

A PROPERTY FOR THE FORMULA OF THE SECTIONAL CLASSES OF CLASSICAL SCROLLS

By

YOSHIAKI FUKUMA

(Received February 23, 2017; Revised August 9, 2017)

Abstract. In this note, we investigate the formula of the sectional class of classical scrolls and we give an answer of a conjecture proposed in a previous paper.

1. Introduction

Let (X, L) be a polarized manifold of dimension n . Assume that L is very ample and let $\varphi : X \hookrightarrow \mathbb{P}^N$ be the morphism defined by $|L|$. Then φ is an embedding. In this situation, its dual variety $X^\vee \rightarrow (\mathbb{P}^N)^\vee$ is a hypersurface of N -dimensional projective space except some special types. Then the *class* $\text{cl}(X, L)$ of (X, L) is defined by the following.

$$\text{cl}(X, L) = \begin{cases} \deg(X^\vee), & \text{if } X^\vee \text{ is a hypersurface in } (\mathbb{P}^N)^\vee \\ 0, & \text{otherwise.} \end{cases}$$

As a generalization of this notion, in [3], we defined the *i th sectional class* $\text{cl}_i(X, L)$ for any ample line bundle L and every integer i with $0 \leq i \leq n$ (see Definition 2.2).

Here we note the following fact: Assume that L is very ample. Then there exists a sequence of smooth subvarieties $X \supset X_1 \supset \cdots \supset X_{n-i}$ such that $X_j \in |L_{j-1}|$ and $\dim X_j = n - j$ for every integer j with $1 \leq j \leq n - i$, where $L_j = L|_{X_j}$ and $L_0 := L$. In particular, X_{n-i} is a smooth projective variety of dimension i and L_{n-i} is a very ample line bundle on X_{n-i} . Then $\text{cl}_i(X, L)$ is equal to the class of (X_{n-i}, L_{n-i}) . This is the reason why we call this invariant the *i th sectional class*.

In [4], we calculated the sectional class of special polarized manifolds. For example, we consider the case where (X, L) is a classical scroll over a smooth

2010 Mathematics Subject Classification: Primary 14C20; Secondary 14J30, 14C17

Key words and phrases: Polarized manifold, sectional class, classical scroll

* This research was partially supported by JSPS KAKENHI Grant Number 24540043.

projective variety Y of dimension m such that $n := \dim X \geq 2m$. Namely, there exists an ample vector bundle \mathcal{E} on Y of rank $r \geq m + 1$ such that $(X, L) \cong (\mathbb{P}_Y(\mathcal{E}), H(\mathcal{E}))$, where $H(\mathcal{E})$ is the tautological line bundle. Here we note that we need the assumption $n \geq 2m$ in order to define and compare $\text{cl}_i(X, L)$ and $\text{cl}_{2m-i}(X, L)$ for every integer i with $0 \leq i \leq m$. Then we get the following:

(i) If $m = 1$, then by [4, Example 2.1 (ix)] we have

$$(1) \quad \text{cl}_i(X, L) = \begin{cases} s_1(\mathcal{E}), & \text{if } i = 0, \\ 2g(C) - 2 + 2c_1(\mathcal{E}), & \text{if } i = 1, \\ c_1(\mathcal{E}), & \text{if } i = 2, \\ 0, & \text{if } i \geq 3 \text{ and } n \geq 3. \end{cases}$$

(ii) If $m = 2$, then by [4, Example 2.1 (x)] we have

$$(2) \quad \text{cl}_i(X, L) = \begin{cases} s_2(\mathcal{E}), & \text{if } i = 0, \\ (s_1(\mathcal{E}) + K_S)s_1(\mathcal{E}) + 2s_2(\mathcal{E}), & \text{if } i = 1, \\ c_2(S) + 3c_1(\mathcal{E})^2 + 2K_S c_1(\mathcal{E}), & \text{if } i = 2, \\ (c_1(\mathcal{E}) + K_S)c_1(\mathcal{E}) + 2c_2(\mathcal{E}), & \text{if } i = 3, \\ c_2(\mathcal{E}), & \text{if } i = 4, \\ 0, & \text{if } i \geq 5 \text{ and } n \geq 5. \end{cases}$$

(iii) If $m = 3$, then by [4, Example 2.1] we have

$$(3) \quad \text{cl}_i(X, L) = \begin{cases} s_3(\mathcal{E}), & \text{if } i = 0, \\ 3s_3(\mathcal{E}) + (s_1(\mathcal{E}) + K_Y)s_2(\mathcal{E}), & \text{if } i = 1, \\ 3s_3(\mathcal{E}) + 12(s_1(\mathcal{E}) + K_Y)s_2(\mathcal{E}) \\ + (s_1(\mathcal{E}) + K_Y)s_1(\mathcal{E})^2 + c_2(Y)s_1(\mathcal{E}), & \text{if } i = 2, \\ -c_3(Y) + 2c_3(\mathcal{E}) - 2c_1(\mathcal{E})c_2(\mathcal{E}) \\ + 4c_1(\mathcal{E})^3 + 3K_Y c_1(\mathcal{E})^2 + 2c_2(Y)c_1(\mathcal{E}), & \text{if } i = 3, \\ 3c_3(\mathcal{E}) + 12(c_1(\mathcal{E}) + K_Y)c_2(\mathcal{E}) \\ + (c_1(\mathcal{E}) + K_Y)c_1(\mathcal{E})^2 + c_2(Y)c_1(\mathcal{E}), & \text{if } i = 4, \\ 3c_3(\mathcal{E}) + (c_1(\mathcal{E}) + K_Y)c_2(\mathcal{E}), & \text{if } i = 5, \\ c_3(\mathcal{E}), & \text{if } i = 6, \\ 0, & \text{if } i \geq 7 \\ & \text{and } n \geq 7. \end{cases}$$

The above equations show that there exists a relation between $\text{cl}_i(X, L)$ and $\text{cl}_{2m-i}(X, L)$. Here we note that for every integer i with $0 \leq i \leq m$, $\text{cl}_i(X, L)$ can be written by the Segre classes $s_1(\mathcal{E}), \dots, s_m(\mathcal{E})$.

DEFINITION 1.1. For every integer i with $0 \leq i \leq m$, we define the polynomial $F_i(t_1, \dots, t_m) \in \mathbb{Z}[t_1, \dots, t_m]$ such that the following equality holds.

$$F_i(s_1(\mathcal{E}), \dots, s_m(\mathcal{E})) = \text{cl}_i(X, L).$$

Then we see from the above that if $m = 1, 2$ and 3 , then

$$\text{cl}_j(X, L) = F_{2m-j}(c_1(\mathcal{E}), \dots, c_m(\mathcal{E}))$$

for $m \leq j \leq 2m$. In general, we can prove the following theorem, which was proposed in [4] and is the main result of this paper.

THEOREM 1.1. *Let a polarized manifold (X, L) be a classical scroll over a smooth projective variety Y with $\dim X = n$ and $\dim Y = m$. Let \mathcal{E} be an ample vector bundle on Y such that $X \cong \mathbb{P}_Y(\mathcal{E})$ and $L = H(\mathcal{E})$. Let $F_i(t_1, \dots, t_m)$ be the polynomial defined in Definition 1.1 for every integer i with $0 \leq i \leq m$. Assume that $n \geq 2m$. Then for any integer j with $m \leq j \leq 2m$ we have*

$$\text{cl}_j(X, L) = F_{2m-j}(c_1(\mathcal{E}), \dots, c_m(\mathcal{E})).$$

In particular

$$F_m(s_1(\mathcal{E}), \dots, s_m(\mathcal{E})) = F_m(c_1(\mathcal{E}), \dots, c_m(\mathcal{E})).$$

By Theorem 1.1 we can easily calculate $\text{cl}_{2m-i}(X, L)$ (resp. $\text{cl}_i(X, L)$) if we are able to calculate $\text{cl}_i(X, L)$ (resp. $\text{cl}_{2m-i}(X, L)$). By this relation we expect that we can get some useful information about $\text{cl}_{2m-i}(X, L)$ (resp. $\text{cl}_i(X, L)$) from several properties of $\text{cl}_i(X, L)$ (resp. $\text{cl}_{2m-i}(X, L)$). Moreover if $i = m$, then we have $\text{cl}_m(X, L) = F_m(c_1(\mathcal{E}), \dots, c_m(\mathcal{E})) = F_m(s_1(\mathcal{E}), \dots, s_m(\mathcal{E}))$ by Theorem 1.1. So $\text{cl}_m(X, L)$ may have special and interesting properties. We will study these on another occasion.

2. Preliminaries

NOTATION 2.1. For a real number m and a non-negative integer n , let

$$[m]_n := \begin{cases} m(m-1) \cdots (m-n+1) & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

For any non-negative integer n ,

$$n! := \begin{cases} [n]_n & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases}$$

Assume that m and n are integers. Then we put

$$\binom{m}{n} := \begin{cases} \frac{[m]_n}{n!} & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases}$$

We note that $\binom{m}{n} = 0$ if $0 \leq m < n$ or $n < 0$, and $\binom{m}{0} = 1$.

DEFINITION 2.1. (See [1, Definition 3.1].) Let (X, L) be a polarized manifold of dimension n , and i an integer with $0 \leq i \leq n$. Then the i th sectional Euler number $e_i(X, L)$ of (X, L) is defined by the following:

$$e_i(X, L) := \sum_{l=0}^i (-1)^l \binom{n-i+l-1}{l} c_{i-l}(X) L^{n-i+l}.$$

DEFINITION 2.2. (See [3, Definitions 2.8 and 2.9]. See also [3, Remark 2.6].) Let (X, L) be a polarized manifold of dimension n and i an integer with $0 \leq i \leq n$. Then the i th sectional class of (X, L) is defined by the following.

$$\text{cl}_i(X, L) = \begin{cases} e_0(X, L), & \text{if } i = 0, \\ (-1)\{e_1(X, L) - 2e_0(X, L)\}, & \text{if } i = 1, \\ (-1)^i\{e_i(X, L) - 2e_{i-1}(X, L) + e_{i-2}(X, L)\}, & \text{if } 2 \leq i \leq n. \end{cases}$$

DEFINITION 2.3. Let Y be a smooth projective variety of dimension m and \mathcal{E} a vector bundle of rank r on Y .

- (i) The Chern polynomial $c_t(\mathcal{E})$ is defined by $c_t(\mathcal{E}) = \sum_{i \geq 0} c_i(\mathcal{E}) t^i$.
- (ii) For every integer j with $j \geq 0$, the j th Segre class $s_j(\mathcal{F})$ of \mathcal{F} is defined by the following equation: $c_t(\mathcal{F}^\vee) s_t(\mathcal{F}) = 1$, where $c_t(\mathcal{F}^\vee)$ is the Chern polynomial of \mathcal{F}^\vee and $s_t(\mathcal{F}) = \sum_{j \geq 0} s_j(\mathcal{F}) t^j$.

REMARK 2.1.

- (i) Let Y be a smooth projective variety and \mathcal{F} a vector bundle on X . Let $\tilde{s}_j(\mathcal{F})$ be the Segre class which is defined in [5, Chapter 3]. Then $s_j(\mathcal{F}) = \tilde{s}_j(\mathcal{F}^\vee)$.
- (ii) For every integer i with $1 \leq i$, $s_i(\mathcal{F})$ can be written by using the Chern classes $c_j(\mathcal{F})$ with $1 \leq j \leq i$. (For example, $s_1(\mathcal{F}) = c_1(\mathcal{F})$, $s_2(\mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, and so on.)

NOTATION 2.2. Let (X, L) be an n -dimensional classical scroll over a smooth projective variety Y of dimension m . Let \mathcal{E} be an ample vector bundle of rank r on Y such that $X = \mathbb{P}_Y(\mathcal{E})$ and $L = H(\mathcal{E})$. Let $p : X \rightarrow Y$ be the projection. Then $n = m + r - 1$. In this paper we assume that $r \geq m + 1$, that is, $n \geq 2m$.

PROPOSITION 2.1. *Let (X, L) be a classical scroll over a smooth projective variety Y of dimension m . We use notations in Notation 2.2. Then for every integer i with $0 \leq i \leq n$ the following holds.*

$$e_i(X, L) = \sum_{t=0}^i \sum_{k=0}^{i-t} (-1)^{i-t} \binom{m-t-2}{i-t-k} c_k(\mathcal{E}) c_t(Y) s_{m-k-t}(\mathcal{E}).$$

Proof. See the first part of the proof in [2, Theorem 3.1]. □

3. Main result

DEFINITION 3.1. Let Y be a smooth projective variety of dimension m and \mathcal{E} a vector bundle on Y . Then for every integer i with $0 \leq i \leq m$ we define the polynomial $t_i(x_0, \dots, x_i) \in \mathbb{Z}[x_0, \dots, x_i]$ which satisfies the following.

$$(4) \quad c_i(\mathcal{E}) = t_i(s_0(\mathcal{E}), \dots, s_i(\mathcal{E})).$$

For example, we see that $t_0(x_0) = 1$, $t_1(x_0, x_1) = x_1$, $t_2(x_0, x_1, x_2) = x_1^2 - x_2$ and so on.

PROPOSITION 3.1. *Let Y be a smooth projective variety of dimension m and \mathcal{E} a vector bundle over Y . For every integer i with $0 \leq i \leq m$, we have $s_i(\mathcal{E}) = t_i(c_0(\mathcal{E}), \dots, c_i(\mathcal{E}))$.*

Proof. We prove this by induction.

(I) If $i = 0$, then this is true because $c_0(\mathcal{E}) = s_0(\mathcal{E}) = 1$.

(II) Assume that the assertion holds for every i with $i \leq k - 1$. So we consider the case $i = k$. Then by Definition 2.3 (ii)

$$(5) \quad \sum_{\substack{i+j=k \\ i \geq 0, j \geq 0}} (-1)^i c_i(\mathcal{E}) s_j(\mathcal{E}) = 0.$$

Hence by (5) we have

$$\begin{aligned} t_k(s_0(\mathcal{E}), \dots, s_k(\mathcal{E})) &= c_k(\mathcal{E}) \\ &= (-1)^{k+1} \sum_{\substack{i+j=k \\ j \geq 1}} (-1)^i c_i(\mathcal{E}) s_j(\mathcal{E}) \end{aligned}$$

$$= (-1)^{k+1} \sum_{\substack{i+j=k \\ j \geq 1}} (-1)^i t_i(s_0(\mathcal{E}), \dots, s_i(\mathcal{E})) s_j(\mathcal{E}).$$

In particular, we have

$$(6) \quad t_k(x_0, \dots, x_k) = (-1)^{k+1} \sum_{\substack{i+j=k \\ j \geq 1}} (-1)^i t_i(x_0, \dots, x_i) x_j.$$

On the other hand, we see from the induction hypothesis and (6) that

$$\begin{aligned} s_k(\mathcal{E}) &= - \sum_{\substack{i+j=k \\ i \geq 1}} (-1)^i c_i(\mathcal{E}) s_j(\mathcal{E}) \\ &= (-1)^{k+1} \sum_{\substack{i+j=k \\ i \geq 1}} (-1)^j c_i(\mathcal{E}) t_j(c_0(\mathcal{E}), \dots, c_j(\mathcal{E})) \\ &= t_k(c_0(\mathcal{E}), \dots, c_k(\mathcal{E})). \end{aligned}$$

So we get the assertion. \square

The following theorem which is Theorem 1.1 in Introduction is the main result of this note.

THEOREM 3.1. *Let (X, L) be an n -dimensional classical scroll over a smooth projective variety Y of dimension m such that $n \geq 2m$. Let $F_i(t_1, \dots, t_m)$ be the polynomial defined in Definition 1.1 for every integer i with $0 \leq i \leq m$. We use notations in Notation 2.2. Then for any integer j with $m \leq j \leq 2m$ we have*

$$\text{cl}_j(X, L) = F_{2m-j}(c_1(\mathcal{E}), \dots, c_m(\mathcal{E})).$$

In particular

$$F_m(s_1(\mathcal{E}), \dots, s_m(\mathcal{E})) = F_m(c_1(\mathcal{E}), \dots, c_m(\mathcal{E})).$$

Proof. First we prove the following.

CLAIM 3.1 *For any integer i with $0 \leq i \leq m$, we have*

$$\begin{aligned} &e_{2m-2-i}(X, L) \\ &= \sum_{t=0}^i \sum_{l=0}^{i-t} (-1)^{i-t} \binom{m-t-2}{i-t-l} c_{m-t-l}(\mathcal{E}^\vee) c_t(Y) s_l(\mathcal{E}) + (m-i-1)c_m(Y), \\ &e_{2m-1-i}(X, L) \\ &= \sum_{t=0}^{i-1} \sum_{l=0}^{i-1-t} (-1)^{i-1-t} \binom{m-t-2}{i-1-t-l} c_{m-t-l}(\mathcal{E}^\vee) c_t(Y) s_l(\mathcal{E}) + (m-i)c_m(Y), \end{aligned}$$

$$e_{2m-i}(X, L) = \sum_{t=0}^{i-2} \sum_{l=0}^{i-2-t} (-1)^{i-2-t} \binom{m-t-2}{i-2-t-l} c_{m-t-l}(\mathcal{E}^\vee) c_t(Y) s_l(\mathcal{E}) + (m-i+1)c_m(Y).$$

(We note that if $i = 0$ (resp. $i = 0, 1$), then $\sum_{t=0}^{i-1} \sum_{l=0}^{i-1-t} (-1)^{i-1-t} \binom{m-t-2}{i-1-t-l} c_{m-t-l}(\mathcal{E}^\vee) c_t(Y) s_l(\mathcal{E}) = 0$ (resp. $\sum_{t=0}^{i-2} \sum_{l=0}^{i-2-t} (-1)^{i-2-t} \binom{m-t-2}{i-2-t-l} c_{m-t-l}(\mathcal{E}^\vee) c_t(Y) s_l(\mathcal{E}) = 0$).

Proof. (A) First we treat $e_{2m-2-i}(X, L)$. Then by Proposition 2.1

$$e_{2m-2-i}(X, L) = \sum_{t=0}^{2m-2-i} \left(\sum_{k=0}^{2m-2-i-t} (-1)^{2m-2-i-t-k} \binom{m-t-2}{2m-2-i-t-k} c_k(\mathcal{E}^\vee) c_t(Y) s_{m-k-t}(\mathcal{E}) \right).$$

Here we note that

$$(7) \quad 2m-2-i-t \geq k.$$

We set

$$E(i, k, t) = (-1)^{2m-2-i-t-k} \binom{m-t-2}{2m-2-i-t-k} c_k(\mathcal{E}^\vee) c_t(Y) s_{m-k-t}(\mathcal{E}).$$

If $E(i, k, t) \neq 0$, then the following two conditions hold by noting (7).

$$(8) \quad 0 \leq k \leq m.$$

$$(9) \quad 0 \leq t \leq m.$$

$$(10) \quad k+t \leq \min\{m, 2m-2-i\}.$$

If $m-t-2 > 0$ and $m-t-2 < 2m-2-i-t-k$, then $\binom{m-t-2}{2m-2-i-t-k} = 0$. Hence if $E(i, k, t) \neq 0$, then $m-t-2 \leq 0$ or $m-t-2 \geq 2m-2-i-t-k$, that is,

$$(11) \quad t \geq m-2 \quad \text{or} \quad k \geq m-i.$$

(A.1) The case where $0 \leq i \leq m-3$.

We see from (8), (9), (10) and (11) that $e_{2m-2-i}(X, L)$ is the sum of $E(i, k, t)$ in the range of the following (k, t) .

$$(A.1.1) \left\{ \begin{array}{l} k = 0, 1, 2, \quad t = m-2, \\ k = 0, 1, \quad t = m-1, \\ k = 0, \quad t = m. \end{array} \right. \quad (A.1.2) \left\{ \begin{array}{l} k = m-i, \quad t = i, i-1, \dots, 1, 0 \\ k = m-i+1, \quad t = i-1, \dots, 1, 0 \\ \vdots \\ k = m, \quad t = 0. \end{array} \right.$$

The sum of $E(i, k, t)$ in the range of the case (A.1.1) is the following.

$$\begin{aligned}
(12) \quad & \sum_{t=m-2}^m \sum_{k=0}^{m-t} E(i, k, t) \\
&= (-1)^{2m-2-i-m+2-0} \binom{m-(m-2)-2}{2m-2-i-(m-2)-0} c_{m-2}(Y) s_2(\mathcal{E}) \\
&\quad + (-1)^{2m-2-i-m+2-1} \binom{m-(m-2)-2}{2m-2-i-(m-2)-1} c_1(\mathcal{E}^\vee) c_{m-2}(Y) s_1(\mathcal{E}) \\
&\quad + (-1)^{2m-2-i-m+2-2} \binom{m-(m-2)-2}{2m-2-i-(m-2)-2} c_2(\mathcal{E}^\vee) c_{m-2}(Y) \\
&\quad + (-1)^{2m-2-i-m+1-0} \binom{m-(m-1)-2}{2m-2-i-(m-1)-0} c_{m-1}(Y) s_1(\mathcal{E}) \\
&\quad + (-1)^{2m-2-i-m+1-1} \binom{m-(m-1)-2}{2m-2-i-(m-1)-1} c_1(\mathcal{E}^\vee) c_{m-1}(Y) \\
&\quad + (-1)^{2m-2-i-m} \binom{m-m-2}{2m-2-i-m-0} c_m(Y) \\
&= (-1)^{m-i} \binom{0}{m-i} c_{m-2}(Y) s_2(\mathcal{E}) \\
&\quad + (-1)^{m-i-1} \binom{0}{m-i-1} c_1(\mathcal{E}^\vee) c_{m-2}(Y) s_1(\mathcal{E}) \\
&\quad + (-1)^{m-i-2} \binom{0}{m-i-2} c_2(\mathcal{E}^\vee) c_{m-2}(Y) \\
&\quad + (-1)^{m-i-1} \binom{-1}{m-1-i} c_{m-1}(Y) s_1(\mathcal{E}) \\
&\quad + (-1)^{m-i-2} \binom{-1}{m-2-i} c_1(\mathcal{E}^\vee) c_{m-1}(Y) \\
&\quad + (-1)^{m-i-2} \binom{-2}{m-i-2} c_m(Y) \\
&= c_{m-1}(Y) s_1(\mathcal{E}) + c_1(\mathcal{E}^\vee) c_{m-1}(Y) + (m-i-1) c_m(Y) \\
&= (m-1-i) c_m(Y).
\end{aligned}$$

On the other hand, the sum of $E(i, k, t)$ in the range of the case (A.1.2) is the following.

$$\begin{aligned}
& \sum_{k=m-i}^m \sum_{t=0}^{m-k} (-1)^{2m-2-i-t-k} \binom{m-t-2}{2m-2-i-t-k} c_k(\mathcal{E}^\vee) c_t(Y) s_{m-k-t}(\mathcal{E}) \\
&= \sum_{t=0}^i \sum_{k=m-i}^{m-t} (-1)^{2m-2-i-t-k} \binom{m-t-2}{2m-2-i-t-k} c_k(\mathcal{E}^\vee) c_t(Y) s_{m-k-t}(\mathcal{E}).
\end{aligned}$$

Here we put $j := k - (m - i)$. Then by $t \leq i \leq m - 3$ we have

$$\begin{aligned}
& \sum_{t=0}^i \sum_{k=m-i}^{m-t} (-1)^{2m-2-i-t-k} \binom{m-t-2}{2m-2-i-t-k} c_k(\mathcal{E}^\vee) c_t(Y) s_{m-k-t}(\mathcal{E}) \\
&= \sum_{t=0}^i \sum_{j=0}^{i-t} (-1)^{m-2-t-j} \binom{m-t-2}{m-2-t-j} c_{j+m-i}(\mathcal{E}^\vee) c_t(Y) s_{i-j-t}(\mathcal{E}) \\
&= \sum_{t=0}^i \sum_{j=0}^{i-t} (-1)^{m-2-t-j} \binom{m-t-2}{j} c_{j+m-i}(\mathcal{E}^\vee) c_t(Y) s_{i-j-t}(\mathcal{E}).
\end{aligned}$$

Hence we have

$$\begin{aligned}
(13) \quad & e_{2m-2-i}(X, L) \\
&= \sum_{t=0}^i \sum_{j=0}^{i-t} (-1)^{m-2-t-j} \binom{m-t-2}{j} c_{j+m-i}(\mathcal{E}^\vee) c_t(Y) s_{i-j-t}(\mathcal{E}) \\
&\quad + (m-1-i) c_m(Y).
\end{aligned}$$

(A.2) The case where $i = m - 2$.

We see from (8), (9), (10) and (11) that $e_{2m-2-i}(X, L)$ is the sum of $E(m-2, k, t)$ in the range of the following (k, t) .

$$(A.2.1) \left\{ \begin{array}{l} k = 0, 1, \quad t = m - 2, \\ k = 0, 1, \quad t = m - 1, \\ k = 0, \quad t = m. \end{array} \right. \quad (A.2.2) \left\{ \begin{array}{l} k = 2, \quad t = m - 2, m - 3, \dots, 1, 0 \\ k = 3, \quad t = m - 3, \dots, 1, 0 \\ \vdots \\ k = m, \quad t = 0. \end{array} \right.$$

First we calculate the sum of $E(m-2, k, t)$ in the range of the case (A.2.1).

$$\begin{aligned}
(14) \quad & \sum_{k=0}^1 E(m-2, k, m-2) + \sum_{t=m-1}^m \sum_{k=0}^{m-t} E(m-2, k, t) \\
&= (-1)^{2m-2-(m-2)-m+2-0} \binom{m-(m-2)-2}{2m-2-(m-2)-(m-2)-0} c_{m-2}(Y) s_2(\mathcal{E}) \\
&\quad + (-1)^{2m-2-(m-2)-m+2-1} \binom{m-(m-2)-2}{2m-2-(m-2)-(m-2)-1} c_1(\mathcal{E}^\vee) c_{m-2}(Y) s_1(\mathcal{E}) \\
&\quad + (-1)^{2m-2-(m-2)-m+1-0} \binom{m-(m-1)-2}{2m-2-(m-2)-(m-1)-0} c_{m-1}(Y) s_1(\mathcal{E})
\end{aligned}$$

$$\begin{aligned}
& + (-1)^{2m-2-(m-2)-m+1-1} \binom{m-(m-1)-2}{2m-2-(m-2)-(m-1)-1} c_1(\mathcal{E}^\vee) c_{m-1}(Y) \\
& + (-1)^{2m-2-(m-2)-m} \binom{m-m-2}{2m-2-(m-2)-m-0} c_m(Y) \\
& = (-1)^2 \binom{0}{2} c_{m-2}(Y) s_2(\mathcal{E}) + (-1)^1 \binom{0}{1} c_1(\mathcal{E}^\vee) c_{m-2}(Y) s_1(\mathcal{E}) \\
& + (-1)^1 \binom{-1}{1} c_{m-1}(Y) s_1(\mathcal{E}) + (-1)^0 \binom{-1}{0} c_1(\mathcal{E}^\vee) c_{m-1}(Y) + (-1)^0 \binom{-2}{0} c_m(Y) \\
& = c_{m-1}(Y) s_1(\mathcal{E}) + c_1(\mathcal{E}^\vee) c_{m-1}(Y) + c_m(Y) \\
& = c_m(Y).
\end{aligned}$$

Next we calculate the sum of $E(m-2, k, t)$ in the range of the case (A.2.2).

$$\begin{aligned}
(15) \quad & \sum_{k=2}^m \sum_{t=0}^{m-k} E(m-2, k, t) \\
& = \sum_{t=0}^{m-2} \sum_{k=2}^{m-t} (-1)^{2m-2-i-t-k} \binom{m-t-2}{2m-2-i-t-k} c_k(\mathcal{E}^\vee) c_t(Y) s_{m-k-t}(\mathcal{E}) \\
& = \sum_{t=0}^{m-2} \sum_{j=0}^{m-2-t} (-1)^{m-2-t-j} \binom{m-t-2}{m-2-t-j} c_{j+2}(\mathcal{E}^\vee) c_t(Y) s_{(m-2)-j-t}(\mathcal{E}) \\
& = \sum_{t=0}^{m-2} \sum_{j=0}^{m-2-t} (-1)^{m-2-t-j} \binom{m-t-2}{j} c_{j+2}(\mathcal{E}^\vee) c_t(Y) s_{(m-2)-j-t}(\mathcal{E}).
\end{aligned}$$

(Here we set $j := k - 2$ in the above equations.)

Since $i = m - 2$, we see from (14) and (15) that

$$\begin{aligned}
(16) \quad & e_{2m-2-i}(X, L) \\
& = \sum_{t=0}^i \sum_{j=0}^{i-t} (-1)^{m-2-t-j} \binom{m-t-2}{j} c_{j+m-i}(\mathcal{E}^\vee) c_t(Y) s_{i-j-t}(\mathcal{E}) \\
& + (m-1-i) c_m(Y).
\end{aligned}$$

(A.3) The case where $i = m - 1$.

We see from (8), (9), (10) and (11) that $e_{2m-2-i}(X, L)$ is the sum of $E(m-1, k, t)$ in the range of the following (k, t) .

$$(A.3.1) \quad \begin{cases} k = 0, & t = m - 2, \\ k = 0, & t = m - 1. \end{cases} \quad (A.3.2) \quad \begin{cases} k = 1, & t = m - 2, m - 3, \dots, 1, 0 \\ k = 2, & t = m - 3, \dots, 1, 0 \\ \vdots \\ k = m - 1, & t = 0. \end{cases}$$

The sum of $E(m-1, k, t)$ in the range of the case (A.3.1) is obtained as follows.

$$\begin{aligned}
(17) \quad & \sum_{t=m-2}^{m-1} E(m-1, 0, t) \\
&= (-1)^{2m-2-(m-1)-m+2-0} \binom{m-(m-2)-2}{2m-2-(m-1)-(m-2)-2} c_2(\mathcal{E}^\vee) c_{m-2}(Y) \\
&\quad + (-1)^{2m-2-(m-1)-m+1-0} \binom{m-(m-1)-2}{2m-2-(m-1)-(m-1)-0} c_{m-1}(Y) s_1(\mathcal{E}) \\
&= (-1)^1 \binom{0}{-1} c_1(\mathcal{E}^\vee) c_{m-2}(Y) s_1(\mathcal{E}) \\
&\quad + (-1)^0 \binom{-1}{0} c_{m-1}(Y) s_1(\mathcal{E}) \\
&= c_{m-1}(Y) s_1(\mathcal{E}).
\end{aligned}$$

On the other hand, we get the sum of $E(m-1, k, t)$ in the range of the case (A.3.2) as follows.

$$\begin{aligned}
(18) \quad & \sum_{k=1}^{m-1} \sum_{t=0}^{m-1-k} E(m-1, k, t) \\
&= \sum_{t=0}^{m-2} \sum_{k=1}^{m-1-t} (-1)^{2m-2-(m-1)-t-k} \binom{m-t-2}{2m-2-(m-1)-t-k} c_k(\mathcal{E}^\vee) c_t(Y) s_{m-k-t}(\mathcal{E}) \\
&= \sum_{t=0}^{m-2} \sum_{j=0}^{m-2-t} (-1)^{m-2-t-j} \binom{m-t-2}{m-2-t-j} c_{j+1}(\mathcal{E}^\vee) c_t(Y) s_{(m-1)-j-t}(\mathcal{E}) \\
&= \sum_{t=0}^{m-1} \sum_{j=0}^{m-1-t} (-1)^{m-2-t-j} \binom{m-t-2}{m-2-t-j} c_{j+1}(\mathcal{E}^\vee) c_t(Y) s_{(m-1)-j-t}(\mathcal{E}).
\end{aligned}$$

We note that in the final step of the above equalities we use $\binom{m-t-2}{m-2-t-j} = 0$ for $(t, j) = (0, m-1), (1, m-2), \dots, (m-1, 0)$. Moreover

$$\begin{aligned}
(19) \quad & \sum_{t=0}^{m-1} \sum_{j=0}^{m-1-t} (-1)^{m-2-t-j} \binom{m-t-2}{m-2-t-j} c_{j+1}(\mathcal{E}^\vee) c_t(Y) s_{(m-1)-j-t}(\mathcal{E}) \\
&= \sum_{t=0}^i \sum_{j=0}^{i-t} (-1)^{m-2-t-j} \binom{m-t-2}{j} c_{j+m-i}(\mathcal{E}^\vee) c_t(Y) s_{i-j-t}(\mathcal{E}) - c_1(\mathcal{E}) c_{m-1}(Y).
\end{aligned}$$

We also note that in the final step of the above equalities

$$\binom{m-t-2}{m-2-t-j} = \begin{cases} \binom{m-t-2}{j} & \text{if } t \leq m-2, \\ \binom{m-t-2}{j} - 1 & \text{if } t = m-1. \end{cases}$$

(Here we note that if $t = m-1$, then $j = 0$ in this case.)

Hence we see from (17), (18) and (19) that for $i = m-1$

$$(20) \quad \begin{aligned} & e_{2m-2-i}(X, L) \\ &= \sum_{t=0}^i \sum_{j=0}^{i-t} (-1)^{m-2-t-j} \binom{m-t-2}{j} c_{j+m-i}(\mathcal{E}^\vee) c_t(Y) s_{i-j-t}(\mathcal{E}) \\ & \quad + (m-1-i) c_m(Y). \end{aligned}$$

(A.4) The case where $i = m$.

We see from (8), (9), (10) and (11) that $e_{2m-2-i}(X, L)$ is the sum of $E(m, k, t)$ in the range of the following (k, t) .

$$\begin{cases} k = 0, & t = m-2, m-3, \dots, 1, 0, \\ k = 1, & t = m-3, \dots, 1, 0, \\ & \vdots \\ k = m-2, & t = 0. \end{cases}$$

By the same argument as above we get

$$(21) \quad \begin{aligned} & e_{2m-2-i}(X, L) \\ &= \sum_{k=0}^{m-2} \sum_{t=0}^{m-2-k} (-1)^{2m-2-m-t-k} \binom{m-t-2}{2m-2-m-t-k} c_k(\mathcal{E}^\vee) c_t(Y) s_{m-k-t}(\mathcal{E}) \\ &= \sum_{t=0}^{m-2} \sum_{j=0}^{m-2-t} (-1)^{m-2-t-j} \binom{m-t-2}{m-2-t-j} c_j(\mathcal{E}^\vee) c_t(Y) s_{m-j-t}(\mathcal{E}) \\ &= \sum_{t=0}^m \sum_{j=0}^{m-t} (-1)^{m-2-t-j} \binom{m-t-2}{m-2-t-j} c_j(\mathcal{E}^\vee) c_t(Y) s_{m-j-t}(\mathcal{E}). \end{aligned}$$

We note that in the final step of the above equalities we use $\binom{m-t-2}{m-2-t-j} = 0$ for $(t, j) = (0, m), (1, m-1), \dots, (m, 0), (0, m-1), (1, m-2), \dots, (m-1, 0)$.

On the other hand

$$(22) \quad \binom{m-t-2}{m-2-t-j} = \begin{cases} \binom{m-t-2}{j} & \text{if } t \leq m-2, \\ \binom{m-t-2}{j} - (-1)^j & \text{if } t = m-1, m. \end{cases}$$

(Here we note that if $t = m - 1$ (resp. $t = m$), then $j = 0, 1$ (resp. $j = 0$) in this case.)

Hence we see from (21) and (22) that for $i = m$

$$\begin{aligned}
(23) \quad & e_{2m-2-i}(X, L) \\
&= \sum_{t=0}^i \sum_{j=0}^{i-t} (-1)^{m-2-t-j} \binom{m-t-2}{m-2-t-j} c_{j+m-i}(\mathcal{E}^\vee) c_t(Y) s_{i-j-t}(\mathcal{E}) \\
&= \sum_{t=0}^i \sum_{j=0}^{i-t} (-1)^{m-2-t-j} \binom{m-t-2}{j} c_{j+m-i}(\mathcal{E}^\vee) c_t(Y) s_{i-j-t}(\mathcal{E}) \\
&\quad + c_{m-1}(Y) s_1(\mathcal{E}) + c_1(\mathcal{E}^\vee) c_{m-1}(Y) - c_m(Y) \\
&= \sum_{t=0}^i \sum_{j=0}^{i-t} (-1)^{m-2-t-j} \binom{m-t-2}{j} c_{j+m-i}(\mathcal{E}^\vee) c_t(Y) s_{i-j-t}(\mathcal{E}) \\
&\quad + (m-i-1) c_m(Y).
\end{aligned}$$

By (13), (16), (20) and (23) for any i with $0 \leq i \leq m$ we have

$$\begin{aligned}
e_{2m-2-i}(X, L) &= \sum_{t=0}^i \sum_{j=0}^{i-t} (-1)^{m-2-t-j} \binom{m-t-2}{j} c_{j+m-i}(\mathcal{E}^\vee) c_t(Y) s_{i-j-t}(\mathcal{E}) \\
&\quad + (m-i-1) c_m(Y).
\end{aligned}$$

Furthermore we set $l := i - t - j$. Then

$$\begin{aligned}
& \sum_{t=0}^i \sum_{j=0}^{i-t} (-1)^{m-2-t-j} \binom{m-t-2}{j} c_{j+m-i}(\mathcal{E}^\vee) c_t(Y) s_{i-j-t}(\mathcal{E}) \\
&= \sum_{t=0}^i \sum_{l=0}^{i-t} (-1)^{m-2-i+l} \binom{m-t-2}{i-t-l} c_{m-t-l}(\mathcal{E}^\vee) c_t(Y) s_l(\mathcal{E}) \\
&= \sum_{t=0}^i \sum_{l=0}^{i-t} (-1)^{i-t} \binom{m-t-2}{i-t-l} c_{m-t-l}(\mathcal{E}) c_t(Y) s_l(\mathcal{E}).
\end{aligned}$$

Hence for every integer i with $0 \leq i \leq m$

$$\begin{aligned}
(24) \quad & e_{2m-2-i}(X, L) \\
&= \sum_{t=0}^i \sum_{l=0}^{i-t} (-1)^{i-t} \binom{m-t-2}{i-t-l} c_{m-t-l}(\mathcal{E}) c_t(Y) s_l(\mathcal{E}) + (m-i-1) c_m(Y).
\end{aligned}$$

(B) Next we consider $e_{2m-1}(X, L)$ and $e_{2m}(X, L)$. Then by [2, Theorem 3.1 (3.1.1)] we have

$$(25) \quad e_{2m-1}(X, L) = m c_m(Y),$$

$$(26) \quad e_{2m}(X, L) = (m+1) c_m(Y).$$

(C) By (24), (25) and (26), we get the assertion of Claim 3.1. \square

Here we set

$$\begin{aligned} & E_i(c_0(\mathcal{E}), \dots, c_i(\mathcal{E}); s_{m-i}(\mathcal{E}), \dots, s_m(\mathcal{E})) \\ & := \sum_{t=0}^i \sum_{k=0}^{i-t} (-1)^{i-t} \binom{m-t-2}{i-t-k} c_k(\mathcal{E}) c_t(Y) s_{m-k-t}(\mathcal{E}). \end{aligned}$$

Then by Proposition 2.1 we have

$$(27) \quad E_i(c_0(\mathcal{E}), \dots, c_i(\mathcal{E}); s_{m-i}(\mathcal{E}), \dots, s_m(\mathcal{E})) = e_i(X, L).$$

Moreover by Claim 3.1 we have

$$(28) \quad \begin{aligned} & e_{2m-2-i}(X, L) \\ & = E_i(s_0(\mathcal{E}), \dots, s_i(\mathcal{E}); c_{m-i}(\mathcal{E}), \dots, c_m(\mathcal{E})) + (m-i-1)c_m(Y), \end{aligned}$$

$$(29) \quad \begin{aligned} & e_{2m-1-i}(X, L) \\ & = E_{i-1}(s_0(\mathcal{E}), \dots, s_{i-1}(\mathcal{E}); c_{m-i+1}(\mathcal{E}), \dots, c_m(\mathcal{E})) + (m-i)c_m(Y), \end{aligned}$$

$$(30) \quad \begin{aligned} & e_{2m-i}(X, L) \\ & = E_{i-2}(s_0(\mathcal{E}), \dots, s_{i-2}(\mathcal{E}); c_{m-i+2}(\mathcal{E}), \dots, c_m(\mathcal{E})) + (m-i+1)c_m(Y) \end{aligned}$$

for every integer i with $0 \leq i \leq m$. By (27) and Definitions 1.1 and 2.2 we get

$$(31) \quad \begin{aligned} & F_i(s_1(\mathcal{E}), \dots, s_m(\mathcal{E})) \\ & = \text{cl}_i(X, L) \\ & = (-1)^i \{ E_i(c_0(\mathcal{E}), \dots, c_i(\mathcal{E}); s_{m-i}(\mathcal{E}), \dots, s_m(\mathcal{E})) \\ & \quad - 2E_{i-1}(c_0(\mathcal{E}), \dots, c_{i-1}(\mathcal{E}); s_{m-i+1}(\mathcal{E}), \dots, s_m(\mathcal{E})) \\ & \quad + E_{i-2}(c_0(\mathcal{E}), \dots, c_{i-2}(\mathcal{E}); s_{m-i+2}(\mathcal{E}), \dots, s_m(\mathcal{E})) \} \\ & = (-1)^i \{ E_i(t_0(s_0(\mathcal{E})), \dots, t_i(s_0(\mathcal{E}), \dots, s_i(\mathcal{E})); s_{m-i}(\mathcal{E}), \dots, s_m(\mathcal{E})) \\ & \quad - 2E_{i-1}(t_0(s_0(\mathcal{E})), \dots, t_{i-1}(s_0(\mathcal{E}), \dots, s_{i-1}(\mathcal{E})); s_{m-i+1}(\mathcal{E}), \dots, s_m(\mathcal{E})) \\ & \quad + E_{i-2}(t_0(s_0(\mathcal{E})), \dots, t_{i-2}(s_0(\mathcal{E}), \dots, s_{i-2}(\mathcal{E})); s_{m-i+2}(\mathcal{E}), \dots, s_m(\mathcal{E})) \} \end{aligned}$$

for every integer i with $0 \leq i \leq m$. Here $t_i(x_0, \dots, x_i)$ denotes the polynomial which was defined in Definition 3.1.

On the other hand we see from Proposition 3.1, (28), (29), (30) and (31) that for every integer i with $0 \leq i \leq m$

$$\begin{aligned} & \text{cl}_{2m-i}(X, L) \\ & = (-1)^{2m-i} \{ e_{2m-i}(X, L) - 2e_{2m-i-1}(X, L) + e_{2m-i-2}(X, L) \} \end{aligned}$$

$$\begin{aligned}
&= (-1)^i \{ E_{i-2}(s_0(\mathcal{E}), \dots, s_{i-2}(\mathcal{E}); c_{m-i+2}(\mathcal{E}), \dots, c_m(\mathcal{E})) \\
&\quad - 2E_{i-1}(s_0(\mathcal{E}), \dots, s_{i-1}(\mathcal{E}); c_{m-i+1}(\mathcal{E}), \dots, c_m(\mathcal{E})) \\
&\quad + E_i(s_0(\mathcal{E}), \dots, s_i(\mathcal{E}); c_{m-i}(\mathcal{E}), \dots, c_m(\mathcal{E})) \\
&\quad + (m-i+1)c_m(Y) - 2(m-i)c_m(Y) + (m-i-1)c_m(Y) \} \\
&= (-1)^i \{ E_i(s_0(\mathcal{E}), \dots, s_i(\mathcal{E}); c_{m-i}(\mathcal{E}), \dots, c_m(\mathcal{E})) \\
&\quad - 2E_{i-1}(s_0(\mathcal{E}), \dots, s_{i-1}(\mathcal{E}); c_{m-i+1}(\mathcal{E}), \dots, c_m(\mathcal{E})) \\
&\quad + E_{i-2}(s_0(\mathcal{E}), \dots, s_{i-2}(\mathcal{E}); c_{m-i+2}(\mathcal{E}), \dots, c_m(\mathcal{E})) \} \\
&= (-1)^i \{ E_i(t_0(c_0(\mathcal{E})), \dots, t_i(c_0(\mathcal{E}), \dots, c_i(\mathcal{E})); c_{m-i}(\mathcal{E}), \dots, c_m(\mathcal{E})) \\
&\quad - 2E_{i-1}(t_0(c_0(\mathcal{E})), \dots, t_{i-1}(c_0(\mathcal{E}), \dots, c_{i-1}(\mathcal{E})); c_{m-i+1}(\mathcal{E}), \dots, c_m(\mathcal{E})) \\
&\quad + E_{i-2}(t_0(c_0(\mathcal{E})), \dots, t_{i-2}(c_0(\mathcal{E}), \dots, c_{i-2}(\mathcal{E})); c_{m-i+2}(\mathcal{E}), \dots, c_m(\mathcal{E})) \} \\
&= F_i(c_1(\mathcal{E}), \dots, c_m(\mathcal{E})).
\end{aligned}$$

Therefore we get the assertion of Theorem 3.1. \square

Finally we note the following.

PROPOSITION 3.2. *Let (X, L) be an n -dimensional classical scroll over a smooth projective variety Y of dimension m such that $n \geq 2m + 1$. Then $\text{cl}_i(X, L) = 0$ for every integer i with $2m + 1 \leq i \leq n$.*

Proof. By [2, Theorem 3.1 (3.1.1)], we see that $e_j(X, L) = (j - m + 1)c_m(Y)$ for every integer j with $j \geq 2m - 1$. Hence

$$\text{cl}_i(X, L) = (-1)^i (e_i(X, L) - 2e_{i-1}(X, L) + e_{i-2}(X, L)) = 0.$$

This completes the proof. \square

Acknowledgement. The author would like to thank the referee for some useful comments and suggestions.

References

- [1] Y. Fukuma, On the sectional invariants of polarized manifolds, *J. Pure Appl. Algebra*, **209** (2007), 99–117.
- [2] Y. Fukuma, Sectional invariants of scroll over a smooth projective variety, *Rend. Sem. Mat. Univ. Padova*, **121** (2009), 93–119.
- [3] Y. Fukuma, Sectional class of ample line bundles on smooth projective varieties, *Riv. Mat. Univ. Parma (N.S.)*, **6** (2015), 215–240.

- [4] Y. Fukuma, *Calculations of sectional classes of special polarized manifolds*, preprint, <http://www.math.kochi-u.ac.jp/fukuma/Cal-SC.html>
- [5] W. Fulton, *Intersection Theory*, Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 2. Springer-Verlag, Berlin, 1998.

Department of Mathematics and Physics,
Faculty of Science and Technology,
Kochi University,
Akebono-cho, Kochi 780-8520,
Japan
E-mail: fukuma@kochi-u.ac.jp