t=0}^{i}(-1)^{i-t}\binom{n-i+t+1}{t}c_{i-t}(X)L^{n-i+t}.$$

Proof. By the definition of the ith sectional Euler number $e_i(X,L)$ of (X,L) (see [13, Definition 3.1 (1)]), we have

$$e_i(X,L)=\sum_{l=0}^{i}(-1)^l\binom{n-i+l-1}{l}c_{i-l}(X)L^{n-i+l}.$$

Hence we get

$$\begin{aligned}&\mathrm{cl}_i(X,L)\\&=(-1)^i\left(\sum_{l=0}^{i}(-1)^l\binom{n-i+l-1}{l}c_{i-l}(X)L^{n-i+l}-2\sum_{l=0}^{i-1}(-1)^l\binom{n-i+l}{l}c_{i-1-l}(X)L^{n-i+l+1}\right.\\&\qquad\left.+\sum_{l=0}^{i-2}(-1)^l\binom{n-i+l+1}{l}c_{i-2-l}(X)L^{n-i+l+2}\right)\end{aligned}$$

$$
\begin{aligned}
&= (-1)^i \left(\sum_{l=2}^{i} (-1)^l \left\{ \binom{n-i+l-1}{l} + 2\binom{n-i+l-1}{l-1} + \binom{n-i+l-1}{l-2} \right\} c_{i-l}(X)L^{n-i+l} \right. \\
&\qquad \left. + c_i(X)L^{n-i} - (n-i)c_{i-1}(X)L^{n-i+1} - 2c_{i-1}(X)L^{n-i+1} \right) \\
&= (-1)^i \left(\sum_{l=2}^{i} (-1)^l \binom{n-i+l+1}{l} c_{i-l}(X)L^{n-i+l} - (n-i+2)c_{i-1}(X)L^{n-i+1} + c_i(X)L^{n-i} \right) \\
&= (-1)^i \left(\sum_{l=0}^{i} (-1)^l \binom{n-i+l+1}{l} c_{i-l}(X)L^{n-i+l} \right).
\end{aligned}
$$

Therefore we get the assertion. □

Remark 3.4 (i) By [2, Lemma 1.6.4] we have $\mathrm{cl}_i(X, L) = c_i(J_1(L))L^{n-i}$, where $J_1(L)$ is the first jet bundle of L.

(ii) Assume that L is very ample. Then there exists a sequence of smooth subvarieties $X \supset X_1 \supset \cdots \supset X_{n-i}$ such that $X_j \in |L_{j-1}|$ and $\dim X_j = n - j$ for every integer j with $1 \leq j \leq n-i$, where $L_j = L_{j-1}|_{X_j}$. In particular, X_{n-i} is a smooth projective variety of dimension i and L_j is a very ample line bundle on X_j. Then $\mathrm{cl}_i(X, L)$ is equal to the class of (X_{n-i}, L_{n-i}).

Remark 3.5 ([21, II-1]) Let X be an n-dimensional smooth projective variety and let L be a very ample line bundle on X. Let $X \hookrightarrow \mathbb{P}^N$ be the embedding defined by $|L|$. For every integer i with $0 \leq i \leq n$, Severi defined the notion of the ith *rank* $r_i(X)$ of X as follows.

$$r_i(X) = \int L^i (L^\vee)^{N-1-i}(CX).$$

Here CX denotes the conormal variety, $X^\vee$ denotes the dual variety of X and $L^\vee = \mathcal{O}_{X^\vee}(1)$. Then we see that $r_i(X) = \mathrm{cl}_{n-i}(X, L)$ (see [21, (6) Theorem in II]). We also note that if $i = 0$, then $r_0(X) = \mathrm{cl}_n(X, L)$ is called the *class* of X.

Definition 3.7 Let X be a smooth projective variety of dimension n, and let $\phi : X \hookrightarrow \mathbb{P}^N$ be an embedding. Assume that X is non-degenerate in $\mathbb{P}^N$. Let $X^\vee \subset (\mathbb{P}^N)^\vee$ be the dual variety of X. Then we set

$$
\begin{aligned}
\mathrm{def}(X, \phi) &:= \mathrm{codim}_{(\mathbb{P}^N)^\vee} X^\vee - 1. \\
\mathrm{codeg}(X, \phi) &:= \deg X^\vee.
\end{aligned}
$$

Next we will give a generalization of these numbers. First we note the following proposition.

Proposition 3.3 *Let (X, L) be a polarized manifold of dimension n. Assume that L is very ample. Let $\phi_L : X \hookrightarrow \mathbb{P}^N$ be the embedding defined by L. Then*

$$
\begin{aligned}
\mathrm{def}(X, \phi_L) &= \min\{\ i \mid 0 \leq i \leq n,\ \mathrm{cl}_{n-i}(X, L) \neq 0\} \\
\mathrm{codeg}(X, \phi_L) &= \mathrm{cl}_{n-\mathrm{def}(X,\phi_L)}(X, L).
\end{aligned}
$$

Proof. By [3, (0.3.1) Lemma and (0.3.2) Remark] and Remark 3.4 (i), we get the assertion. □

Definition 3.8 Let X be a smooth projective variety of dimension n, and let L, A_1 and A_2 ample line bundles on X.

(i) The *deficiency* of $(X, L; A_1, A_2)$ is defined by the following.

$$\mathrm{def}(X, L; A_1, A_2) := \min\{\ i \mid 0 \leq i \leq n,\ \mathrm{cl}_{n-i}(X, L, \ldots, L; A_1, A_2) \neq 0\}$$

(ii) The *codegree* of $(X, L; A_1, A_2)$ is defined by the following.

$$\mathrm{codeg}(X, L; A_1, A_2) := \mathrm{cl}_{n-k}(X, L, \dots, L; A_1, A_2),$$

where $k = \mathrm{def}(X, L; A_1, A_2)$.

(iii) If $A_1 = A_2 = L$, then

$$\begin{aligned} \mathrm{def}(X, L) &:= \mathrm{def}(X, L; L, L) \\ \mathrm{codeg}(X, L) &:= \mathrm{codeg}(X, L; L, L). \end{aligned}$$

Here we note that the following holds if L is very ample.

Proposition 3.4 *Let X be a smooth projective variety of dimension n, and let L be a very ample line bundle on X. Let $\phi_L : X \hookrightarrow \mathbb{P}^N$ be the embedding defined by L. Then $\mathrm{def}(X, L) = \mathrm{def}(X, \phi_L)$ and $\mathrm{codeg}(X, L) = \mathrm{codeg}(X, \phi_L)$.*

Proof. By Proposition 3.3, we have $\mathrm{def}(X, \phi_L) = \min\{\ i \mid 0 \le i \le n,\ \mathrm{cl}_{n-i}(X, L) \neq 0\}$. Hence

$$\begin{aligned} \mathrm{def}(X, L) = \mathrm{def}(X, L; L, L) &= \min\{\ i \mid 0 \le i \le n,\ \mathrm{cl}_{n-i}(X, L, \dots, L; L, L) \neq 0\} \\ &= \min\{\ i \mid 0 \le i \le n,\ \mathrm{cl}_{n-i}(X, L) \neq 0\} \\ &= \mathrm{def}(X, \phi_L) \end{aligned}$$

Therefore we get the first assertion.
By Definition 3.8 and Proposition 3.3, we have

$$\begin{aligned} \mathrm{codeg}(X, L) &= \mathrm{codeg}(X, L; L, L) \\ &= \mathrm{cl}_{n-\mathrm{def}(X,L;L,L)}(X, L, \dots, L; L, L) \\ &= \mathrm{cl}_{n-\mathrm{def}(X,L)}(X, L) \\ &= \mathrm{cl}_{n-\mathrm{def}(X,\phi_L)}(X, L) \\ &= \mathrm{codeg}(X, \phi_L). \end{aligned}$$

So we get the second assertion. □

When L is very ample, the following shows that we can calculate $\mathrm{codeg}(X, L)$ by using the sectional Euler numbers.

Proposition 3.5 *Let X be a smooth projective variety of dimension n, and let L be a very ample line bundle on X. Then the following equality holds.*

$$\mathrm{codeg}(X, L) = (-1)^{n-k}\left((k+1)e_n(X, L) - (k+2)e_{n-1}(X, L) + e_{n-k-2}(X, L)\right),$$

where $k = \mathrm{def}(X, L)$.

Proof. By Proposition 3.4 we have $\mathrm{codeg}(X, L) = \mathrm{codeg}(X, \phi_L)$. On the other hand, by [32, Proposition 2] or [33, Theorem 10.6] we see

$$\mathrm{codeg}(X, \phi_L) = (-1)^{n-k}\left((k+1)e(X) - (k+2)e(X_1) + e(X_{k+2})\right),$$

where $k = \mathrm{def}(X, L)$ and we use notation in Remark 3.4 (ii). Since $e_n(X, L) = e(X)$, $e_{n-1}(X, L) = e(X_1)$ and $e_{n-k-2}(X, L) = e(X_{k+2})$, we get the assertion. □

Problem 3.1 *Does the equality in Proposition 3.5 hold for any ample line bundle L?*

Next we will prove the following which is useful in order to classify (X, L) by the value of $\mathrm{cl}_{n,r}(X, \mathcal{E}; \mathcal{F}_1, \mathcal{F}_2; \mathcal{G})$ (see also [21, (18) Lemma in II]).

Proposition 3.6 *Let X be a smooth projective variety of dimension n and let i be an integer. Let $\mathcal{E}$, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$ be ample vector bundles on X with* rank $\mathcal{E} = r$, rank $\mathcal{F}_1 = r+1$, rank $\mathcal{F}_2 = r+1$ *and* rank $\mathcal{G} = r+2$. *Assume that $1 \leq r \leq n$. Then the following holds.*

$$\begin{aligned}
&\mathrm{cl}_{n,r}(X, \mathcal{E}; \mathcal{F}_1, \mathcal{F}_2; \mathcal{G}) \\
&:= \begin{cases}
b_{n,r}(X,\mathcal{E}) - b_{n-r-2}(X) + b_{n,r+1}(X,\mathcal{F}_1) - b_{n-r-1}(X) & \\
+b_{n,r+1}(X,\mathcal{F}_2) - b_{n-r-1}(X) + b_{n,r+2}(X,\mathcal{G}) - b_{n-r-2}(X), & \text{if } 1 \leq r \leq n-2, \\
& \\
b_{n,n-1}(X,\mathcal{E}) + b_{n,n}(X,\mathcal{F}_1) - b_0(X) + b_{n,n}(X,\mathcal{F}_2) - b_0(X), & \text{if } r = n-1, \\
& \\
b_{n,n}(X,\mathcal{E}), & \text{if } r = n.
\end{cases}
\end{aligned}$$

Proof. Here we note the following.

$$e_{n,r}(X,\mathcal{E}) = 2\sum_{j=0}^{n-r-1}(-1)^j b_j(X) + (-1)^{n-r} b_{n,r}(X,\mathcal{E}), \tag{1}$$

$$e_{n,r+1}(X,\mathcal{F}_k) = 2\sum_{j=0}^{n-r-2}(-1)^j b_j(X) + (-1)^{n-r-1} b_{n,r+1}(X,\mathcal{F}_k) \tag{2}$$

$$e_{n,r+2}(X,\mathcal{G}) = 2\sum_{j=0}^{n-r-3}(-1)^j b_j(X) + (-1)^{n-r-2} b_{n,r+2}(X,\mathcal{G}). \tag{3}$$

Since

$$\begin{aligned}
&(-1)^i \left(2\sum_{j=0}^{i-1}(-1)^j b_j(X) - 4\sum_{j=0}^{i-2}(-1)^j b_j(X) + 2\sum_{j=0}^{i-3}(-1)^j b_j(X) \right) \\
&= (-1)^i \left((-1)^{i-1} 2b_{i-1}(X) + (-1)^{i-2} 2b_{i-2}(X) - (-1)^{i-2} 4b_{i-2}(X) \right) \\
&= -2b_{i-1}(X) - 2b_{i-2}(X),
\end{aligned}$$

we get the assertion by substituting the above three equations (1), (2) and (3) for the formula in Definition 3.4. □

Corollary 3.1 *Let X be a smooth projective variety of dimension n and let i be an integer with $0 \leq i \leq n-1$. Let $L_1, \dots, L_{n-i}, A_1, A_2$ be ample line bundles on X. Then the following holds.*

$$\begin{aligned}
&\mathrm{cl}_i(X, L_1, \dots, L_{n-i}; A_1, A_2) \\
&:= \begin{cases}
b_0(X, L_1, \dots, L_n), & \text{if } i = 0, \\
& \\
b_1(X, L_1, \dots, L_{n-1}) + b_0(X, L_1, \dots, L_n, A_1) - b_0(X) & \\
+b_0(X, L_1, \dots, L_n, A_2) - b_0(X), & \text{if } i = 1, \\
& \\
b_i(X, L_1, \dots, L_{n-i}) - b_{i-2}(X) + b_{i-1}(X, L_1, \dots, L_{n-i}, A_1) - b_{i-1}(X) & \\
+b_{i-1}(X, L_1, \dots, L_{n-i}, A_2) - b_{i-1}(X) & \\
+b_{i-2}(X, L_1, \dots, L_{n-i}, A_1, A_2) - b_{i-2}(X), & \text{if } 2 \leq i \leq n-1.
\end{cases}
\end{aligned}$$

Proof. By setting $\mathcal{E} = L_1 \oplus \cdots \oplus L_{n-i}$, $\mathcal{F}_1 = L_1 \oplus \cdots \oplus L_{n-i} \oplus A_1$, $\mathcal{F}_2 = L_1 \oplus \cdots \oplus L_{n-i} \oplus A_2$ and $\mathcal{G} = L_1 \oplus \cdots \oplus L_{n-i} \oplus A_1 \oplus A_2$, we get the assertion from Proposition 3.6. □

Next we study the non-negativity of the generalized sectional class.

Theorem 3.1 *Let X be a smooth projective variety of dimension n and let i be an integer with $0 \leq i \leq n-1$. Let $\mathcal{E}$, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{G}$ be ample and spanned vector vector bundles on X with* rank $\mathcal{E} = r$, rank $\mathcal{F}_1 = r+1$, rank $\mathcal{F}_2 = r+1$ *and* rank $\mathcal{G} = r+2$.
(i) *Assume that $r \leq n-1$. Then $\mathrm{cl}_{n,r}(X, \mathcal{E}; \mathcal{F}_1, \mathcal{F}_2; \mathcal{G}) \geq 0$.*
(ii) *Assume that $r = n$. Then $\mathrm{cl}_{n,n}(X, \mathcal{E}) > 0$.*

Proof. (i) First we assume that $r \leq n-2$. Then by Proposition 3.6, we get

$$\begin{aligned}\mathrm{cl}_{n,r}(X, \mathcal{E}; \mathcal{F}_1, \mathcal{F}_2; \mathcal{G}) &= b_{n,r}(X, \mathcal{E}) - b_{n-r-2}(X) + b_{n,r+1}(X, \mathcal{F}_1) - b_{n-r-1}(X) \\ &\quad + b_{n,r+1}(X, \mathcal{F}_2) - b_{n-r-1}(X) + b_{n,r+2}(X, \mathcal{G}) - b_{n-r-2}(X).\end{aligned}$$

Since $\mathcal{E}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{G}$ are ample and spanned, by [16, Proposition 4.1] we have

$$\begin{aligned}b_{n,r}(X, \mathcal{E}) &\geq b_{n-r}(X) \\ b_{n,r+1}(X, \mathcal{F}_k) &\geq b_{n-r-1}(X) \\ b_{n,r+2}(X, \mathcal{G}) &\geq b_{n-r-2}(X).\end{aligned}$$

On the other hand, we obtain $b_{n-r}(X) \geq b_{n-r-2}(X)$ by the hard Lefschetz theorem [28, Corollary 3.1.40]. Therefore we get the assertion.

Next we assume that $r = n-1$. Then by definition we have $\mathrm{cl}_{n,n-1}(X, \mathcal{E}; \mathcal{F}_1, \mathcal{F}_2; \mathcal{G}) = (K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}) + c_n(\mathcal{F}_1) + c_n(\mathcal{F}_2)$. Since $(K_X + c_1(\mathcal{E}))c_{n-1}(\mathcal{E}) \geq -2$ by [29, Theorem 1], and the ampleness of $\mathcal{F}_k$ implies $c_n(\mathcal{F}_k) \geq 1$, we have $\mathrm{cl}_{n,n-1}(X, \mathcal{E}; A_1, A_2) \geq 0$.

(ii) Assume that $r = n$. Then $\mathrm{cl}_{n,n}(X, \mathcal{E}) = c_n(\mathcal{E}) > 0$ since $\mathcal{E}$ is ample. Therefore we get the assertion. □

Remark 3.6 We do not need the assumption that $\mathcal{E}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ are spanned when we consider the case where $r = n-1$ or n.

By Definition 3.6 and Theorem 3.1 the following holds.

Corollary 3.2 *Let (X, L) be a polarized manifold of dimension n and let i be an integer with $0 \leq i \leq n-1$. Assume that L is base point free. Then $\mathrm{cl}_i(X, L) \geq 0$.*

Next we consider the value of the sectional class of a reduction of multi-polarized manifolds.

Definition 3.9 Let k be a positive integer.

(1) Let $(X, L_1, \cdots, L_k)$ and $(Y, H_1, \cdots, H_k)$ be n-dimensional multi-polarized manifolds of type k. Then $(X, L_1, \cdots, L_k)$ is called a *simple blowing up of a multi-polarized manifold* $(Y, H_1, \cdots, H_k)$ *of type* k if there exists a blowing up $\pi : X \to Y$ at a point $y \in Y$ such that $L_j = \pi^*(H_j) - E$ and $E|_E \cong \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ for every integer j with $1 \leq j \leq k$, where $E \cong \mathbb{P}^{n-1}$ is the exceptional effective divisor.

(2) A multi-polarized manifold $(\widetilde{X}, \widetilde{L_1}, \cdots, \widetilde{L_k})$ of type k is called a *reduction of* $(X, L_1, \cdots, L_k)$ if there exists a birational morphism $\pi : X \to \widetilde{X}$ such that π is a composite of simple blowing ups and $(\widetilde{X}, \widetilde{L_1}, \cdots, \widetilde{L_k})$ is not a simple blowing up of another multi-polarized manifold of type k. This π is called the *reduction map*.

Proposition 3.7 *Let $(X, L_1, \cdots, L_{n-i})$ be a multi-polarized manifold of type $n-i$ with* $\dim X = n$, *where i is an integer with $0 \leq i \leq n-1$. Let $(Y, H_1, \cdots, H_{n-i})$ be a multi-polarized manifold of type $n-i$ such that $(X, L_1, \cdots, L_{n-i})$ is a composite of simple blowing ups of $(Y, H_1, \cdots, H_{n-i})$ and let γ be the number of its simple blowing ups. Then*

$$\begin{aligned}e_i(X, L_1, \cdots, L_{n-i}) &= e_i(Y, H_1, \cdots, H_{n-i}) + (i-1)\gamma, \\ e(X) &= e(Y) + (n-1)\gamma.\end{aligned}$$

Proof. See [16, Proposition 5.3.1] and its proof. □

Proposition 3.8 *Let $(X, L_1, \cdots, L_{n-i}, A_1, A_2)$ be a multi-polarized manifold of type $n-i+2$ with $\dim X = n$, where i is an integer with $0 \leq i \leq n-1$. Let $(Y, H_1, \cdots, H_{n-i}, B_1, B_2)$ be a multi-polarized manifold of type $n-i+2$ such that $(X, L_1, \cdots, L_{n-i}, A_1, A_2)$ is a composite of simple blowing ups of $(Y, H_1, \cdots, H_{n-i}, B_1, B_2)$ and let γ be the number of its simple blowing ups. Then*

$$\begin{aligned} &\mathrm{cl}_i(X, L_1, \ldots, L_{n-i}; A_1, A_2) \\ &:= \begin{cases} \mathrm{cl}_0(Y, H_1, \ldots, H_n; B_1, B_2) - \gamma, & \text{if } i = 0, \\ \mathrm{cl}_1(Y, H_1, \ldots, H_{n-1}; B_1, B_2) - 2\gamma, & \text{if } i = 1, \\ \mathrm{cl}_i(Y, H_1, \ldots, H_{n-i}; B_1, B_2), & \text{if } 2 \leq i \leq n-1 \text{ or } i = n \geq 2. \end{cases} \end{aligned}$$

Proof. By Definition 3.5, Remark 3.3 and Proposition 3.7, we get the assertion. □

Corollary 3.3 *Let (X, L) be a polarized manifold of dimension $n \geq 2$ and let (Y, H) be a polarized manifold such that (X, L) is a composite of simple blowing ups of (Y, H) and let γ be the number of its simple blowing ups. Then for every integer i with $0 \leq i \leq n-1$, we have*

$$\mathrm{cl}_i(X, L) \quad := \quad \begin{cases} \mathrm{cl}_0(Y, H) - \gamma, & \text{if } i = 0, \\ \mathrm{cl}_1(Y, H) - 2\gamma, & \text{if } i = 1, \\ \mathrm{cl}_i(Y, H), & \text{if } 2 \leq i \leq n-1 \text{ or } i = n \geq 2. \end{cases}$$

Proof. By putting $L_1 := L, \cdots, L_{n-i} := L$, $A_1 := L$, $A_2 := L$, $H_1 := H, \cdots, H_{n-i} := H$, $B_1 := H$ and $B_2 := H$, we get the assertion by Proposition 3.8. □

4 Calculations on the sectional class of some special polarized manifolds

Here we are going to calculate the ith sectional class $\mathrm{cl}_i(X, L)$ of some special polarized manifolds (X, L) with $n = \dim X \geq 3$ by using its ith sectional Euler number $e_i(X, L)$. By Remark 3.3 (1) and Definitions 3.5 and 3.6, we have

$$\mathrm{cl}_i(X, L) \quad := \quad \begin{cases} e_0(X, L), & \text{if } i = 0, \\ (-1)\{e_1(X, L) - 2e_0(X, L)\}, & \text{if } i = 1, \\ (-1)^i\{e_i(X, L) - 2e_{i-1}(X, L) + e_{i-2}(X, L)\}, & \text{if } 2 \leq i \leq n-1. \end{cases}$$

Example 4.1 (i) The case where (X, L) is $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.
Then by [17, Example 3.1] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{if } i \geq 1. \end{cases}$$

(ii) The case where (X, L) is $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.
Then by [17, Example 3.2] we have $\mathrm{cl}_i(X, L) = 2$ for $0 \leq i \leq n$. In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 2$.
(iii) The case where (X, L) is $(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.
Then by [17, Example 3.3] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 16, & \text{if } i = 0, \\ 40, & \text{if } i = 1, \\ 40, & \text{if } i = 2, \\ 20, & \text{if } i = 3, \\ 5, & \text{if } i = 4. \end{cases}$$

In this case, $\mathrm{def}(X,L)=0$ and $\mathrm{codeg}(X,L)=5$.
(iv) The case where (X,L) is $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.
Then by [17, Example 3.4] we have

$$\mathrm{cl}_i(X,L) = \begin{cases} 16, & \text{if } i=0, \\ 40, & \text{if } i=1, \\ 40, & \text{if } i=2, \\ 20, & \text{if } i=3. \end{cases}$$

In this case, $\mathrm{def}(X,L)=0$ and $\mathrm{codeg}(X,L)=20$.
(v) The case where (X,L) is $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.
Then by [17, Example 3.5] we have

$$\mathrm{cl}_i(X,L) = \begin{cases} 27, & \text{if } i=0, \\ 72, & \text{if } i=1, \\ 72, & \text{if } i=2, \\ 32, & \text{if } i=3. \end{cases}$$

In this case, $\mathrm{def}(X,L)=0$ and $\mathrm{codeg}(X,L)=32$.
(vi) The case where (X,L) is a Veronese fibration over a smooth curve C.
Here we use Notation 2.1 (2). Then by [17, Example 3.6] we have

$$\mathrm{cl}_i(X,L) = \begin{cases} 8e+12b, & \text{if } i=0, \\ 20e+28b, & \text{if } i=1, \\ 36e+47b, & \text{if } i=2, \\ 41e+52b, & \text{if } i=3. \end{cases}$$

First we note that

$$8e+12b = L^3 \tag{4}$$
$$2g(C)-2+e+2b = 0 \tag{5}$$
$$g(X,L) = 1+2e+2b \tag{6}$$

Here we set $L^3 = 4m$. Then m is an integer with $m \geq 1$. We see from (4) and (5) that $b = 4(1-g(C))-m$ and $e = 6(g(C)-1)+2m$. Therefore

$$\mathrm{cl}_1(X,L) = 20e+28b = 12m+8(g(C)-1) > 0.$$

Next we consider $\mathrm{cl}_2(X,L)$. Then

$$\mathrm{cl}_2(X,L) = 36e+47b = 25m+28(g(C)-1).$$

If $g(C)=0$ and $m=1$, then we have $e=-4$ and $b=3$. But then by (6) we have $g(X,L)=-1<0$ and this is impossible. Hence $g(C)\geq 1$ or $m\geq 2$, and we get

$$\mathrm{cl}_2(X,L) \geq 25m+28(g(C)-1) \geq 22.$$

Finally we consider $\mathrm{cl}_3(X,L)$. Then

$$\mathrm{cl}_3(X,L) = 41e+52b = 30m+38(g(C)-1).$$

By the same argument as above, the case where $g(C)=0$ and $m=1$ does not occur. Hence $g(C)\geq 1$ or $m\geq 2$, and we get

$$\mathrm{cl}_3(X,L) \geq 30m+38(g(C)-1) \geq 22.$$

Therefore $\mathrm{def}(X,L)=0$ and $\mathrm{codeg}(X,L)=30m+38(g(C)-1)$.
(vii) The case where (X,L) is a Del Pezzo manifold with $n=\dim X\geq 3$.
Here we note that by [7, (8.11) Theorem], we have $L^n\leq 8$ and (X,L) is one of the following:

(vii.1) $(X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$.
Then by [17, Example 3.7 (3.7.1)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 8, & \text{if } i = 0, \\ 16, & \text{if } i = 1, \\ 12, & \text{if } i = 2, \\ 4, & \text{if } i = 3. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 4$.

(vii.2) X is the blowing up of $\mathbb{P}^3$ at a point and $L = \pi^*(\mathcal{O}_{\mathbb{P}^3}(2)) - E$, where $\pi : X \to \mathbb{P}^3$ is its birational morphism and E is the exceptional divisor. Then by [17, Example 3.7 (3.7.2)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 7, & \text{if } i = 0, \\ 14, & \text{if } i = 1, \\ 12, & \text{if } i = 2, \\ 4, & \text{if } i = 3. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 4$.

(vii.3) (X, L) is either

$$(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^* \mathcal{O}_{\mathbb{P}^1}(1)), (\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^2 p_i^* \mathcal{O}_{\mathbb{P}^2}(1)) \text{ or } (\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}), H(T_{\mathbb{P}^2}))$$

where p_i is the ith projection and $T_{\mathbb{P}^2}$ is the tangent bundle of $\mathbb{P}^2$.

(vii.3.1) The case where $(X, L) \cong (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^* \mathcal{O}_{\mathbb{P}^1}(1))$.
Then by [17, Example 3.7 (3.7.3.1)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 6, & \text{if } i = 0, \\ 12, & \text{if } i = 1, \\ 12, & \text{if } i = 2, \\ 4, & \text{if } i = 3. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 4$.

(vii.3.2) The case where $(X, L) \cong (\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^2 p_i^* \mathcal{O}_{\mathbb{P}^2}(1))$.
Then by [17, Example 3.7 (3.7.3.2)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 6, & \text{if } i = 0, \\ 12, & \text{if } i = 1, \\ 12, & \text{if } i = 2, \\ 6, & \text{if } i = 3, \\ 3, & \text{if } i = 4. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 3$.

(vii.3.3) The case where $(X, L) \cong (\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2}), H(T_{\mathbb{P}^2}))$.
Then by [17, Example 3.7 (3.7.3.3)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 6, & \text{if } i = 0, \\ 12, & \text{if } i = 1, \\ 12, & \text{if } i = 2, \\ 6, & \text{if } i = 3. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 6$.

(vii.4) The case where (X, L) is a linear section of the Grassmann variety $\mathrm{Gr}(5, 2)$ parametrizing lines in $\mathbb{P}^4$, embedded in $\mathbb{P}^9$ via the Plücker embedding. Then $3 \leq n \leq 6$ and $L^n = 5$.

By [17, Example 3.7 (3.7.4)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 5, & \text{if } i = 0, \\ 10, & \text{if } i = 1, \\ 12, & \text{if } i = 2, \\ 10, & \text{if } i = 3, \\ 5, & \text{if } i = 4 \text{ and } 4 \leq n \leq 6, \\ 0, & \text{if } i = 5 \text{ and } 5 \leq n \leq 6, \\ 0, & \text{if } i = 6 \text{ and } n = 6. \end{cases}$$

In this case, if $n = 6$ (resp. 5, 4, 3), then $\mathrm{def}(X, L) = 2$ (resp. 1, 0, 0) and $\mathrm{codeg}(X, L) = 5$ (resp. 5, 5, 10).

(vii.5) The case where (X, L) is a complete intersection of two hyperquadrics in $\mathbb{P}^{n+2}$.

Then by [17, Example 3.7 (3.7.5)] we have

$$\mathrm{cl}_i(X, L) = 4i + 4.$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 4n + 4$.

(vii.6) The case where X is a hypercubic in $\mathbb{P}^{n+1}$ and $L = \mathcal{O}_X(1)$.

Then by [17, Example 3.7 (3.7.6)] we have

$$\mathrm{cl}_i(X, L) = 3 \cdot 2^i.$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 3 \cdot 2^n$.

In general, the following holds by Definitions 3.5, 3.6 and [17, Lemma 3.3] (see also [21, (9) Proposition in II]).

Proposition 4.1 *If X is a hypersurface of degree m in $\mathbb{P}^{n+1}$, then*

$$\mathrm{cl}_i(X, L) = m(m-1)^i, \ \mathrm{def}(X, L) = 0 \textit{ and } \mathrm{codeg}(X, L) = m(m-1)^n.$$

(vii.7) The case where X is a double covering of $\mathbb{P}^n$ branched along a smooth hypersurface of degree 4, and L is the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$.

Then by [17, Example 3.7 (3.7.7)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 2, & \text{if } i = 0, \\ 4 \cdot 3^{i-1}, & \text{if } i \geq 1. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 4 \cdot 3^{n-1}$.

In general, we can prove the following by using [17, Lemma 3.4].

Proposition 4.2 *If X is a double covering of $\mathbb{P}^n$ branched along a smooth hypersurface of degree m, and L is the pull back of $\mathcal{O}_{\mathbb{P}^n}(1)$, then for $i \geq 1$ we have*

$$\mathrm{cl}_i(X, L) = m(m-1)^{i-1}, \ \mathrm{def}(X, L) = 0 \textit{ and } \mathrm{codeg}(X, L) = m(m-1)^{n-1}.$$

(vii.8) The case where (X, L) is a weighted hypersurface of degree 6 in the weighted projective space $\mathbb{P}(3, 2, 1, \ldots, 1)$.

Then by [17, Example 3.7 (3.7.8)] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 1, & \text{if } i = 0, \\ 2, & \text{if } i = 1, \\ 12 \cdot 5^{i-2}, & \text{if } i \geq 2. \end{cases}$$

In this case, $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 12 \cdot 5^{n-2}$.

(viii) The case where (X, L) is a hyperquadric fibration over a smooth curve C.
Here we use notation in Notation 2.1 (1). Then by [17, Example 3.8] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 2e + b, & \text{if } i = 0, \\ 6e + 4b + 4(g(C) - 1), & \text{if } i = 1, \\ 8e + 4ib + 4(g(C) - 1), & \text{if } i \geq 2. \end{cases}$$

Here we consider a lower bound of $\mathrm{cl}_i(X, L)$ for $i \geq 1$.

Proposition 4.3 *Let (X, L) be a hyperquadric fibration over a smooth curve C. If $i \geq 1$, then $\mathrm{cl}_i(X, L) \geq 4$.*

Proof. Then we use the following inequalities.

$$2e + b > 0 \tag{7}$$
$$2e + (n + 1)b \geq 0 \tag{8}$$

(A) First we consider the case $i = 1$. Then $g(X, L) \geq 2$ holds because (X, L) is a hyperquadric fibration over a smooth curve. Hence by definition we have $\mathrm{cl}_1(X, L) = 2(g(X, L) + L^n - 1) \geq 4$.
(B) Next we consider the case $i \geq 2$.
(B.1) If $b < 0$, then by (8) we have

$$\begin{aligned} 2e + ib &= 2e + (n + 1)b - (n + 1 - i)b \\ &\geq -(n + 1 - i)b \\ &\geq n + 1 - i. \end{aligned} \tag{9}$$

Hence

$$\begin{aligned} \mathrm{cl}_i(X, L) &= 8e + 4ib + 4(g(C) - 1) \\ &= 4(2e + ib) + 4(g(C) - 1) \\ &\geq 4(n + 1 - i) + 4(g(C) - 1) \\ &= 4(n - i) + 4g(C) \\ &\geq 0. \end{aligned}$$

If $\mathrm{cl}_i(X, L) = 0$, then $i = n$ and $g(C) = 0$. Then by (8) we have $0 = \mathrm{cl}_i(X, L) = 4(2e + (n + 1)b) - 4b - 4 \geq -4b - 4 \geq 0$ and we get $2e + (n + 1)b = 0$ and $b = -1$. Since $g(C) = 0$, we see that $\mathcal{E}$ can be expressed as

$$\mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{O}(e_i).$$

We may assume that $e_0 \leq \cdots \leq e_n$. Since $b = -1$, we see that $e_0 \geq 1$ by the same argument as in the proof of [5, Lemma (3.19)]. Hence

$$e = \sum_{i=0}^{n} e_i \geq n + 1.$$

But this is impossible because

$$e = -\frac{(n+1)}{2} b = \frac{(n+1)}{2}.$$

Hence $\mathrm{cl}_i(X, L) > 0$ in this case.

(B.2) If $b \geq 0$, then by (7) we have $2e + ib = 2e + b + (i-1)b \geq 1 + (i-1)b$. Hence

$$\begin{aligned} \mathrm{cl}_i(X, L) &= 8e + 4ib + 4(g(C) - 1) \\ &\geq 4(i-1)b + 4g(C) \\ &\geq 0. \end{aligned}$$

If $\mathrm{cl}_i(X, L) = 0$, then $b = 0$ and $g(C) = 0$. Then we have $\mathrm{cl}_i(X, L) = 8e - 4$. But since $\mathrm{cl}_i(X, L) = 0$, we have $e = \frac{1}{2}$ and this is impossible. Therefore $\mathrm{cl}_i(X, L) > 0$ holds in this case, too.

Since $\mathrm{cl}_i(X, L)$ for $i \geq 2$ is divided by 4, we see that $\mathrm{cl}_i(X, L) \geq 4$. □

Hence we see from Proposition 4.3 that $\mathrm{def}(X, L) = 0$ and $\mathrm{codeg}(X, L) = 8e + 4nb + 4(g(C) - 1)$.

(ix) The case where (X, L) is a scroll over a smooth curve C with $n = \dim X \geq 3$. Then there exists an ample vector bundle $\mathcal{E}$ on C of rank n such that $X = \mathbb{P}_S(\mathcal{E})$ and $L = H(\mathcal{E})$.
Then by [17, Example 3.9] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} s_1(\mathcal{E}), & \text{if } i = 0, \\ 2g(C) - 2 + 2c_1(\mathcal{E}), & \text{if } i = 1, \\ c_1(\mathcal{E}), & \text{if } i = 2, \\ 0, & \text{if } i \geq 3. \end{cases}$$

In this case, $\mathrm{def}(X, L) = n - 2$ and $\mathrm{codeg}(X, L) = c_1(\mathcal{E})$.

(x) The case where (X, L) is $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth surface and $\mathcal{E}$ is an ample vector bundle of rank $n - 1$. Then by [17, Example 3.10] we have

$$\mathrm{cl}_i(X, L) = \begin{cases} s_2(\mathcal{E}), & \text{if } i = 0, \\ (s_1(\mathcal{E}) + K_S)s_1(\mathcal{E}) + 2s_2(\mathcal{E}), & \text{if } i = 1, \\ c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}), & \text{if } i = 2, \\ 2c_2(\mathcal{E}) + (c_1(\mathcal{E}) + K_S)c_1(\mathcal{E}), & \text{if } i = 3, \\ c_2(\mathcal{E}), & \text{if } i = 4 \text{ and } n \geq 4, \\ 0, & \text{if } i \geq 5 \text{ and } n \geq 5. \end{cases}$$

(x.1) Assume that $K_S + c_1(\mathcal{E})$ is not nef. Here we note that $\mathrm{rank}\, \mathcal{E} \geq 2 = \dim S$. Then by a result of [35, Theorem 1] we see that $(S, \mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. In this case, $c_2(S) = 3$, $c_1(\mathcal{E})^2 = 4$, $K_S c_1(\mathcal{E}) = -6$, $c_2(\mathcal{E}) = 1$, $s_2(\mathcal{E}) = 3$. So we get the following.

$$\mathrm{cl}_i(X, L) = \begin{cases} 3, & \text{if } i = 0, \\ 4, & \text{if } i = 1, \\ 3, & \text{if } i = 2, \\ 0, & \text{if } i = 3. \end{cases}$$

Hence in this case $\mathrm{def}(X, L) = 1$ and $\mathrm{codeg}(X, L) = 3$.

Remark 4.1 Here we note that if $(S,\mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$, then $(X,L) = (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ is a scroll over $\mathbb{P}^1$.

(x.2) Next we consider the case where $K_S + c_1(\mathcal{E})$ is nef. Then the following holds.

Claim 4.1 *$cl_i(X,L) > 0$ for every $0 \leq i \leq \min\{4, n\}$.*

Proof. First of all, since $\mathcal{E}$ is ample, we see from [20, Example 12.1.7] and Remark 2.1 that $\mathrm{cl}_0(X,L) = s_2(\mathcal{E}) > 0$. Next we consider the case of $i \geq 1$. $(K_S + c_1(\mathcal{E}))c_1(\mathcal{E}) \geq 0$ because $K_S + c_1(\mathcal{E})$ is nef. Moreover $c_2(\mathcal{E}) > 0$ since $\mathcal{E}$ is ample. Hence $\mathrm{cl}_1(X,L) > 0$, $\mathrm{cl}_3(X,L) > 0$ and $\mathrm{cl}_4(X,L) > 0$ for $n \geq 4$. (Here we note that $c_1(\mathcal{E}) = s_1(\mathcal{E})$.) Finally we consider the case of $\mathrm{cl}_2(X,L)$. We note the following.

(a) If $\kappa(S) \geq 0$, then $c_2(S) \geq 0$.

(b) If $\kappa(S) = -\infty$ and $q(S) = 0$, then $c_2(S) \geq 3$.

(c) If $\kappa(S) = -\infty$ and $q(S) \geq 1$, then $c_2(S) \geq 4(1 - q(S))$.

So if $\kappa(S) \geq 0$ or $\kappa(S) = -\infty$ and $q(S) = 0$, then

$$\begin{aligned} \mathrm{cl}_2(X,L) &= c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}) \\ &\geq c_1(\mathcal{E})^2 > 0. \end{aligned}$$

If $\kappa(S) = -\infty$ and $q(S) \geq 1$, then

$$\begin{aligned} \mathrm{cl}_2(X,L) &= c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}) \\ &\geq c_1(\mathcal{E})^2 + 4(g(S, c_1(\mathcal{E})) - q(S)). \end{aligned}$$

Since $\kappa(S) = -\infty$, we have $g(S, c_1(\mathcal{E})) \geq q(S)$ by [8, Theorem 2.1]. Therefore we get $\mathrm{cl}_2(X,L) \geq c_1(\mathcal{E})^2 > 0$. $\square$

Therefore, in this case, we get $\mathrm{def}(X,L) = \max\{0, 4-n\}$ and

$$\mathrm{codeg}(X,L) = \begin{cases} 2c_2(\mathcal{E}) + (c_1(\mathcal{E}) + K_S)c_1(\mathcal{E}), & \text{if } n = 3, \\ c_2(\mathcal{E}), & \text{if } n \geq 4. \end{cases}$$

In general, if X is a projective bundle over a smooth projective variety Y of dimension m with $\dim X \geq 2m$ and L is the tautological line bundle $H(\mathcal{E})$, then we can calculate $\mathrm{def}(X,L)$ and $\mathrm{codeg}(X,L)$.

Proposition 4.4 *Let X be an n-dimensional projective bundle $P_Y(\mathcal{E})$ over a smooth projective variety Y of dimension m and let $H(\mathcal{E})$ be the tautological line bundle. Assume that $n \geq 2m$. Then* $\mathrm{def}(X, H(\mathcal{E})) = n - 2m$ *and* $\mathrm{codeg}(X, H(\mathcal{E})) = c_m(\mathcal{E})$.

Proof. If $j - 2 \geq 2m - 1$, that is, $j \geq 2m + 1$, then by [15, Theorem 3.1 (3.1.1)] we have

$$\begin{aligned} \mathrm{cl}_j(P_Y(\mathcal{E}), H(\mathcal{E})) &= (-1)^j (e_j(P_Y(\mathcal{E}), H(\mathcal{E})) - 2e_{j-1}(P_Y(\mathcal{E}), H(\mathcal{E})) + e_{j-2}(P_Y(\mathcal{E}), H(\mathcal{E}))) \\ &= (-1)^j ((j-m+1)c_m(Y) - 2(j-m)c_m(Y) + (j-m-1)c_m(Y)) \\ &= 0. \end{aligned}$$

If $j = 2m$, then by [15, Theorem 3.1 (3.1.1) and (3.1.2)]

$$\begin{aligned} \mathrm{cl}_{2m}(P_Y(\mathcal{E}), H(\mathcal{E})) &= (-1)^{2m} (e_{2m}(P_Y(\mathcal{E}), H(\mathcal{E})) - 2e_{2m-1}(P_Y(\mathcal{E}), H(\mathcal{E})) + e_{2m-2}(P_Y(\mathcal{E}), H(\mathcal{E}))) \\ &= ((m+1)c_m(Y) - 2mc_m(Y) + (m-1)c_m(Y) + c_m(\mathcal{E})) \\ &= c_m(\mathcal{E}) > 0. \end{aligned}$$

Hence by Definition 3.8 we have

$$\begin{aligned}\operatorname{def}(X,H(\mathcal{E})) &= \min\{\ i \mid \mathrm{cl}_{n-i}(X,H(\mathcal{E}))\neq 0\} = n-2m.\\ \operatorname{codeg}(X,H(\mathcal{E})) &= c_m(\mathcal{E}).\end{aligned}$$

This completes the proof. $\square$

Assume that (X,L) is a $\mathbb{P}^{n-3}$-bundle over a smooth projective variety Y with $n\geq 4$ and $\dim Y=3$. Let $\mathcal{E}$ be an ample vector bundle on Y such that $X\cong\mathbb{P}_Y(\mathcal{E})$ and $L=H(\mathcal{E})$. Then by [15, Theorem 3.1] $\mathrm{cl}_i(X,L)$ is the following.

$$\mathrm{cl}_i(X,L)=\begin{cases} s_3(\mathcal{E}), & \text{if } i=0,\\ 3s_3(\mathcal{E})+(s_1(\mathcal{E})+K_Y)s_2(\mathcal{E}), & \text{if } i=1,\\ 3s_3(\mathcal{E})+12(s_1(\mathcal{E})+K_Y)s_2(\mathcal{E}) \\ \quad +(s_1(\mathcal{E})+K_Y)s_1(\mathcal{E})^2+c_2(Y)s_1(\mathcal{E}), & \text{if } i=2,\\ -c_3(Y)+2c_3(\mathcal{E})-2c_1(\mathcal{E})c_2(\mathcal{E})+4c_1(\mathcal{E})^3 \\ \quad +3K_Yc_1(\mathcal{E})^2+2c_2(Y)c_1(\mathcal{E}), & \text{if } i=3,\\ 3c_3(\mathcal{E})+12(c_1(\mathcal{E})+K_Y)c_2(\mathcal{E}) \\ \quad +(c_1(\mathcal{E})+K_Y)c_1(\mathcal{E})^2+c_2(Y)c_1(\mathcal{E}), & \text{if } i=4,\\ 3c_3(\mathcal{E})+(c_1(\mathcal{E})+K_Y)c_2(\mathcal{E}), & \text{if } i=5 \text{ and } n\geq 5,\\ c_3(\mathcal{E}), & \text{if } i=6 \text{ and } n\geq 6,\\ 0, & \text{if } i\geq 7 \text{ and } n\geq 7.\end{cases}$$

By considering the above results, we can propose the following conjecture.

Conjecture 4.1 *Assume that (X,L) is a $\mathbb{P}^{n-m}$-bundle over a smooth projective variety Y with $\dim Y=m$. Let $\mathcal{E}$ be an ample vector bundle on Y such that $X\cong\mathbb{P}_Y(\mathcal{E})$ and $L=H(\mathcal{E})$. Assume that $n\geq 2m$. For any integer i with $0\leq i\leq m$ we set*

$$F_i(s_1(\mathcal{E}),\dots,s_m(\mathcal{E})):=\mathrm{cl}_i(X,L).$$

Then for any integer j with $m\leq j\leq 2m$ we have

$$\mathrm{cl}_j(X,L)=F_{2m-j}(c_1(\mathcal{E}),\dots,c_m(\mathcal{E})).$$

In particular

$$F_m(s_1(\mathcal{E}),\dots,s_m(\mathcal{E}))=F_m(c_1(\mathcal{E}),\dots,c_m(\mathcal{E})).$$

Remark 4.2 This conjecture is true for the case where $m=1$, 2 and 3.

By looking at the above examples, we see that $\mathrm{cl}_{i+1}(X,L)=0$ if $\mathrm{cl}_i(X,L)=0$. So we can propose the following problem.

Problem 4.1 *Let (X,L) be a polarized manifold of dimension n and let i be an integer with $0\leq i\leq n-1$. Is it true that $\mathrm{cl}_{i+1}(X,L)=0$ if $\mathrm{cl}_i(X,L)=0$?*

Remark 4.3 Let (X,L) be a polarized manifold of dimension $n\geq 3$. Assume that L is spanned and $g(X,L)\leq q(X)+2$. Then (X,L) is one of the following types (see [9], [10] and [11]).

(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(b) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(c) A scroll over a smooth curve.

(d) A Del Pezzo manifold[1] with $L^n \geq 2$.

(e) X is a double covering of $\mathbb{P}^n$ branched along a smooth hypersurface of degree 6, and L is the pull-back of $\mathcal{O}_{\mathbb{P}^n}(1)$.

(f) A scroll over a smooth surface S and (X, L) satisfies one of the types (2-1), (2-2) and (2-3) in [11, Theorem 3.3].

(g) A hyperquadric fibration over a smooth curve C and (X, L) satisfies one of the types (3-1) and (3-2) in [11, Theorem 3.3].

Here we calculate the ith sectional class of the above (X, L).

If (X, L) is the type (a) (resp. (b), (c) and (d)), then we have already calculated the ith sectional classes (see Example 4.1 (i), (ii), (vii), (ix)).

If (X, L) is the type (e), then by (vii.7) in Example 4.1, we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 2, & \text{if } i = 0, \\ 6 \cdot 5^{i-1}, & \text{if } i \geq 1. \end{cases}$$

Next we consider the case (f). Here we use the same notation as in [11, Theorem 3.3]. First we assume that (X, L) is the type (2-1) in [11, Theorem 3.3]. Then we have $K_S = -2H_\alpha - 2H_\beta$, $c_1(\mathcal{E}) = 2H_\alpha + 3H_\beta$ and $c_2(\mathcal{E}) = (H_\alpha + 2H_\beta)(H_\alpha + H_\beta) = 3$. Hence $K_S^2 = 8$, $K_S c_1(\mathcal{E}) = -10$, $c_1(\mathcal{E})^2 = 12$ and $L^n = s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 9$. On the other hand since $c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2 = 4$, by [15, Corollary 3.1 (3.1.2)] we have

$$e_i(X, L) = \begin{cases} 9, & \text{if } i = 0, \\ -2, & \text{if } i = 1, \\ 7, & \text{if } i = 2, \\ 8, & \text{if } i = 3. \end{cases}$$

Therefore

$$\mathrm{cl}_i(X, L) = \begin{cases} 9, & \text{if } i = 0, \\ 20, & \text{if } i = 1, \\ 20, & \text{if } i = 2, \\ 8, & \text{if } i = 3. \end{cases}$$

Next we consider the type (2-2) in [11, Theorem 3.3]. Then $K_S = -3H + E$ and $\mathcal{E} = (2H - E)^{\oplus 2}$. Hence $K_S^2 = 8$, $c_1(\mathcal{E})^2 = (4H - 2E)^2 = 12$, $c_2(\mathcal{E}) = (2H - E)^2 = 3$, $K_S c_1(\mathcal{E}) = -10$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 9$. We also note that $c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2 = 4$. Hence we have

$$e_i(X, L) = \begin{cases} 9, & \text{if } i = 0, \\ -2, & \text{if } i = 1, \\ 7, & \text{if } i = 2, \\ 8, & \text{if } i = 3. \end{cases}$$

Therefore

$$\mathrm{cl}_i(X, L) = \begin{cases} 9, & \text{if } i = 0, \\ 20, & \text{if } i = 1, \\ 20, & \text{if } i = 2, \\ 8, & \text{if } i = 3. \end{cases}$$

[1]Here we assume that L is spanned. So we see that $L^n \geq 2$

Next we consider the type (2-3) in [11, Theorem 3.3]. Then $K_S = -2H(\mathcal{F}) + c_1(\mathcal{F})F = -2H(\mathcal{F}) + F$, $\mathcal{E} = H(\mathcal{F}) \otimes p^*\mathcal{G}$, $\deg \mathcal{G} = 1$ and $H(\mathcal{G})^2 = 1$. Hence $K_S^2 = 4H(\mathcal{F})^2 - 4 = 0$, $c_1(\mathcal{E})^2 = (2H(\mathcal{F}) + F)^2 = 8$, $c_2(\mathcal{E}) = c_2(p^*\mathcal{G}) + H(\mathcal{G})c_1(p^*\mathcal{G}) + H(\mathcal{G})^2 = 2$, $K_S c_1(\mathcal{E}) = -4H(\mathcal{G})^2 = -4$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 6$. We also note that $c_2(S) = 12\chi(\mathcal{O}_S) - K_S^2 = 0$. Hence we have

$$e_i(X, L) = \begin{cases} 6, & \text{if } i = 0, \\ -4, & \text{if } i = 1, \\ 2, & \text{if } i = 2, \\ 0, & \text{if } i = 3. \end{cases}$$

Therefore

$$\mathrm{cl}_i(X, L) = \begin{cases} 6, & \text{if } i = 0, \\ 16, & \text{if } i = 1, \\ 16, & \text{if } i = 2, \\ 8, & \text{if } i = 3. \end{cases}$$

Finally we consider the case (g).
First we assume that (X, L) is the type in the type (3-1) in [11, Theorem 3.3]. Then by Example 4.1 (viii) we have

$$\mathrm{cl}_i(X, L) = \begin{cases} 6, & \text{if } i = 0, \\ 16, & \text{if } i = 1, \\ 16, & \text{if } i = 2, \\ 8, & \text{if } i = 3. \end{cases}$$

Next we consier the type (3-2) in [11, Theorem 3.3]. Then $e = d - 3$ and $b = 6 - d$. So by Example 4.1 (viii) we have

$$\mathrm{cl}_i(X, L) = \begin{cases} d, & \text{if } i = 0, \\ 2d + 2, & \text{if } i = 1, \\ 4(6 - d)(i - 1) + 4(d - 1), & \text{if } 2 \leq i \leq n. \end{cases}$$

Here we note that $3 \leq d \leq 9$ holds in this case, and if $d = 8$ (resp. $d \neq 8$), then $3 \leq n \leq 4$ (resp. $n = 3$).

Remark 4.4 Let (X, L) be a polarized manifold of dimension $n \geq 3$. Assume that $q(X) = 0$, L is spanned and $g(X, L) = 3$. Then (X, L) is one of (I-2), (III), (IV), (IV′) and (V) in [19, Theorem 2.1]. Here we calculate the second sectional class of (X, L), which will be used in Theorem 6.3.

(A) First we consider the case (I-2) in [19, Theorem 2.1]. Then by Example 4.1 (viii) we have $\mathrm{cl}_2(X, L) = 8e + 8b + 4(g(C) - 1) = 8e + 8b - 4 = 28$.

(B) Next we consider the case (III) in [19, Theorem 2.1].
(B.1a) If (X, L) is the type (III-1a), then $n = 5$ and $\mathrm{cl}_2(X, L) = c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}) = 27$ by Example 4.1 (x).
(B.1b) If (X, L) is the type (III-1b), then $n = 4$. If $(S, \mathcal{E}) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^n}(2))$, then by Example 4.1 (x) we have $\mathrm{cl}_2(X, L) = c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}) = 27$.
If $(S, \mathcal{E}) = (\mathbb{P}^2, T_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^n}(1))$, then by Example 4.1 (x) we have $\mathrm{cl}_2(X, L) = c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}) = 27$.
(B.1c) If (X, L) is the type (III-1c), then $S \cong \mathbb{P}^2$, $\mathrm{rank}\mathcal{E} = 2$ and $c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^2}(4)$. Hence $\mathrm{cl}_2(X, L) = c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}) = 27$.
(B.2) If (X, L) is the type (III-2), then S is a Del Pezzo surface with $K_S^2 = 2$ and $\mathcal{E}$ is an ample vector bundle of rank two on S with $c_1(\mathcal{E})^2 = 8$ and $K_S c_1(\mathcal{E}) = -4$. Hence $\mathrm{cl}_2(X, L) = c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}) = 26$.

(C) Next we consider the case (IV) in [19, Theorem 2.1]. By Proposition 4.1 we have $\mathrm{cl}_2(X, L) = 4 \cdot 3^2 = 36$.

(D) Next we consider the case (IV′) in [19, Theorem 2.1]. Since $\mathrm{cl}_2(X, L)$ and $\mathrm{cl}_3(X, L)$ are invariant under simple blowing ups by Corollary 3.3, we have $\mathrm{cl}_2(X, L) = 4 \cdot 3^2 = 36$.

(E) Next we consider the case (V) in [19, Theorem 2.1].
(E.1) If (X, L) is the type (V-1), then by Proposition 4.2 we have $\mathrm{cl}_2(X, L) = 8 \cdot 7^1 = 56$.
(E.2) If (X, L) is the type (V-2), then (X, L) is a Mukai manifold, that is, $\mathcal{O}_X(K_X+(n-2)L) = \mathcal{O}_X$. Hence by [12, Example 2.10 (7)] we have $g_2(X, L) = 1$ and $\chi_2^H(X, L) = 1 - h^1(\mathcal{O}_X) + g_2(X, L) = 2$, where $\chi_2^H(X, L)$ is the second sectional H-arithmetic genus of (X, L) (see [13, Definition 2.2 and Remark 2.1 (5)]). Furthermore by [14, Proposition 3.1] we have

$$\begin{aligned} h_2^{1,1}(X, L) &= 10\chi_2^H(X, L) - (K_X + (n-2)L)^2 L^{n-2} + 2h^1(\mathcal{O}_X) \\ &= 20. \end{aligned}$$

Here $h_2^{1,1}(X, L)$ denotes the second sectional Hodge number of type $(1, 1)$ (see [13, Definition 3.1 (3)]). Hence by [13, Theorem 3.1 (3.1.1), (3.1.3) and (3.1.4)] we get $b_2(X, L) = 2g_2(X, L) + h_2^{1,1}(X, L) = 22$. Since $b_1(X, L) = 2g_1(X, L) = 6$ and $b_0(X, L) = L^n$, we have

$$\begin{aligned} e_2(X, L) &= 2b_0(X) - 2b_1(X) + b_2(X, L) \\ &= 2 - 2 \cdot 0 + 22 = 24, \\ e_1(X, L) &= 2b_0(X) - b_1(X, L) \\ &= 2 - 6 = -4, \\ e_0(X, L) &= b_0(X, L) \\ &= 4. \end{aligned}$$

Therefore we get $\mathrm{cl}_2(X, L) = 24 - 2(-4) + 4 = 36$.

Remark 4.5 Let (X, L) be a polarized manifold of dimension $n \geq 3$ such that $h^0(L) \geq n + 1$ and $L^n \leq 2$. Then we see that $\Delta(X, L) \leq 1$, and (X, L) is one of the following types.

(i) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(ii) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(iii) X is a double covering of $\mathbb{P}^n$ whose branch locus is of degree $2g(X, L) + 2$ and L is the pull back of $\mathcal{O}_{\mathbb{P}^n}(1)$. In this case we see that $g(X, L) \geq 1$, and if $g(X, L) = 1$, then (X, L) is a Del Pezzo manifold.

Here we calculate $\mathrm{cl}_i(X, L)$. If (X, L) is the type (i) (resp. (ii)), then we get values of $\mathrm{cl}_i(X, L)$ from Example 4.1 (i) (resp. Example 4.1 (ii)).

If (X, L) is the type (iii), then by Proposition 4.2 we have $\mathrm{cl}_i(X, L) = (2g(X, L)+2)(2g(X, L)+1)^{i-1}$ for $i \geq 1$ and $\mathrm{cl}_0(X, L) = 2$.

Remark 4.6 Here we calculate $\mathrm{cl}_i(X, L)$ if (X, L) is the type (e) in Theorem 2.1.
(i) If $(S, \mathcal{E})$ is the type (e.1), then $c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{P}^2}(3)$, $c_1(\mathcal{E})^2 = 9$, $c_2(S) = 3$, $c_2(\mathcal{E}) = 2$, $K_S^2 = 9$, $K_S c_1(\mathcal{E}) = -9$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 7$. Hence by Example 4.1 (x)

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	7	14	12	4

In this case, $(X, L) = (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ is a Del Pezzo manifold with $L^3 = 7$.

(ii) If $(S, \mathcal{E})$ is the type (e.2), then $c_1(\mathcal{E}) = \mathcal{O}_{\mathbb{Q}^2}(2)$, $c_1(\mathcal{E})^2 = 8$, $c_2(S) = 4$, $c_2(\mathcal{E}) = 2$, $K_S^2 = 8$, $K_S c_1(\mathcal{E}) = -8$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 6$. Hence by Example 4.1 (x)

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	6	12	12	4

In this case, $(X, L) = (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ is a Del Pezzo manifold with $L^3 = 6$.

(iii) If $(S, \mathcal{E})$ is the type (e.3), then $c_1(\mathcal{E}) = 2H(\mathcal{F}) + \pi^* c_1(\mathcal{G})$, $c_1(\mathcal{E})^2 = 8$, $c_2(S) = 0$, $c_2(\mathcal{E}) = 2$, $K_S^2 = 0$, $K_S c_1(\mathcal{E}) = -4$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 6$. Hence by Example 4.1 (x)

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	6	16	16	8

(iv) Assume that $(S, \mathcal{E})$ is the type (e.4). Then there exists a line bundle $\mathcal{O}_{\mathbb{P}^2}(2b)$ such that the branch locus $C \in |\mathcal{O}_{\mathbb{P}^2}(2b)|$. Hence by Example 4.1 (x) $c_1(\mathcal{E}) = f^*\mathcal{O}_{\mathbb{P}^2}(2)$, $c_1(\mathcal{E})^2 = 8$, $c_2(S) = 2c_2(\mathbb{P}^2) + 2g(C) - 2 = 4b^2 - 6b + 6$, $c_2(\mathcal{E}) = 2$, $K_S^2 = 2(b-3)^2$, $K_S c_1(\mathcal{E}) = 4(b-3)$ and $s_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 6$. Hence by Example 4.1 (x)

i	0	1	2	3
$\mathrm{cl}_i(X, L)$	6	$4b+8$	$4b^2+2b+6$	$4b$

If $b = 1$, then $(X, L) = (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ is a Del Pezzo manifold with $L^3 = 6$.

5 The case where $i = 1$

In this section, we consider the case where $i = 1$. Here we assume that $n \geq 3$. In this case we have

$$\begin{aligned} \mathrm{cl}_1(X, L) &= -e_1(X, L) + 2e_0(X, L) \\ &= 2g_1(X, L) - 2 + 2L^n. \end{aligned} \tag{10}$$

Since $g_1(X, L) \geq 0$ and $L^n \geq 1$, we see that $\mathrm{cl}_1(X, L) \geq 0$. We also note that $c_1(X, L)$ is even.

Next we consider a classification of (X, L) with small $\mathrm{cl}_1(X, L)$.

(1) First we consider the case where $\mathrm{cl}_1(X, L) = 0$.

Proposition 5.1 *Let (X, L) be a polarized manifold of dimension $n \geq 3$. If $\mathrm{cl}_1(X, L) = 0$, then (X, L) is isomorphic to $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.*

Proof. If $\mathrm{cl}_1(X, L) = 0$, then we have $g_1(X, L) = 0$ and $L^n = 1$ from (10). Therefore we see from [7, (12.1) Theorem and (5.10) Theorem] that (X, L) is isomorphic to $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. □

(2) Next we consider the case where $\mathrm{cl}_1(X, L) = 2$.

Proposition 5.2 *Let (X, L) be a polarized manifold of dimension $n \geq 3$. If $\mathrm{cl}_1(X, L) = 2$, then (X, L) is one of the following types.*

(a) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(b) *A Del Pezzo manifold and $L^n = 1$.*

(c) *A scroll over an elliptic curve and $L^n = 1$.*

Proof. Then by (10) we have $(g_1(X,L), L^n) = (0,2)$ or $(1,1)$. If (X,L) is the first type, then by [7, (12.1) Theorem and (5.10) Theorem] (X,L) is the type (a) above. If (X,L) is the last type, then we see from [7, (12.3) Theorem] that (X,L) is either the type (b) or the type (c) above. $\square$

(3) Next we consider the case where $\mathrm{cl}_1(X,L) = 4$.

Proposition 5.3 *Let (X,L) be a polarized manifold of dimension $n \geq 3$. If $\mathrm{cl}_1(X,L) = 4$, then (X,L) is one of the following types.*

- (a) $(\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}), H(\mathcal{E}))$, *where* $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.
- (b) *A Del Pezzo manifold and* $L^n = 2$.
- (c) *A scroll over an elliptic curve and* $L^n = 2$.
- (d) $K_X = (3-n)L$ *and* $L^n = 1$.
- (e) (X,L) *is a simple blowing up of* (M,A), *where M is a double covering of $\mathbb{P}^n$ with branch locus being a smooth hypersurface of degree* 6 *and* $A = \pi^*(\mathcal{O}_{\mathbb{P}^n}(1))$, *where* $\pi : M \to \mathbb{P}^n$ *is its double covering.*
- (f) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, *where* $(S,\mathcal{E})$ *is one of the types* 1), 2-i) *and* 4-b) *in* [6, (2.25) Theorem].
- (g) *A hyperquadric fibration over a smooth curve C. In this case C is one of the following types (here we use the notation in Definition* **??** (ii))*:*
  - (g.1) *C is an elliptic curve,* $b = 1$ *and* $e = 0$.
  - (g.2) *C is* $\mathbb{P}^1$ *and* $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ *and* $b = 5$.
- (h) $(\mathbb{P}_C(\mathcal{E}), H(\mathcal{E}))$, *where C is a smooth curve of genus two and $\mathcal{E}$ is an ample vector bundle of rank n on C with* $c_1(\mathcal{E}) = 1$.

Proof. By (10) we have $(g_1(X,L), L^n) = (0,3)$, $(1,2)$ or $(2,1)$. If $(g_1(X,L), L^n) = (0,3)$, then by [7, (12.1) Theorem and (5.10) Theorem] (X,L) is the type (a) above. If $(g_1(X,L), L^n) = (1,2)$, then we see from [7, (12.3) Theorem] that (X,L) is either the type (b) or (c) above. If $(g_1(X,L), L^n) = (2,1)$, then by using a list of a classification of polarized manifolds with $g_1(X,L) = 2$ and $L^n = 1$ (see [5, (1.10) Theorem, (3.7) and (3.30) Theorem]) we see that (X,L) is one of the types from (c) to (h) above. $\square$

6 The case where $i = 2$

If $i = 2$ and $\kappa(X) \geq 0$, then we can get the following lower bound.

Theorem 6.1 *Let X be a smooth projective variety of dimension n with $\kappa(X) \geq 0$ and Let $L_1, \ldots, L_{n-2}, A_1, A_2$ be ample line bundles on X. Then the following inequality holds.*

$$\begin{aligned}
\mathrm{cl}_2(X, L_1, \ldots, L_{n-2}; A_1, A_2) \quad \geq \quad & \frac{1}{2n}\left(\sum_{j=1}^{n-2} L_j\right)^2 L_1 \cdots L_{n-2} + \frac{1}{2}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2} \\
& + (L_1 + \cdots + L_{n-2} + A_1) L_1 \cdots L_{n-2} A_1 \\
& + (L_1 + \cdots + L_{n-2} + A_2) L_1 \cdots L_{n-2} A_2 + L_1 \cdots L_{n-2} A_1 A_2.
\end{aligned}$$

Proof. First we note that

$$\begin{aligned}
&\mathrm{cl}_2(X, L_1, \dots, L_{n-2}; A_1, A_2) \\
&= b_2(X, L_1, \dots, L_{n-2}) - b_0(X) + b_1(X, L_1, \dots, L_{n-2}, A_1) - b_1(X) \\
&\quad + b_1(X, L_1, \dots, L_{n-2}, A_2) - b_1(X) + b_0(X, L_1, \dots, L_{n-2}, A_1, A_2) - b_0(X) \\
&= e_2(X, L_1, \dots, L_{n-2}) + 2g_1(X, L_1, \dots, L_{n-2}, A_1) \\
&\quad + 2g_1(X, L_1, \dots, L_{n-2}, A_2) + L_1 \dots L_{n-2}A_1A_2 - 4.
\end{aligned}$$

From [16, Theorem 5.3.1], we have

$$e_2(X, L_1, \dots, L_{n-2}) \geq \frac{1}{2n}\left(\sum_{j=1}^{n-2} L_j\right)^2 L_1 \cdots L_{n-2} + \frac{1}{2}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2}.$$

Hence

$$\begin{aligned}
&\mathrm{cl}_2(X, L_1, \dots, L_{n-2}; A_1, A_2) \\
&\geq \frac{1}{2n}\left(\sum_{j=1}^{n-2} L_j\right)^2 L_1 \cdots L_{n-2} + \frac{1}{2}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2} \\
&\quad + 2g_1(X, L_1, \dots, L_{n-2}, A_1) + 2g_1(X, L_1, \dots, L_{n-2}, A_2) + L_1 \cdots L_{n-2}A_1A_2 - 4 \\
&= \frac{1}{2n}\left(\sum_{j=1}^{n-2} L_j\right)^2 L_1 \cdots L_{n-2} + \frac{1}{2}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2} \\
&\quad + (K_X + L_1 + \dots + L_{n-2} + A_1)L_1 \cdots L_{n-2}A_1 \\
&\quad + (K_X + L_1 + \dots + L_{n-2} + A_2)L_1 \cdots L_{n-2}A_2 + L_1 \cdots L_{n-2}A_1A_2 \\
&\geq \frac{1}{2n}\left(\sum_{j=1}^{n-2} L_j\right)^2 L_1 \cdots L_{n-2} + \frac{1}{2}\left(\sum_{j=1}^{n-2} L_j^2\right) L_1 \cdots L_{n-2} \\
&\quad + (L_1 + \dots + L_{n-2} + A_1)L_1 \cdots L_{n-2}A_1 + (L_1 + \dots + L_{n-2} + A_2)L_1 \cdots L_{n-2}A_2 \\
&\quad + L_1 \cdots L_{n-2}A_1A_2.
\end{aligned}$$

Therefore we get the assertion. □

Here we classify polarized manifolds (X, L) such that L is spanned and $\mathrm{cl}_2(X, L) \leq 15$.

Theorem 6.2 *Let (X, L) be a polarized manifold (X, L) with $\dim X = n \geq 3$. Assume that L is spanned and $\mathrm{cl}_2(X, L) \leq 15$. Then (X, L) is one of the following.*

(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. *In this case* $\mathrm{cl}_2(X, L) = 0$.

(b) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$. *In this case* $\mathrm{cl}_2(X, L) = 2$.

(c) *A scroll over a smooth curve. In this case* $3 \leq \mathrm{cl}_2(X, L) \leq 15$.

(d) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, *where* $(S, \mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. *In this case* $\mathrm{cl}_2(X, L) = 3$.

(e) *A Del Pezzo manifold (X, L) with $L^n \geq 2$. In this case* $\mathrm{cl}_2(X, L) = 12$.

Proof. We note that

$$\begin{aligned}
\mathrm{cl}_2(X, L) &= e_2(X, L) - 2e_1(X, L) + e_0(X, L) \\
&= b_2(X, L) - b_0(X) + 2(b_1(X, L) - b_1(X)) + b_0(X, L) - b_0(X) \\
&= (b_2(X, L) - b_2(X)) + (b_2(X) - b_0(X)) + 4(g_1(X, L) - h^1(\mathcal{O}_X)) + (b_0(X, L) - b_0(X)).
\end{aligned}$$

We also note that $b_0(X,L) \geq 1 = b_0(X)$ and $b_2(X) \geq b_0(X)$. Since L is spanned, we have $b_2(X,L) \geq b_2(X)$ and $g_1(X,L) \geq h^1(\mathcal{O}_X)$ by [13, Proposition 3.3 (2)] and [2, Theorem 7.2.10]. Hence we get the following.

- If $0 \leq \mathrm{cl}_2(X,L) \leq 3$, then $g_1(X,L) = h^1(\mathcal{O}_X)$ holds.
- If $4 \leq \mathrm{cl}_2(X,L) \leq 7$, then $g_1(X,L) \leq h^1(\mathcal{O}_X) + 1$ holds.
- If $8 \leq \mathrm{cl}_2(X,L) \leq 11$, then $g_1(X,L) \leq h^1(\mathcal{O}_X) + 2$ holds.
- If $\mathrm{cl}_2(X,L) = 12$, then $g_1(X,L) \leq h^1(\mathcal{O}_X) + 2$ or $L^n = 1$ holds.
- If $\mathrm{cl}_2(X,L) = 13$, then $g_1(X,L) \leq h^1(\mathcal{O}_X) + 2$ or $L^n \leq 2$ holds.
- If $\mathrm{cl}_2(X,L) = 14$, then $g_1(X,L) \leq h^1(\mathcal{O}_X) + 2$ or $L^n \leq 2$ or $b_2(X,L) = b_2(X)$ holds.
- If $\mathrm{cl}_2(X,L) = 15$, then $g_1(X,L) \leq h^1(\mathcal{O}_X) + 2$ or $L^n \leq 2$ or $b_2(X,L) \leq b_2(X) + 1$ holds.

Hence by [14, Theorem 4.1], Theorem 2.1, Remarks 4.3, 4.5 and 4.6 and Example 4.1, we get the assertion. $\square$

Next we consider the case where $\mathrm{cl}_2(X,L) = 16$.

Theorem 6.3 *Let (X,L) be a polarized manifold (X,L) with $\dim X = n \geq 3$. Assume that L is spanned and $\mathrm{cl}_2(X,L) = 16$. Then (X,L) is one of the following.*

(a) *A scroll over a smooth curve with $c_1(\mathcal{E}) = 16$.*

(b) *A hyperquadric fibration over an elliptic curve with $e = 4$, $b = -2$ and $\mathcal{E}$ is ample. (Here we use the notation in Notation 2.1 (1)).*

(c) *$(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ and $(S,\mathcal{E}) \cong (\mathbb{P}_C(\mathcal{F}), \pi^*(\mathcal{G}) \otimes H(\mathcal{F}))$, where C is an elliptic curve, $\mathcal{F}$ and $\mathcal{G}$ are indecomposable vector bundles of rank two on C with $\deg \mathcal{F} = 1$ and $\deg \mathcal{G} = 1$, and $\pi : \mathbb{P}_C(\mathcal{F}) \to C$ is the projection map.*

Proof. By the same argument as the proof of Theorem 6.2, one of the following types occurs.

(i) $g_1(X,L) \leq h^1(\mathcal{O}_X) + 2$.

(ii) $L^n \leq 2$.

(iii) $b_2(X,L) \leq b_2(X) + 1$.

(iv) $g_1(X,L) = h^1(\mathcal{O}_X) + 3$, $L^n = 3$ and $b_2(X,L) = b_2(X) + 2$.

If (X,L) satisfies one of the cases (i), (ii), or (iii), then we see from [14, Theorem 4.1], Theorem 2.1, Remarks 4.3 and 4.5, and Example 4.1 that (X,L) is one of the types (a), (b) and (c) in Theorem 6.3. So we may assume that the case (iv) occurs. Then $\Delta(X,L) = n + L^n - h^0(L) \leq n + 3 - (n+1) \leq 2$.

Claim 6.1 *$h^1(\mathcal{O}_X) = 0$ holds.*

Proof. If $\Delta(X,L) \leq 1$, then by [7, (5.10) Theorem and (6.7) Corollary] we get the assertion. So we may assume that $\Delta(X,L) = 2$. Since L is spanned, $h^0(L) = n+1$ and $L^n = 3$, the morphism $X \to \mathbb{P}^n$ defined by $|L|$ is a triple covering. So by [28, Theorem 7.1.15], we get the assertion. $\square$

Hence $g_1(X,L) = 3$. Since $\mathrm{Bs}|L| = \emptyset$, we see from Remark 4.4 that this case (iv) cannot occur. $\square$

7 The case where $i = 3$

Here we consider a classification of (X, L) such that L is spanned and $\mathrm{cl}_3(X, L) \le 8$.

Theorem 7.1 *Let (X, L) be a polarized manifold with $\dim X = n \ge 3$. Assume that L is spanned and $\mathrm{cl}_3(X, L) \le 8$. Then (X, L) is one of the following.*

(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 0$.

(b) *A scroll over a smooth curve. In this case* $\mathrm{cl}_3(X, L) = 0$.

(c) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 2$.

(d) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$. *In this case* $\mathrm{cl}_3(X, L) = 4$.

(e) *A simple blowing up of* $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$. *In this case* $\mathrm{cl}_3(X, L) = 4$.

(f) $(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \otimes_{i=1}^3 p_i^* \mathcal{O}_{\mathbb{P}^1}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 4$.

(g) $(\mathbb{P}^2 \times \mathbb{P}^2, \otimes_{i=1}^2 p_i^* \mathcal{O}_{\mathbb{P}^2}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 6$.

(h) *A hyperquadric fibration over a smooth curve C.*

  (h.1) $g(C) = 1$, $n = 3$, $L^3 = 6$, $e = 4$, $b = -2$, *and $\mathcal{E}$ is ample. In this case* $\mathrm{cl}_3(X, L) = 8$.

  (h.2) $g(C) = 0$, $n = 3$, $L^3 = 9$, $e = 6$, $b = -3$ *and* $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ (*see* [5, (3.30) Theorem 9)]). *In this case* $\mathrm{cl}_3(X, L) = 8$.

(i) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ *and $(S, \mathcal{E})$ is one of the following.*

  (i.1) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 0$.

  (i.2) $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$. *In this case* $\mathrm{cl}_3(X, L) = 4$.

  (i.3) $(\mathbb{Q}^2, \mathcal{O}_{\mathbb{Q}^2}(1) \oplus \mathcal{O}_{\mathbb{Q}^2}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 4$.

  (i.4) *S is a double covering $f : S \to \mathbb{P}^2$ branched along a smooth hypersurface of degree* 2, *and* $\mathcal{E} = f^*(\mathcal{O}_{\mathbb{P}^n}(1)) \oplus f^*(\mathcal{O}_{\mathbb{P}^n}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 4$.

  (i.5) $(\mathbb{P}^2, T_{\mathbb{P}^2})$. *In this case* $\mathrm{cl}_3(X, L) = 6$.

  (i.6) $(\mathbb{P}_C(\mathcal{F}), \pi^*(\mathcal{G}) \otimes H(\mathcal{F}))$, *where C is an elliptic curve, $\mathcal{F}$ and $\mathcal{G}$ are indecomposable vector bundles of rank two on C with* $\deg \mathcal{F} = 1$ *and* $\deg \mathcal{G} = 1$, *and* $\pi : \mathbb{P}_C(\mathcal{F}) \to C$ *is the projection map. In this case* $\mathrm{cl}_3(X, L) = 8$.

  (i.7) *S is a double covering $f : S \to \mathbb{P}^2$ branched along a smooth hypersurface of degree* 4, *and* $\mathcal{E} \cong f^*(\mathcal{O}_{\mathbb{P}^2}(1)) \oplus f^*(\mathcal{O}_{\mathbb{P}^2}(1))$. *In this case* $\mathrm{cl}_3(X, L) = 8$.

  (i.8) $(\mathbb{P}^1_\alpha \times \mathbb{P}^1_\beta, [H_\alpha + 2H_\beta] \oplus [H_\alpha + H_\beta])$ *and H_α (resp. H_β) is the ample generator of* $\mathrm{Pic}(\mathbb{P}_\alpha)$ (*resp.* $\mathrm{Pic}(\mathbb{P}_\beta)$). *In this case* $\mathrm{cl}_3(X, L) = 8$.

  (i.9) *S is the blowing up of $\mathbb{P}^2$ at a point and* $\mathcal{E} \cong [2H - E]^{\oplus 2}$, *where H is the pull back of* $\mathcal{O}_{\mathbb{P}^2}(1)$ *and E is the exceptional divisor. In this case* $\mathrm{cl}_3(X, L) = 8$.

Proof. We note that

$$\begin{aligned}
\mathrm{cl}_3(X, L) &= -e_3(X, L) + 2e_2(X, L) - e_1(X, L) \\
&= b_3(X, L) - b_1(X) + 2(b_2(X, L) - b_2(X)) + b_1(X, L) - b_1(X) \\
&= (b_3(X, L) - b_3(X)) + (b_3(X) - b_1(X)) + 2(b_2(X, L) - b_2(X)) + 2(g_1(X, L) - h^1(\mathcal{O}_X)).
\end{aligned}$$

We also note that $\mathrm{cl}_3(X, L)$ is even and $b_3(X) \ge b_1(X)$. Since L is spanned, we have $b_3(X, L) \ge b_3(X)$, $b_2(X, L) \ge b_2(X)$ and $g_1(X, L) \ge h^1(\mathcal{O}_X)$ by [13, Proposition 3.3 (2)] and [2, Theorem 7.2.10]. Hence we get the following.

- If $0 \leq \mathrm{cl}_3(X,L) \leq 2$, then $b_2(X,L) \leq b_2(X)+1$ holds.
- If $\mathrm{cl}_3(X,L) = 4$, then $b_2(X,L) \leq b_2(X)+1$ or $g_1(X,L) = h^1(\mathcal{O}_X)$ holds.
- If $\mathrm{cl}_3(X,L) = 6$, then $b_2(X,L) \leq b_2(X)+1$ or $g_1(X,L) \leq h^1(\mathcal{O}_X)+1$ holds.
- If $\mathrm{cl}_3(X,L) = 8$, then $b_2(X,L) \leq b_2(X)+1$ or $g_1(X,L) \leq h^1(\mathcal{O}_X)+2$ holds.

By [14, Theorem 4.1], Theorem 2.1, Remarks 4.3 and 4.6, and Example 4.1 we get the assertion[2]. $\square$

8 The case where $i = 4$

Here we consider a classification of (X,L) such that L is spanned (resp. very ample) and $\mathrm{cl}_4(X,L) \leq 1$ (resp. $\mathrm{cl}_4(X,L) = 2$).

Theorem 8.1 *Let (X,L) be a polarized manifold (X,L) with $\dim X = n \geq 4$. Assume that L is spanned and $\mathrm{cl}_4(X,L) \leq 1$. Then (X,L) is one of the following.*

(a) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. *In this case* $\mathrm{cl}_4(X,L) = 0$.

(b) *A scroll over a smooth curve. In this case* $\mathrm{cl}_4(X,L) = 0$.

Proof. We note that the following equality holds.

$$\begin{aligned}
\mathrm{cl}_4(X,L) &= b_4(X,L) - b_2(X) + 2(b_3(X,L) - b_3(X)) + b_2(X,L) - b_2(X) \\
&= b_4(X,L) - b_4(X) + (b_4(X) - b_2(X)) + 2(b_3(X,L) - b_3(X)) + b_2(X,L) - b_2(X).
\end{aligned}$$

Since L is spanned, we see from [13, Proposition 3.3 (2)] that $b_4(X,L) \geq b_4(X)$, $b_3(X,L) \geq b_3(X)$ and $b_2(X,L) \geq b_2(X)$ hold. Furthermore by the strong Lefschetz theorem, we have $b_4(X) \geq b_2(X)$. Hence if $\mathrm{cl}_4(X,L) \leq 1$, then $b_2(X,L) \leq b_2(X)+1$. Since $n \geq 4$, we can easily check that (X,L) is one of the above types by [14, Theorem 4.1], Theorem 2.1 and Example 4.1. $\square$

Remark 8.1 If L is spanned, then there does not exists (X,L) with $\mathrm{cl}_4(X,L) = 1$.

Theorem 8.2 *Let (X,L) be a polarized manifold (X,L) with $\dim X = n \geq 5$. Assume that L is very ample and $\mathrm{cl}_4(X,L) = 2$. Then (X,L) is $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.*

Proof. By the same argument as above, (X,L) with $\mathrm{cl}_4(X,L) = 2$ satisfies one of the following types.

(I) $b_2(X,L) \leq b_2(X)+1$.

(II) $b_4(X,L) = b_4(X)$.

(I) If $b_2(X,L) \leq b_2(X)+1$ holds, then by [14, Theorem 4.1], Theorem 2.1 and Example 4.1 we see that (X,L) with $\mathrm{cl}_4(X,L) = 2$ is $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.
(II) Next we assume that $b_4(X,L) = b_4(X)$ holds. Then by [14, Theorem 4.2], we see that (X,L) is one of the following types since we assume that $n \geq 5$.

(II.1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(II.2) A scroll over a smooth projective curve.

[2] We note that the type (2-1) (resp. (2-2), (2-3), (3-1) and (3-2)) in [11, Theorem 3.3] corresponds to (i.8) (resp. (i.9), (i.6), (h.1) and (h.2)).

(II.3) $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$, where S is a smooth projective surface and $\mathcal{E}$ is an ample vector bundle of rank $n-1$ on S.

(II.4) X is the Plücker embedding of $G(2,5)$ and $L = \mathcal{O}_X(1)$. In this case $n = 6$.

(II.5) X is a nonsingular hyperplane section of the Plücker embedding of $G(2,5)$ in $\mathbb{P}^9$ and $L = \mathcal{O}_X(1)$. In this case $n = 5$.

Then by calculating $\mathrm{cl}_4(X, L)$, we see from Example 4.1 that $\mathrm{cl}_4(X, L) = 0$ (resp. 0, $c_2(\mathcal{E})$, 5 and 5) if (X, L) is the type (II.1) (resp. (II.2), (II.3), (II.4) and (II.5)). Hence we find that the type (II.3) is possible and in this case $c_2(\mathcal{E}) = 2$. But by [30, Theorem 6.1] and [24], the rank of $\mathcal{E}$ is two and so we have $n = 3$. This contradicts the assumption that $n \geq 5$. So we get the assertion. $\square$

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