A NOTE ON THE PRODUCTS $\alpha_1\beta_2\gamma_t$ AND $\beta_1^{r+1}\beta_2\gamma_t$ IN THE STABLE HOMOTOPY OF SPHERES

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Abstract. In the stable homotopy groups of spheres, we have Greek letter elements due to J. F. Adams [2], L. Smith [12] and H. Toda [13]. Here we study the non-triviality of products of the first alpha element, the first and the second beta elements and a gamma element in the homotopy groups.

1. Introduction

Let $\mathcal{S}_{(p)}$ denote the stable homotopy category of spectra localized at a prime number $p > 5$, and $S^0 \in \mathcal{S}_{(p)}$ be the sphere spectrum localized at p. Since S^0 is a generator of $\mathcal{S}_{(p)}$ in a sense, the homotopy groups $\pi_*(S^0)$ play an important role to understand the category $\mathcal{S}_{(p)}$. The homotopy groups $\pi_*(S^0)$ form a commutative graded algebra with multiplication given by composition. Unfortunately, the structure of $\pi_*(S^0)$ is little known. G. Nishida showed that every element in $\pi_t(S^0)$ for $t > 0$ is nilpotent. We have generators of the groups called Greek letter elements. In this paper, we study whether or not a product of the Greek letter elements $\alpha_1 \in \pi_{q-1}(S^0)$, $\beta_1 \in \pi_{pq-2}(S^0)$, $\beta_2 \in \pi_{(2p+1)q-2}(S^0)$ and $\gamma_t \in \pi_{(tp^2+(t-1)p+t-2)q-3}(S^0)$ for $t \geq 1$ is trivial. Hereafter, we put $q = 2p-2$ as usual.

In [1], M. Aubry determined the homotopy groups $\pi_*(S^0)$ through total degree less than $(3p^2+4p)q$. In particular, we have the following:

Theorem 1.1 ([1]). *$\alpha_1\beta_2\gamma_2$ and $\beta_1^r\beta_2\gamma_2$ for $r < p$ are non-trivial, and $\alpha_1\beta_1\beta_2\gamma_2 = 0$.*

X. Liu showed the theorems:

Theorem 1.2 ([5]). *The products $\alpha_1\beta_2\gamma_s$ are non-trivial for $2 < s < p$.*

Theorem 1.3 ([14]). *The products $\alpha_1\beta_1\beta_2\gamma_s$ are non-trivial for $2 < s < p$.*

These two theorems are shown by use of the classical Adams spectral sequence. Thus, the subscript s of γ_s must be greater than two.

Consider the Adams-Novikov spectral sequence $\{E_r^{*,*}(X)\}$ converging to the homotopy groups $\pi_*(X)$ of a spectrum X, and let

$$\overline{\alpha}_1 \in E_2^{1,q}(S^0),\ \overline{\beta}_1 \in E_2^{2,pq}(S^0),\ \overline{\beta}_2 \in E_2^{2,(2p+1)q}(S^0) \text{ and}$$
$$\overline{\gamma}_t \in E_2^{3,(tp^2+(t-1)p+t-2)q}(S^0) \quad (t \geq 1)$$

be the elements detecting the Greek letter elements α_1, β_1, β_2 and γ_t, respectively. Observing products of these elements in the E_2-term, we obtained the following theorems:

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Theorem 1.4 ([11, Th. 1.1]). *The products* $\alpha_1\beta_1^r\gamma_{up+t} \neq 0$ *if* $1 < t < t+u < p$ *and* $r \leq p-2$.

Theorem 1.5 ([3, Th. 1.4]). *Let* t *be a positive integer with* $p \nmid t(t^2-1)$. *Then,* $\beta_2\gamma_t \neq 0 \in \pi_*(S^0)$.

C. -N. Lee showed that

Theorem 1.6 ([4, Th. 4.1, Th. 4.4]). *Let* $p \geq 7$. *The products* $\beta_1^r\gamma_t$ *and* $\beta_1^{r-1}\beta_2\gamma_t$ *are non-trivial if* $0 < t < p$ *and* $r \leq p-1$. *The product* $\alpha_1\beta_1^r\gamma_t$ *is non-trivial if* $2 \leq t < p$ *and* $r \leq p-2$.

By using a result $\beta_1^{p-2}\beta_2\gamma_2 \neq 0$ of Lee's, we deduce the non-triviality of the product $\beta_1^{p-2}\beta_2\gamma_{p+2}$:

Theorem 1.7. *Let* t *be an integer with* $1 < t < p$ *or* $t = p+2$. *Then, the products* $\beta_1^r\beta_2\gamma_t$ *are non-trivial for* $0 \leq r \leq p-2$.

Consider spectra $V(2)_k$ for $k \geq 1$ characterized by the Brown-Peterson homology $BP_*(V(2)_k) = BP_*/(p, v_1, v_2^k)$ (see (2.6)). The spectrum $V(2) = V(2)_1$ is the second Smith-Toda spectrum. It is well known that $\overline{\gamma}_1 = \overline{\alpha}_1\overline{\beta}_{p-1}$, and so $\overline{\alpha}_1\overline{\gamma}_1 = 0$ as well as $\alpha_1\gamma_1 = 0$. If $t = p, p+1$, then $\overline{\gamma}_t = 0 \in E_2^{3,(tp^2+(t-1)p+t-2)q}(V(2))$ (see (3.5), *cf.* [4, Lemma 4.3]).

For products $\overline{\alpha}_1\overline{\beta}_2\overline{\gamma}_t$ in the Adams-Novikov E_2-term for computing $\pi_*(V(2))$, we have

Theorem 1.8. $\overline{\alpha}_1\overline{\beta}_2\overline{\gamma}_t = 0 \in E_2^{6,(tp^2+(t+1)p+t)q}(V(2))$ *for* $t \geq p$.

By use of the May and the Novikov spectral sequences together with Toda's calculation [13] on the May E_1-term, we show the non-triviality of an element $\overline{\alpha}_1\overline{\beta}_2\overline{\gamma}_{p+2} \neq 0 \in E_2^{6,(p^3+3p^2+4p+2)q}(V(2)_3)$ in Lemma 2.20. From this, we extend non-triviality of products of Theorems 1.1 and 1.2 to the following:

Theorem 1.9. *Let* t *be an integer with* $1 < t < p$ *or* $t = p+2$. *Then,* $\alpha_1\beta_2\gamma_t \neq 0 \in \pi_*(S^0)$.

In the next section, we study the Adams-Novikov E_2-term by use of the May and the Novikov spectral sequences with Toda's calculation [13] on the May E_1-term. We then show the non-triviality of $\alpha_1\beta_2\gamma_{p+2}$ in Theorem 1.9 and the triviality of the products in Theorem 1.8 in Section 3. The last section is devoted to show the non-triviality of the composite $\beta_1^{p-2}\beta_2\gamma_{p+2}$ in Theorem 1.7.

2. The Adams-Novikov E_2-terms

We fix a prime number $p \geq 7$. Let BP denote the Brown-Peterson spectrum at the prime p, and we have a Hopf algebroid

$$(BP_*, BP_*(BP)) = (\mathbb{Z}_{(p)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots])$$

with structure maps: the left and the right units $\eta_L, \eta_R\colon BP_* \to BP_*(BP)$, the coproduct $\Delta\colon BP_*(BP) \to BP_*(BP) \otimes_{BP_*} BP_*(BP)$, the counit $\varepsilon\colon BP_*(BP) \to BP_*$ and the conjugation $c\colon BP_*(BP) \to BP_*(BP)$. Here, v_i and t_i are generators of degree $2p^i - 2 = e(i)q$ for $e(i) = \frac{p^i-1}{p-1}$ and $q = 2p-2$. We notice here the following action of the structure maps on the generators:

$$
(2.1)\quad
\begin{array}{rcl}
\eta_R(v_n) & \equiv & v_n + v_{n-1}t_1^{p^{n-1}} - v_{n-1}^p t_1 \mod I_{n-1} \quad (n \geq 2),\\
\eta_R(v_3) & \equiv & v_3 + v_2 t_1^{p^2} + v_1 t_2^p - t_1\eta_R(v_2^p) + v_1 w_1(v_2) - v_1^{p^2} t_2 \mod (p),\\
\eta_R(v_4) & \equiv & v_4 + v_3 t_1^{p^3} + v_2 t_2^{p^2} - \eta_R(v_3^p)t_1 - v_2^{p^2} t_2 \mod I_2,\\
\Delta(t_n) & \equiv & \sum_{i=0}^n t_i \otimes t_{n-i}^{p^i} + v_{n-1}b_{1,n-2} \mod I_{n-1} \quad (n \geq 1),\\
\Delta(t_4) & \equiv & \sum_{i=0}^4 t_i \otimes t_{4-i}^{p^i} + v_3 b_{1,2} + v_2 b_{2,1} \mod I_2,\\
c(t_1) & = & -t_1, \quad c(t_2) = t_1^{p+1} - t_2 \quad \text{and}\\
\Delta(c(x)) & = & (c\otimes c)T\Delta(x) \quad \text{for } x \in BP_*(BP).
\end{array}
$$

(*cf.* [10, Ch. 4]). Here, $T\colon BP_*(BP)\otimes BP_*(BP) \to BP_*(BP)\otimes BP_*(BP)$ denotes the switching map given by $T(x\otimes y) = y\otimes x$, I_{n-1} denotes the invariant ideal of BP_* generated by $n-1$ elements $v_0 = p$, v_1, $\dots$, v_{n-2} ($I_0 = 0$), $w_1(v_2) = \left(v_2^p + v_1^p t_1^{p^2} - v_1^{p^2} t_1^p - (v_2 + v_1 t_1^p - v_1^p t_1)^p\right)/p$, and $b_{1,k}$, $b_{2,k}$ and $b_{3,k} \in BP*(BP)\otimes_{BP_*} BP_*(BP)$ for $k \geq 0$ are the elements fitting in the following equalities

$$
(2.2)\quad
\begin{array}{l}
d(t_1^{p^{k+1}}) = pb_{1,k}, \quad d(t_2^{p^{k+1}}) = -t_1^{p^{k+1}} \otimes t_1^{p^{k+2}} - v_1^{p^{k+1}} b_{1,0}^{p^{k+1}} + pb_{2,k} \quad \text{and}\\
d(t_3^{p^{k+1}}) = -t_1^{p^{k+1}} \otimes t_2^{p^{k+2}} - t_2^{p^{k+1}} \otimes t_1^{p^{k+3}} - v_2^{p^{k+1}} b_{1,1}^{p^{k+1}} - v_1^{p^{k+1}} b_{2,0}^{p^{k+1}} + pb_{3,k},
\end{array}
$$

in which $d(x) = 1\otimes x + x\otimes 1 - \Delta(x) \in BP_*(BP)\otimes_{BP_*} BP_*(BP)$. By the definition (2.2) and the formulas on $\Delta(t_1)$ and $\Delta(t_2)$ in (2.1), we see that

$$
(2.3)\quad
\begin{array}{rcl}
d(b_{2,i}) & = & b_{1,i}\otimes t_1^{p^{i+2}} - t_1^{p^{i+1}} \otimes b_{1,i+1} \quad \text{for } i \geq 0, \text{ and}\\
d(b_{3,0}) & \equiv & b_{1,0}\otimes t_2^{p^2} - t_1^p \otimes b_{2,1} + b_{2,0}\otimes t_1^{p^3} - t_2^p \otimes b_{1,2} \mod (p).
\end{array}
$$

We have the Adams-Novikov spectral sequence:

$$E_2^{s,t}(W) = \mathrm{Ext}_{BP_*(BP)}^{s,t}(BP_*, BP_*(W)) \Longrightarrow \pi_{t-s}(W)$$

for a spectrum W. In this paper, we use the cobar complex $\Omega^{*,*}BP_*(W)$ for studying elements of the E_2-term: $E_2^{s,t}(W) = H^{s,t}(BP_*(W))$ (*cf.* [7], [4]). Here,

$$(2.4)\quad H^{s,t}(M) = \mathrm{Ext}_{BP_*(BP)}^{s,t}(BP_*, M)$$

for a $BP_*(BP)$-comodule M. Furthermore, we consider the k-th Smith-Toda spectrum $V(k)$ for $k = 0, 1, 2$ defined by the cofiber sequences

$$
(2.5)\quad
\begin{array}{c}
S^0 \xrightarrow{p} S^0 \xrightarrow{i} V(0) \xrightarrow{j} S^1, \quad \Sigma^q V(0) \xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1}V(0) \text{ and}\\
\Sigma^{(p+1)q}V(1) \xrightarrow{\beta} V(1) \xrightarrow{i_2} V(2) \xrightarrow{j_2} \Sigma^{(p+1)q+1}V(1)
\end{array}
$$

for the maps p, α and β, which induces a multiplication by p, v_1 and v_2 on the BP_*-homologies, respectively ([2], [12], *cf.* [10]). We also consider similar spectra $V(2)_k$ for $k \geq 2$ defined by the cofiber sequences

$$(2.6)\quad \Sigma^{k(p+1)q}V(1) \xrightarrow{\beta^k} V(1) \xrightarrow{\widetilde{i}_k} V(2)_k \xrightarrow{\widetilde{j}_k} \Sigma^{k(p+1)q+1}V(1).$$

We notice that $V(2)_k$ is a ring spectrum if $k \leq (p-2)/2$ ([9, Lemma 4.1], where it is denoted by L_k). Note that $BP_*(V(k)) = BP_*/I_{k+1}$, and $BP_*(V(2)_k) = BP_*/(p, v_1, v_2^k)$.

Consider a Hopf algebra $\mathcal{T} = \mathbb{Z}/p[t_1, t_2, \dots] = BP_*(BP)/(p, v_1, v_2, \dots)$ with structure maps obtained from $BP_*(BP)$ under the projection $BP_*(BP) \to \mathcal{T}$. May [6] constructed spectral sequences:

$$E_1 = H^*(V(L)) \Rightarrow H^*(\mathcal{T}) \quad \text{and} \quad E_2 = P(b_{i,j}) \otimes H^*(U(L)) \Rightarrow H^*(V(L)). \tag{2.7}$$

Here, L denotes the restricted Lie algebra associated to the Hopf algebra $\mathcal{T}$ and $U(L)$ and $V(L) = U(L)/(\xi(x) - x^p)$ are the enveloping algebras of L (ξ is the "p operation"). The bidegree of the generator $b_{i,j}$ is $(2, p^{j+1}e(i)q)$, and $b_{i,j}$'s correspond to those given above for $i = 1, 2, 3$. The cohomology $H^*(U(L))$ is isomorphic to the cohomology of the exterior complex $E(t_{i,j} : i \geq 1, j \geq 0)$ over generators $t_{i,j}$ with bidegree $(1, p^j e(i)q)$ along with the differential given by

$$d(t_{i,j}) = \sum_{k=1}^{i-1} t_{i-k,j+k} t_{k,j}. \tag{2.8}$$

In [13], Toda determined $H^{s,t}(U(L))$ for $t - s \leq (p^3 + 3p^2 + 2p + 1)q - 4$, which is additively generated by the unit element 1 and the elements in the table:

h_0	h_1	g_0	k_0	k_0h_0	h_2
1	p	$p+2$	$2p+1$	$2p+2$	p^2
h_2h_0	g_1	l_1	l_2	l_1h_1	k_1
p^2+1	p^2+2p	p^2+2p+3	p^2+3p+1	p^2+3p+3	$2p^2+p$
l_3	k_1h_1	l_1h_2	m_1	m_1h_0	l_4
$2p^2+p+2$	$2p^2+2p$	$2p^2+2p+3$	$2p^2+4p+2$	$2p^2+4p+3$	$3p^2+2p+1$
l_4h_0	l_4h_1	l_4g_0	l_4k_0	$l_4k_0h_0$	h_3
$3p^2+2p+2$	$3p^2+3p+1$	$3p^2+3p+3$	$3p^2+4p+2$	$3p^2+4p+3$	p^3
h_3h_0	h_3h_1	h_3g_0	h_3k_0	$h_3k_0h_0$	g_2
p^3+1	p^3+p	p^3+p+2	p^3+2p+1	p^3+2p+2	p^3+2p^2
g_2h_0	l_5	m_2	m_3	l_6	m_4
p^3+2p^2+1	p^3+2p^2+3p	p^3+2p^2 $+3p+4$	p^3+2p^2 $+4p+1$	p^3+3p^2+p	p^3+3p^2 $+p+2$

Table 2.9

Here, the integer under each element is the degree of it divided by q, and

$$\begin{gathered} h_i = [t_{1,i}], \quad g_i = [t_{1,i}t_{2,i}], \quad k_i = [t_{1,i+1}t_{2,i}], \quad (i \geq 0); \\ l_1 = [t_{3,0}t_{2,0}t_{1,0}], \quad l_2 = [t_{2,1}t_{2,0}t_{1,1}], \quad l_3 = [t_{3,0}t_{1,2}t_{1,0}], \\ l_4 = [t_{3,0}t_{2,1}t_{1,2}], \quad l_5 = [t_{3,1}t_{2,1}t_{1,1}], \quad l_6 = [t_{2,2}t_{2,1}t_{1,2}]; \\ m_1 = [t_{3,0}t_{2,1}t_{2,0}t_{1,1}], \quad m_2 = [t_{4,0}t_{3,0}t_{2,0}t_{1,0}], \\ m_3 = [t_{3,1}t_{2,1}t_{2,0}t_{1,1}], \quad \text{and} \quad m_4 = [t_{2,2}t_{3,0}t_{1,2}t_{1,0}]. \end{gathered} \tag{2.10}$$

Lemma 2.11. *The cohomology* $H^{5,(p^3+3p^2+3p+1)q}(\mathcal{T})$ *is a subquotient of* $\mathbb{Z}/p\{l_4h_3h_1\}$, *and* $H^{5,(p^3+3p^2+4p+2)q}(\mathcal{T}) = 0$.

Proof. We consider the May spectral sequences (2.7). The module $(E(t_{i,j}))^{5,tq}$ for $t = (p^3 + 3p^2 + ap + a - 2)$ with $a = 3$ or $a = 4$ is generated by the monomials of the form

$$t_{1,0}^{\varepsilon_{1,0}} t_{1,1}^{\varepsilon_{1,1}} t_{1,2}^{\varepsilon_{1,2}} t_{1,3}^{\varepsilon_{1,3}} t_{2,0}^{\varepsilon_{2,0}} t_{2,1}^{\varepsilon_{2,1}} t_{2,2}^{\varepsilon_{2,2}} t_{3,0}^{\varepsilon_{3,0}} t_{3,1}^{\varepsilon_{3,1}} t_{4,0}^{\varepsilon_{4,0}}$$

with $\varepsilon_{i,j} \in \{0,1\}$ satisfying equations

(1) $5 = \varepsilon_{1,0}+\varepsilon_{1,1}+\varepsilon_{1,2}+\varepsilon_{1,3}+\varepsilon_{2,0}+\varepsilon_{2,1}+\varepsilon_{2,2}+\varepsilon_{3,0}+\varepsilon_{3,1}+\varepsilon_{4,0}$,

(2) $1 = \varepsilon_{1,3}+\varepsilon_{2,2}+\varepsilon_{3,1}+\varepsilon_{4,0}$,

(3) $3 = \varepsilon_{1,2}+\varepsilon_{2,1}+\varepsilon_{2,2}+\varepsilon_{3,0}+\varepsilon_{3,1}+\varepsilon_{4,0}$,

(4) $a = \varepsilon_{1,1}+\varepsilon_{2,0}+\varepsilon_{2,1}+\varepsilon_{3,0}+\varepsilon_{3,1}+\varepsilon_{4,0}$ and

(5) $a-2 = \varepsilon_{1,0}+\varepsilon_{2,0}+\varepsilon_{3,0}+\varepsilon_{4,0}$.

These equations implies

(6) $4 = \varepsilon_{1,0}+\varepsilon_{1,1}+\varepsilon_{1,2}+\varepsilon_{2,0}+\varepsilon_{2,1}+\varepsilon_{3,0}$ by (1) and (2),

(7) $2 = \varepsilon_{1,0}+\varepsilon_{1,1}+\varepsilon_{1,3}+\varepsilon_{2,0}$ by (1) and (3),

(8) $2 = \varepsilon_{1,2}+\varepsilon_{2,1}+\varepsilon_{3,0}-\varepsilon_{1,3}$ by (2) and (3), and

(9) $2 = \varepsilon_{1,1}+\varepsilon_{2,1}+\varepsilon_{3,1}-\varepsilon_{1,0}$ by (4) and (5).

The case for $\varepsilon_{3,1}=0$: In this case, we see that $\varepsilon_{1,1}=\varepsilon_{2,1}=1$ and $\varepsilon_{1,0}=0$ by (9). Then,

$$2=\varepsilon_{1,2}+\varepsilon_{2,0}+\varepsilon_{3,0} \quad \text{by (6) and} \quad \varepsilon_{1,3}+\varepsilon_{2,0}=1 \quad \text{by (7)}.$$

- If $\varepsilon_{1,3}=1$, then $\varepsilon_{2,0}=0$, and so $\varepsilon_{1,2}=\varepsilon_{3,0}=1$, and obtain a monomial $t_{1,1}t_{2,1}t_{1,2}t_{3,0}t_{1,3}$ at degree $(p^3+3p^2+3p+1)q$, which yields the element $l_4h_1h_3$.
- If $\varepsilon_{1,3}=0$, then $\varepsilon_{2,0}=1$, and so $\varepsilon_{1,2}+\varepsilon_{3,0}=1$.
  - If $\varepsilon_{1,2}=1$, then the monomial has a factor $t_{1,1}t_{2,1}t_{2,0}t_{1,2}$ of degree $(2p^2+3p+1)q$, and so we obtain
    $t_{1,1}t_{2,1}t_{2,0}t_{1,2}t_{2,2}$ at $a=3$, and
    $t_{1,1}t_{2,1}t_{2,0}t_{1,2}t_{4,0}$ at $a=4$.
    The first monomial gives us the element $l_2g_2=l_6k_0\in H^{5,tq}(U(L))$. We name the second monomial x_1.
  - If $\varepsilon_{1,2}=0$, then $\varepsilon_{3,0}=1$, and the monomial has a factor $t_{1,1}t_{2,1}t_{2,0}t_{3,0}$ of degree $(2p^2+4p+2)q$, and so the monomial is $t_{1,1}t_{2,1}t_{2,0}t_{3,0}t_{2,2}$ at degree $(p^3+3p^2+4p+2)q$. We name it x_2.

The case for $\varepsilon_{3,1}=1$: In this case, $\varepsilon_{1,3}=\varepsilon_{2,2}=\varepsilon_{4,0}=0$ by (2). By (9), $1=\varepsilon_{1,1}+\varepsilon_{2,1}-\varepsilon_{1,0}$.

- If $\varepsilon_{1,0}=1$, then $\varepsilon_{1,1}=\varepsilon_{2,1}=1$, and the monomial has a factor $t_{1,0}t_{1,1}t_{2,1}t_{3,1}$ of degree (p^3+2p^2+3p+1). Therefore, we have monomials $t_{1,0}t_{1,1}t_{2,1}t_{1,2}t_{3,1}$ at $a=3$ and $t_{1,0}t_{1,1}t_{2,1}t_{3,0}t_{3,1}$ at $a=4$. The first monomial corresponds $l_5h_2h_0$. By Table 2.9, we see that $l_5h_0=0$ and the monomial yields nothing. We name the second one x_3.
- If $\varepsilon_{1,0}=0$, then $1=\varepsilon_{1,1}+\varepsilon_{2,1}$. This together with (6) implies $3=\varepsilon_{1,2}+\varepsilon_{2,0}+\varepsilon_{3,0}$, and we obtain $\varepsilon_{1,2}=\varepsilon_{2,0}=\varepsilon_{3,0}=1$. By (8), $\varepsilon_{2,1}=0$, and so $\varepsilon_{1,1}=1$. Therefore, we have $t_{1,1}t_{1,2}t_{2,0}t_{3,0}t_{3,1}$ at degree $(p^3+3p^2+4p+2)q$. We name it x_4.

Now put

$$\widetilde{x}_1 = t_{1,1}t_{2,1}t_{2,0}t_{1,2}t_{3,1}t_{1,0} \quad \widetilde{x}_2 = t_{1,1}t_{2,1}t_{2,0}t_{1,2}t_{1,3}t_{3,0}$$

Then,

$$d(x_1)=\widetilde{x}_1+\widetilde{x}_2, \quad d(x_2)=-\widetilde{x}_2, \quad d(x_3)=-\widetilde{x}_1 \quad \text{and} \quad d(x_4)=-\widetilde{x}_1+\widetilde{x}_2,$$

and

$$d(t_{1,1}t_{2,1}t_{3,0}t_{4,0}) = -x_1 - x_3 - x_2 \quad \text{and} \quad d(t_{2,1}t_{2,0}t_{3,0}t_{3,1}) = -x_2 + x_3 - x_4.$$

Thus, the elements x_i for $i = 1, 2, 3, 4$ yield no element of $H^{5,(p^3+3p^2+4p+2)q}(U(L))$. We also have

$$\begin{aligned} & d(t_{1,1}t_{2,1}t_{1,2}t_{4,0} - t_{2,0}t_{2,1}t_{1,2}t_{3,1}) \\ = \ & -t_{1,1}t_{2,1}t_{1,2}(t_{3,1}t_{1,0} + t_{2,2}t_{2,0} + t_{1,3}t_{3,0}) - t_{1,1}t_{1,0}t_{2,1}t_{1,2}t_{3,1} \\ & +t_{2,0}t_{2,1}t_{1,2}t_{2,2}t_{1,1} \ = \ -2l_2g_2 + l_4h_3h_1. \end{aligned}$$

$H^{5,tq}(V(L))$ for $t = (p^3+3p^2+ap+a-2)$ with $a = 3$ or 4 also contains elements obtained from the E_1-term of the May spectral sequence (2.7):

$$\begin{array}{ll} b_{1,0}H^{3,t'q}(U(L)) & \text{for } t' = t - p = (p^3 + 3p^2 + (a-1)p + a - 2), \text{ and} \\ b_{1,0}^2H^{1,t''q}(U(L)) & \text{for } t'' = t - 2p = (p^3 + 3p^2 + (a-2)p + a - 2). \end{array}$$

The latter module is trivial. We have a monomial of the complex $(E(t_{i,j}))^{3,t'q}$:

$$t_{2,1}t_{3,0}t_{4,0} \quad (t' = p^3 + 3p^2 + 3p + 2).$$

on which the differential acts by $d(t_{2,1}t_{3,0}t_{4,0}) = t_{2,1}t_{2,0}t_{1,2}t_{4,0} + \cdots \neq 0$, and this monomial yields no element of $H^{3,t'q}(U(L))$. Thus there is no element in these modules.

From Table (2.9), we find no element of the form $xb_{i,j}b_{k,l}$ or $xb_{i,j}$ for $x \in H^*(U(L))$ in our degree. □

For studying the Adams-Novikov E_2-term, we consider the Novikov spectral sequences

(2.12) $$E_1 = \mathrm{Ext}_{\mathcal{T}}(\mathbb{Z}/p, Q) \Longrightarrow E_2^{*,*}(V(0))$$

(*cf.* [1, Lemme in p. 61]) and

(2.13) $$E_1 = \mathbb{Z}/p[v_n] \otimes \mathrm{Ext}_{\mathcal{T}}(\mathbb{Z}/p, Q(n+1)) \Longrightarrow \mathrm{Ext}_{\mathcal{T}}(\mathbb{Z}/p, Q(n))$$

(*cf.* [1, (1.4.3)]). Here,

(2.14) $$Q = \mathbb{Z}/p[v_1, v_2, \dots] \quad \text{and} \quad Q(n) = Q/(v_1, \dots, v_{n-1})$$

are comodules with coactions given by

(2.15) $$\eta(v_n) = \sum_{i=0}^{n} v_i t_{n-i}^{p^i}.$$

We note that

$$\mathrm{Ext}_{\mathcal{T}}(\mathbb{Z}/p, Q(5)) = H^*(\mathcal{T})$$

in our range.

Among the generators (2.10) of $H^*(U(L))$, the elements g_i and k_i for $i \geq 0$, l_2, l_4 and l_6 survive to the Adams-Novikov E_2-term, $E_2^*(V(2)_p)$ by the Massey products

(2.16) $$\begin{gathered} g_i = \langle h_i, h_i, h_{i+1} \rangle, \quad k_i = \langle h_i, h_{i+1}, h_{i+1} \rangle, \\ l_2 = \langle h_0, h_1, g_1 \rangle, \quad l_4 = -2\langle h_2, h_2, h_2, k_0 \rangle \quad \text{and} \quad l_6 = \langle h_1, h_2, g_2 \rangle. \end{gathered}$$

These satisfy

(2.17) $$g_i = \langle h_{i+1}, h_i, h_i \rangle, \quad 2g_i = -\langle h_i, h_{i+1}, h_i \rangle \quad \text{and} \quad 2k_i = -\langle h_{i+1}, h_i, h_{i+1} \rangle$$

for $i \geq 0$. By a juggling theorem of the Massey products, we also see that

$$h_ig_i = 0, \quad h_{i+1}g_i = h_ik_i \quad \text{and} \quad g_ih_{i+2} = 0.$$

We moreover have elements of the $E_2^{*,*}(V(2)_p)$:

$$v_3h_2 = \langle v_2, h_2, h_2\rangle \quad \text{and} \quad xb_{2,0} = \left\langle x, (h_1, h_2), \begin{pmatrix} -b_1 \\ b_0 \end{pmatrix} \right\rangle \tag{2.18}$$

for an element $x \in E_2^{*,*}(V(2)_p)$ with $xh_1 = 0 = xh_2$. Hereafter, we write b_i for the homology class of $b_{1,i}$ (see also (3.3)). For example, $x = h_1$, h_2, g_2 and k_1b_2. Indeed, $k_1b_2h_1 = g_1h_2b_2 = g_1h_3b_1 = 0$.

Lemma 2.19. *For the spectra $V(2)_k$ in (2.6), some of the Adams-Novikov E_2-terms are given as follows:*

$$E_2^{3,(2p^2+p)q}(V(2)_3) = \mathbb{Z}/p\{h_2b_{2,0}\} \quad \textit{and} \quad E_2^{2p,(3p^2+p)q}(V(2)_{p-1}) = 0.$$

Proof. For $t \le 2p^2+3p+2$, $E_2^{*,tq}(V(2)_3)$ is a subquotient of $\mathbb{Z}/p[v_2, v_3] \otimes H^*(\mathcal{T})$ by the spectral sequences (2.12) and (2.13), and $H^*(\mathcal{T})$ is a subquotient of $P(b_{i,j}) \otimes H^*(U(L))$ by the May spectral sequence.

We pick generators with given bidegrees out of the module $\mathbb{Z}/p[v_2, v_3] \otimes P(b_{i,j}) \otimes H^*(U(L))$ as in the following table, where a, $b \in \{0, 1, 2\}$ and $x \in H^{*,*}(U(L))$.

bidegree		a, b	$\dim x$	x	generators
$(3, (2p^2+p)q)$	$v_2^av_3^bx$	$a = b = 0$	3	$-$	$-$
	$v_2^av_3^bxb_{i,j}$	$a = b = 0$	1	h_2	$h_2b_{2,0}$

By (2.18), the element $h_2b_{2,0}$ yields an element of the Adams-Novikov E_2-term. We easily find only one element k_1 of bidegree $(2, (2p^2+p)q)$ in $\mathbb{Z}/p[v_2, v_3] \otimes P(b_{i,j}) \otimes H^*(U(L))$. This is an element of $E_2^{2,(2p^2+p)q}(V(2)_3)$, and no differential hit $h_2b_{2,0}$ in any above spectral sequences. Therefore, $h_2b_{2,0}$ survives to the E_2-term $E_2^{3,(2p^2+p)q}(V(2)_3)$.

Turn to the second. A monomial of bidegree $(2p, (3p^2+p)q)$ of $\mathbb{Z}/p[v_2, v_3] \otimes P(b_{i,j}) \otimes H^*(U(L))$ has one of the forms $v_2^av_3^bxb_{2,0}^2b_{1,0}^{p-2-\frac12\dim x}$, $v_2^av_3^bxb_{2,0}b_{1,1}b_{1,0}^{p-2-\frac12\dim x}$, $v_2^av_3^bxb_{1,1}^2b_{1,0}^{p-2-\frac12\dim x}$, $v_2^av_3^bxb_{2,0}b_{1,0}^{p-1-\frac12\dim x}$, $v_2^av_3^bxb_{1,1}b_{1,0}^{p-1-\frac12\dim x}$ and $v_2^av_3^bxb_{1,0}^{p-\frac12\dim x}$. The degrees of these elements are

monomials	degrees
$v_2^av_3^bxb_{2,0}^2b_{1,0}^{p-2-\frac12\dim x}$	$q\left((p+1)a + (p^2+p+1)b + \deg x + 3p^2 - \frac{p}{2}\dim x\right)$
$v_2^av_3^bxb_{2,0}b_{1,1}b_{1,0}^{p-2-\frac12\dim x}$	$q\left((p+1)a + (p^2+p+1)b + \deg x + 3p^2 - p - \frac{p}{2}\dim x\right)$
$v_2^av_3^bxb_{1,1}^2b_{1,0}^{p-2-\frac12\dim x}$	$q\left((p+1)a + (p^2+p+1)b + \deg x + 3p^2 - 2p - \frac{p}{2}\dim x\right)$
$v_2^av_3^bxb_{2,0}b_{1,0}^{p-1-\frac12\dim x}$	$q\left((p+1)a + (p^2+p+1)b + \deg x + 2p^2 - \frac{p}{2}\dim x\right)$
$v_2^av_3^bxb_{1,1}b_{1,0}^{p-1-\frac12\dim x}$	$q\left((p+1)a + (p^2+p+1)b + \deg x + 2p^2 - p - \frac{p}{2}\dim x\right)$
$v_2^av_3^bxb_{1,0}^{p-\frac12\dim x}$	$q\left((p+1)a + (p^2+p+1)b + \deg x + p^2 - \frac{p}{2}\dim x\right)$

Since the degree is $(3p^2+p)q$, we see that $\deg x/q \equiv -a-b \bmod p$, and deduce that $a = b = 0$. Indeed, $\deg x/q \equiv d \bmod p$ with $0 \le d \le 3$, $0 \le a < p-1$ and $0 \le b \le 2$. Thus, $x = g_1$, k_1, and we have a candidate $g_1b_{2,0}b_{1,0}^{p-2}$ for a generator. Note that $d_{2p-1}(g_1b_{2,0}b_{1,0}^{p-2}) = g_1h_2b_{1,0}^{p-1} = h_1k_1b_{1,0}^{p-1}$ in the second May spectral sequence in (2.7). Since $h_1k_1 \neq 0$ by Table 2.9, we have no generator at the degree. □

Lemma 2.20. *We have a non-zero element $v_2^2v_3^pb_0b_1^2 \in E_2^{6,(p^3+3p^2+4p+2)q}(V(2)_3)$.*

Proof. Put $t_0 = p^3+3p^2+4p+2$. We consider the element $v_2^2v_3^pb_0b_1^2 \in E_2^{6,t_0q}(V(2)_3)$ by the spectral sequences (2.7), (2.12) and (2.13). For this sake, we compute the Ext group $E = \mathrm{Ext}_{\mathcal{T}}^{5,t_0q}(\mathbb{Z}/p, Q(2))$ for the comodule $Q(2)$ in (2.14). We study whether or not the element $v_2^2v_3^pb_0b_1^2$ is in the image of a differential of the spectral sequences, and so it suffices to consider modules

$$M(a,b,c) = \left(v_2^av_3^bv_4^cH^{5,*}(V(L))\right)^{5,t_0q} \subset \left(P(v_2,v_3,v_4)/(v_2^3)\otimes H^{5,*}(V(L))\right)^{5,t_0q}.$$

We read off from Table 2.9 and Lemma 2.11, the module

$$M(a,b,c) \sqsubseteq \begin{cases} \mathbb{Z}/p\{v_4l_2b_1\} & (a,b,c)=(0,0,1)\\ \mathbb{Z}/p\{v_3v_4h_2b_0^2, v_3v_4h_1b_0b_1\} & (a,b,c)=(0,1,1)\\ \mathbb{Z}/p\{v_2v_4h_2b_0b_{2,0}, v_2v_4h_1b_1b_{2,0}\} & (a,b,c)=(1,0,1)\\ \mathbb{Z}/p\{v_3l_2b_{2,1}\} & (a,b,c)=(0,1,0)\\ \mathbb{Z}/p\{v_2v_3h_3b_{2,0}^2, v_2v_3h_1b_{2,0}b_{2,1}, v_2v_3k_1h_1b_2, & \\ \qquad v_2v_3h_1b_1b_{3,0}, v_2v_3h_2b_0b_{3,0}\} & (a,b,c)=(1,1,0)\\ \mathbb{Z}/p\{v_3^2h_3b_0b_{2,0}, v_3^2h_1b_2b_{2,0}, v_3^2h_1b_0b_{2,1}\} & (a,b,c)=(0,2,0)\\ \mathbb{Z}/p\{v_2v_3^ph_0b_{2,0}^2\} & (a,b,c)=(1,p,0)\\ \mathbb{Z}/p\{v_2^2v_3^ph_2b_0b_1, v_2^2v_3^ph_1b_1^2\} & (a,b,c)=(2,p,0)\\ \mathbb{Z}/p\{v_3^{p+1}h_0b_0b_{2,0}\} & (a,b,c)=(0,p+1,0)\\ \mathbb{Z}/p\{v_2l_4h_3h_1\} & (a,b,c)=(1,0,0)\\ \mathbb{Z}/p\{v_2^2l_6b_0, v_2^2k_1h_1b_{2,1}, v_2^2h_2b_{3,0}b_{2,0}\} & (a,b,c)=(2,0,0)\\ 0 & \text{otherwise.} \end{cases}$$

Here, we write $A \sqsubseteq B$ if A is a subquotient of B. Let $E(a,b,c)$ denote a submodule of E generated by elements detected by elements of $M(a,b,c)$. We first verify which of the elements on the right hand side of the above relation yields an element of $M(a,b,c)$, and then evaluate $E(a,b,c)$ by the spectral sequences (2.13).

We consider the second spectral sequence (2.7). Note that the May filtration of the elements $h_{i,j}$ and $b_{i,j}$ are $2i-1$ and $p(2i-1)$, respectively. Then, the May differential $d_{2p-1}\colon E_{2p-1}^{s,t,u} \to E_{2p-1}^{s+1,t,u-2p+1}$ of the spectral sequence acts as

$$\begin{array}{rcll} d_{2p-1}(b_{2,i}) & = & b_{1,i}h_{i+2} - h_{i+1}b_{1,i+1} & \text{for } i\ge 0\text{, and}\\ d_{2p-1}(b_{3,0}) & = & -h_1b_{2,1} + b_{2,0}h_3 & \end{array} \tag{2.21}$$

by (2.3).

We start from the modules $M(0,1,1)$, $M(1,0,1)$, $M(1,1,0)$ and $M(2,p,0)$. By (2.21), $h_2b_0^2 = h_1b_0b_1$, $h_2b_0b_{2,0} = h_1b_1b_{2,0}$ and $h_2b_0b_1 = h_1b_1^2$ in $H^*(V(L))$, and

$$\begin{array}{rcl} d_{2p-1}(h_3b_{2,0}^2) & = & -2h_3(b_{1,0}h_2 - h_1b_{1,1})b_{2,0} \;=\; 2h_3h_1b_{1,1}b_{2,0},\\ d_{2p-1}(h_1b_{2,0}b_{2,1}) & = & -h_1(b_{1,0}h_2 - h_1b_{1,1})b_{2,1} - h_1b_{2,0}(b_{1,1}h_3 - h_2b_{1,2})\\ & = & h_3h_1b_{1,1}b_{2,0},\\ d_{2p-1}(h_1b_{1,1}b_{3,0}) & = & -h_1b_{1,1}(-h_1b_{2,1}+b_{2,0}h_3) \;=\; h_3h_1b_{1,1}b_{2,0},\\ d_{2p-1}(h_2b_{1,0}b_{3,0}) & = & -h_2b_{1,0}(-h_1b_{2,1}+b_{2,0}h_3) \;=\; 0, \quad \text{and}\\ d_{2p-1}(b_{2,0}b_{3,0}) & = & (b_{1,0}h_2 - h_1b_{1,1})b_{3,0} + b_{2,0}(-h_1b_{2,1}+b_{2,0}h_3)\\ & = & h_2b_{1,0}b_{3,0} - h_1b_{1,1}b_{3,0} - h_1b_{2,0}b_{2,1} + h_3b_{2,0}^2. \end{array}$$

These differentials imply that the rank of the module $M(1,1,0)$ is not greater than three. Therefore, $M(0,1,1) \sqsubseteq \mathbb{Z}/p\{v_3v_4h_2b_0^2\}$, $M(1,0,1) \sqsubseteq \mathbb{Z}/p\{v_2v_4h_2b_{2,0}b_0\}$, $M(1,1,0) \sqsubseteq v_2v_3\mathbb{Z}/p\{h_2b_0b_{3,0}, h_1b_{2,0}b_{2,1} - h_1b_1b_{3,0}, k_1h_1b_2\}$ and $M(2,p,0) \sqsubseteq$

$\mathbb{Z}/p\{v_2^2v_3^ph_2b_0b_1\}$. Furthermore, we have $d_{4p-3}(h_2b_{1,0}b_{3,0}) = -h_2b_{1,0}(b_{1,0}h_{2,2} - h_{2,1}b_{1,2}) = -g_2b_{1,0}^2 + k_1b_{1,0}b_{1,2}$, and $d_{4p-3}(h_1b_{2,0}b_{2,1} - h_1b_{1,1}b_{3,0}) = h_1b_{1,1}(b_{1,0}h_{2,2} - h_{2,1}b_{1,2}) = g_2b_{1,0}^2 - g_1b_{1,1}b_{1,2}$. Therefore, we obtain $M(1,1,0) \sqsubseteq \mathbb{Z}/p\{v_2v_3k_1h_1b_2\}$.

Consider the spectral sequence (2.13). The differentials of the spectral sequences are read off from the structure map (2.15). For example, $d_1(v_4) = v_3h_3$ for $n = 3$ and $d_1(v_3) = v_2h_2$ for $n = 2$. For $M(0,1,1)$, noticing that v_4h_2 is represented by a cocycle $v_4t_1^{p^2} + v_3c(t_2^{p^2}) + v_2t_1^{p^2}t_2^{p^2}$ in the cobar complex $Q(2) \otimes \mathcal{T}$, we compute

$$\begin{aligned} & d(v_4t_1^{p^2} + v_3c(t_2^{p^2}) + v_2t_1^{p^2}t_2^{p^2}) \\ = \;& \underline{v_3t_1^{p^3} \otimes t_1^{p^2}}_1 + \underline{v_2t_2^{p^2} \otimes t_1^{p^2}}_2 + v_2t_1^{p^2} \otimes c(t_2^{p^2}) - \underline{v_3t_1^{p^3} \otimes t_1^{p^2}}_1 \\ & -v_2t_1^{p^2} \otimes t_2^{p^2} - \underline{v_2t_2^{p^2} \otimes t_1^{p^2}}_2 - v_2t_1^{2p^2} \otimes t_1^{p^3} - v_2t_1^{p^2} \otimes t_1^{p^3+p^2} \\ = \;& -2v_2t_1^{p^2} \otimes t_2^{p^2} - v_2t_1^{2p^2} \otimes t_1^{p^3}, \end{aligned}$$

in which the underlined terms with a subscript cancel each other out. The cocycle $2t_1^{p^2} \otimes t_2^{p^2} + t_1^{2p^2} \otimes t_1^{p^3}$ appearing in the right hand side of the above computation represents $2g_2 \neq 0 \in \mathrm{Ext}_{\mathcal{T}}(\mathbb{Z}/p, Q(3))$ (see (2.14) for $Q(3)$). It follows that v_4h_2 does not survive to $\mathrm{Ext}_{\mathcal{T}}(\mathbb{Z}/p, Q(2))$ in (2.13). Thus, $E(0,1,1) = 0$.

For $M(1,0,1)$, we compute

$$\begin{aligned} h_3h_2b_{2,0} &= h_3\left\langle h_2, (h_1,h_2), \begin{pmatrix} -b_1 \\ b_0 \end{pmatrix} \right\rangle \\ &= (\langle h_3,h_2,h_1\rangle, \langle h_3,h_2,h_2\rangle) \begin{pmatrix} -b_1 \\ b_0 \end{pmatrix} = g_2b_0 \end{aligned} \tag{2.22}$$

by the juggling theorem in the E_{2p}-term of the second spectral sequence in (2.7) by (2.18) and (2.17). We also note that $\langle h_3,h_2,h_1\rangle = 0$ by considering $d(t_3^p)$. Therefore, $d_1(v_4h_2b_{2,0}b_0) = v_3g_2b_0^2$ in the spectral sequence (2.13) for $n = 3$, and $E(1,0,1) = 0$ follows.

In the spectral sequence (2.13) for $n = 2$, we compute

$$\begin{aligned} d_1(v_3^2g_1b_2) &= 2v_2v_3h_2g_1b_2 = 2v_2v_3k_1h_1b_2 \quad \text{and} \\ d_1(v_2v_3^{p+1}b_0b_1) &= v_2^2v_3^ph_2b_0b_1, \end{aligned}$$

where we use the well known relation $g_1h_2 = h_1k_1$. Therefore, the triviality of $E(1,1,0)$ and $E(2,p,0)$ follows.

Since $h_2l_2 = 0 = h_3l_2$ by Table 2.9, we see that

$$l_2b_{2,1} = \left\langle l_2, (h_2,h_3), \begin{pmatrix} -b_2 \\ b_1 \end{pmatrix} \right\rangle$$

in $H^*(V(L))$ in the same manner as (2.18). Note that $\langle h_2,l_2,h_2\rangle = 2l_4h_1$ and $\langle h_2,l_2,h_3\rangle = 0$ in $H^*(V(L))$. Therefore, in the spectral sequence (2.13) for $n = 2$, we compute $d_1(v_3l_2b_{2,1}) = -2v_2l_4h_1b_2 \neq 0$ and so $E(0,1,0) = 0$.

Since $d_{2p-1}(b_{3,0}b_{1,0}) = (-h_1b_{2,1} + b_{2,0}h_3)b_{1,0}$ and

$$d_{2p-1}(h_1b_{2,1}b_{1,0}) = -h_1(b_{1,1}h_3 - h_2b_{1,2})b_{1,0} = -h_3h_1b_{1,1}b_{1,0},$$

we see that $M(0,2,0) \sqsubseteq \mathbb{Z}/p\{v_3^2h_1b_{2,0}b_2\}$. In the spectral sequence (2.13) for $n = 2$,

$$\begin{aligned} d_1(v_3^2h_1b_{2,0}) &= 2v_2v_3h_2\left\langle h_1, (h_1,h_2), \begin{pmatrix} -b_1 \\ b_0 \end{pmatrix} \right\rangle \\ &= 2v_2v_3\langle h_2,h_1,(h_1,h_2)\rangle \begin{pmatrix} -b_1 \\ b_0 \end{pmatrix} = 2v_2v_3(g_1b_1 - 2k_1b_0) \end{aligned}$$

by (2.17) and (2.18). It follows that $E(0,2,0) = 0$.

In the spectral sequence in (2.7), $d_{2p-1}(k_1b_{3,0}) = k_1(-h_1b_{2,1}+b_{2,0}h_3) = -k_1h_1b_{2,1}$ and $k_1h_1b_{2,1} = 0 \in H^*(V(L))$. By (2.3), we compute the differential $d(t_1^{p^2} \otimes b_{2,0} \otimes b_{3,0})$ in the cobar complex for computing $H^*(V(L))$, and deduce that

$$d_{4p-3}(h_2b_{2,0}b_{3,0}) = h_2b_{2,0}(b_{1,0}h_{2,2} - h_{2,1}b_{1,2}) = g_2b_{2,0}b_{1,0} - k_1b_{1,2}b_{2,0}$$

in the spectral sequence. Here, $xb_{2,0}$ for $x = g_2$, k_1b_2 are given in (2.18). Thus, $M(2,0,0) \sqsubseteq \mathbb{Z}/p\{v_2^2l_6b_0\}$.

We have $M(1,p,0) = 0$ and $M(0,p+1,0) = 0$, since

$$d_{2p-1}(h_0b_{2,0}) = -h_0(b_{1,0}h_2 - h_1b_{1,1}) = h_2h_0b_{1,0}.$$

Therefore, $E(1,p,0) = 0$ and $E(0,p+1,0) = 0$.

Therefore, $\mathrm{Ext}_{\mathcal{T}}^{5,t_0q}(\mathbb{Z}/p, Q(2))$ is a subquotient of the module

$$\mathbb{Z}/p\{v_4l_2b_1, v_2l_4h_3h_1, v_2^2l_6b_0\}.$$

We consider the element v_4l_2. By (2.16), $l_2 \in E_2^{*,*}(V(2)_2)$. Let $\bar{l}_2$ denote a cocycle representing l_2 in the cobar complex for computing $E_2^{*,*}(V(2)_2)$. By Table 2.9 together with (2.16), we see that $h_0l_2 = 0$ and $h_3l_2 = 0$, and so we have cochains y_i such that $d(y_i) = t_1^{p^i} \otimes \bar{l}_2$ for $i = 0, 3$ in the cobar complex. Then,

$$\begin{aligned} d(v_4\bar{l}_2 - v_3y_3 + v_3^py_0) &\equiv v_3t_1^{p^3} \otimes \bar{l}_2 - v_3^pt_1 \otimes \bar{l}_2 + v_2t_2^{p^2} \otimes \bar{l}_2 - v_2t_1^{p^2} \otimes y_3 \\ &\quad -v_3t_1^{p^3} \otimes \bar{l}_2 + v_3^pt_1 \otimes \bar{l}_2 \\ &\equiv v_2(t_2^{p^2} \otimes \bar{l}_2 - t_1^{p^2} \otimes y_3) \mod (p, v_1, v_2^3). \end{aligned}$$

Since $t_2^{p^2} \otimes \bar{l}_2 - t_1^{p^2} \otimes y_3$ represents an element of the Massey product $\langle h_2, h_3, l_2\rangle$, which belongs to $H^{4,(p^3+2p^2+3p+1)q}(U(L))$. Therefore, we deduce that $\langle h_2, h_3, l_2\rangle = 0$ by Table 2.9, and so we have a cochain z such that $d(z) = t_2^{p^2} \otimes \bar{l}_2 - t_1^{p^2} \otimes y_3$. Now the element $v_4l_2b_1$ yields an element of $E_2^{*,*}(V(2)_3)$ represented by $(v_4\bar{l}_2 - v_3y_3 + v_3^py_0 - v_2z) \otimes b_{1,1}$.

The other generators of the module are represented by the Massey products

$$-2v_2\langle h_2, h_2, h_2, k_0\rangle h_3h_1 \quad \text{and} \quad v_2^2\langle h_1, h_2, g_2\rangle b_0$$

in the Adams-Novikov E_2-term $E_2^{*,*}(V(2)_3)$ (*cf.* (2.16)). Therefore, the differentials of (2.12) on these generators act trivially, and $v_2^2v_3^pb_0b_1^2$ is not in the image of any differentials of the spectral sequences. □

3. On the product $\alpha_1\beta_2\gamma_{p+2}$

We recall the definition of the Greek letter elements. The Greek letter elements in the homotopy groups $\pi_*(S^0)$ are defined by composites

(3.1) $$\alpha_s = j\alpha^si, \qquad \beta_s = jj_1\beta^si_1i \qquad \text{and} \qquad \gamma_s = jj_1j_2\gamma^si_2i_1i$$

for the maps in (2.5) and a map $\gamma\colon \Sigma^{(p^2+p+1)q}V(2) \to V(2)$ inducing a multiplication by v_3 on BP_*-homologies given by Toda [13]. We notice that $(\iota \wedge V(0))\alpha^si = v_1^s \in BP_*/(p)$, $\beta^si_1i = (\iota \wedge V(1))v_2^s \in BP_*/I_2$ and $(\iota \wedge V(2))\gamma^si_2i_1i = v_3^s \in BP_*/I_3$ for the unit map $\iota\colon S^0 \to BP$ of the ring spectrum BP. Then by the Geometric Boundary Theorem (*cf.* [10, Th. 2.3.4]), the Greek letter elements (3.1) are detected by those in the Adams-Novikov E_2-term defined by

(3.2) $$\begin{aligned} &\overline{\alpha}_s = \delta_0(v_1^s) \in E_2^{1,sq}(S^0), \quad \overline{\beta}_s = \delta_0\delta_1(v_2^s) \in E_2^{2,(sp+s-1)q}(S^0) \quad \text{and} \\ &\overline{\gamma}_s = \delta_0\delta_1\delta_2(v_3^s) \in E_2^{3,(sp^2+(s-1)p+s-2)q}(S^0). \end{aligned}$$

Here $\delta_k\colon E_2^{*,*}(V(k)) \to E_2^{*+1,*}(V(k-1))$ denotes the connecting homomorphism associated to the cofiber sequences in (2.5) ($V(-1) = S^0$). Traditionally we put

$$(3.3) \qquad h_i = [t_1^{p^i}] \in E_2^{1,p^i q}(S^0) \quad \text{and} \quad b_i = [b_{1,i}] \in E_2^{2,p^{i+1}q}(S^0),$$

where $[c]$ denotes the cohomology class of a cocycle $c \in \Omega^{*,*}BP_*$. We note that h_i corresponds to h_i in Table 2.9. Then, by definition, we have well known relations (*cf.* [4], [10]):

$$(3.4). \qquad \overline{\alpha}_1 = h_0, \quad \overline{\beta}_1 \equiv b_0 \quad \text{and} \quad \overline{\beta}_2 \equiv 2v_2 b_0 + k_0 \mod I_2$$

in the E_2-term. Furthermore, it is also shown in [4, Lemma 4.3] that

$$(3.5) \qquad \overline{\gamma}_t = 2\binom{t}{2} v_3^{t-2} h_2 b_{2,0} + 3\binom{t}{3} v_3^{t-3} l_4 \mod I_3 = (p, v_1, v_2)$$

in $E_2^{3,(tp^2+(t-1)p+t-2)q}(S^0) = H^{3,(tp^2+(t-1)p+t-2)q}(BP_*)$, where $h_2b_{2,0}$ and l_4 are given in (2.18) and (2.16). By Lemma 2.19, we have

Lemma 3.6. $\overline{\gamma}_2 = 2h_2b_{2,0} \neq 0 \in E_2^{3,(2p^2+p)q}(V(2)_3)$.

Lemma 3.7. *The element* $\overline{\gamma}_{p+2} \in E_2^{3,(p^3+3p^2+2p)q}(S^0)$ *satisfies that* $\overline{\gamma}_{p+2} \equiv v_3^p \overline{\gamma}_2$ mod (p, v_1, v_2^3).

Proof. The relation $\overline{\gamma}_{p+2} \equiv v_3^p \overline{\gamma}_2$ follows from computation:

$$\begin{aligned}
\delta_2(v_3^{p+2}) &\equiv v_3^p \delta_2(v_3^2) \mod (v_2^6). \\
\delta_1\delta_2(v_3^{p+2}) &= \delta_1(v_3^p\delta_2(v_3^2) + v_2^5 x) \equiv v_3^p \delta_1\delta_2(v_3^2) \mod (v_1^2, v_2^4). \\
\delta_0\delta_1\delta_2(v_3^{p+2}) &= \delta_0(v_3^p\delta_1\delta_2(v_3^2) + v_1^2 y + v_2^4 z) \equiv v_3^p \delta_0\delta_1\delta_2(v_3^2) \mod (p, v_1, v_2^3),
\end{aligned}$$

for elements $x \in E_2^{1,*}(V(1))$, and y, $z \in E_2^{2,*}(V(0))$. □

Lemma 3.8. *For the spectrum* $V(2)_3$ *in (2.6), we have*

$$h_0 k_0 \overline{\gamma}_2 = 0 \in E_2^{6,(2p^2+3p+2)q}(V(2)_3).$$

Proof. By the juggling Theorem of the Massey products, (2.18) and Lemma 3.6, we compute

$$\begin{aligned}
h_0 k_0 \overline{\gamma}_2 &= g_0 h_1 \overline{\gamma}_2 = 2g_0(\langle h_1, h_2, h_1\rangle, \langle h_1, h_2, h_2\rangle)\begin{pmatrix} -b_1 \\ b_0 \end{pmatrix} \\
&= 4g_0g_1b_1 + 2g_0k_1b_1 = 0
\end{aligned}$$

in $E_2^{6,(2p^2+3p+2)q}(V(2)_3)$. Indeed, $\langle h_1, h_2, h_1\rangle = -2g_1$ by (2.17), and $g_0g_1 = 0 = g_0k_1$. Therefore, the lemma follows. □

Lemma 3.9. *In the Adams-Novikov* E_2*-term,*

$$\overline{\alpha}_1 \overline{\beta}_2 \overline{\gamma}_2 = 4v_2^2 b_0 b_1^2 \in E_2^{6,(2p^2+3p+2)q}(V(2)_3).$$

Proof. By (3.4) and Lemma 3.8, we see that $\overline{\alpha}_1\overline{\beta}_2\overline{\gamma}_2 = 2v_2\overline{\alpha}_1\overline{\beta}_1\overline{\gamma}_2$, which is congruent to $4v_2h_0b_0h_2b_{2,0}$ modulo (p, v_1, v_2^3) by Lemma 3.6. We compute

$$\begin{aligned}
\tfrac{1}{4}v_2\overline{\alpha}_1\overline{\beta}_1\overline{\gamma}_2 &\equiv v_2h_0b_0\left\langle h_2, (h_1, h_2), \begin{pmatrix}-b_1\\ b_0\end{pmatrix}\right\rangle \\
&\equiv h_0b_0\left\langle v_2, h_2, (h_1, h_2)\right\rangle\begin{pmatrix}-b_1\\ b_0\end{pmatrix} \\
&\equiv h_0b_0(\langle v_2, h_2, h_1\rangle, \langle v_2, h_2, h_2\rangle)\begin{pmatrix}-b_1\\ b_0\end{pmatrix} \\
&\equiv h_0b_0\left(-\langle v_2, h_2, h_1\rangle b_1 + \langle v_2, h_2, h_2\rangle b_0\right) \\
&\equiv -v_2\langle h_2, h_1, h_0\rangle b_0b_1 + \langle v_2, h_2, h_2\rangle h_0b_0b_0 \\
&\equiv v_2^2b_0b_1^2 + v_3h_2h_0b_0^2.
\end{aligned}$$

Here, the differential $d(c(t_3))$ (see (2.1)) gives us a relation $\langle h_2, h_1, h_0\rangle \equiv v_2b_1$ mod I_2 in the E_2-term. We further see that $h_2h_0b_0^2 = 0 \in H^*(V(L))$, since $d_{2p-1}(h_0b_{1,0}b_{2,0}) = h_0h_2b_{1,0}^2$ in the May spectral sequence. □

Theorem 3.10. $\overline{\alpha}_1\overline{\beta}_2\overline{\gamma}_{p+2} \neq 0 \in E_2^{6,(p^3+3p^2+4p+2)q}(S^0)$.

Proof. By Lemma 3.7, we have $\overline{\gamma}_{p+2} = v_3^p\overline{\gamma}_2 \in E_2^{3,(p^3+3p^2+2p)q}(V(2)_3)$, and so

$$\overline{\alpha}_1\overline{\beta}_2\overline{\gamma}_{p+2} = v_3^p\overline{\alpha}_1\overline{\beta}_2\overline{\gamma}_2 = 4v_2^2v_3^pb_0b_1^2 \in E_2^{6,(p^3+3p^2+4p+2)q}(V(2)_3)$$

by Lemma 3.9. Now the theorem follows from Lemma 2.20. □

Proof of Theorem 1.8. For $t = p$ and $= p+1$, $\overline{\gamma}_t = 0$ by (3.5), and so the proposition holds in these cases. Suppose now $t \geq p+2$. Note that $\overline{\beta}_2 = [\widetilde{k}_0] = k_0$ and $\overline{\gamma}_t = 2\binom{t}{2}v_3^{t-2}h_2b_{2,0} + 3\binom{t}{3}v_3^{t-3}l_4$ for $t \geq 2$ in $E_2^*(V(2))$ by (3.4) and (3.5) (*cf.* [4, p. 234], [4, Lemma 4.3]). Here, $BP_*(V(2)) = BP_*/I_3$ and l_4 denotes the generator given in [13, p. 55]. This implies that $\overline{\gamma}_t = v_3^p\overline{\gamma}_{t-p}$ for $t \geq p+2$ in $E_2^*(V(2))$, and we also see $v_3^ph_0 = v_3h_3$ in $E_2^1(V(2))$ by $d(v_4)$, where $h_i \in E_2^{1,p^iq}(V(2))$ is an element represented by a cocycle $t_1^{p^i}$. Therefore, $\overline{\alpha}_1\overline{\beta}_2\overline{\gamma}_t$ is represented by $v_3^{t-p-2}h_3k_0(2\binom{t}{2}v_3h_2b_{2,0} + 3\binom{t}{3}l_4)$. Here, we see that $h_3k_0h_2b_{2,0} = k_0g_2b_{1,0}$ by (2.22). We also see that $h_3k_0l_4 = h_3h_2m_1$ for the generators in Toda's calculation [13, p. 55]. Since both of k_0g_2 and h_3h_2 are zero by Toda's calculation (see Table 2.9), these imply the triviality of $\overline{\alpha}_1\overline{\beta}_2\overline{\gamma}_t$ for $t \geq p+2$. □

4. Non-triviality of $\beta_1^{p-2}\beta_2\gamma_{p+2}$

We begin with a recollection of some results from [4]: $\Omega^{*,*}BP_*\{a\}$ denotes a quotient complex of the cobar complex $\Omega^{*,*}BP_*$ by a subcomplex generated by monomials $m \otimes t^{E_1} \otimes \cdots \otimes t^{E_n}$ with $\sum_{i=1}^n E_i > (a, 0, \dots)$. Here, t^E for a sequence $E = (e_1, e_2, \dots)$ denotes the monomial $t_1^{e_1}t_2^{e_2}\cdots \in BP_*(BP)$, and the set of sequences admits the lexicographical ordering (*cf.* [4, p.235]).

Then, the gamma elements $\overline{\gamma}_t$ for $t \geq 2$ in the Adams-Novikov E_2-term are represented by a cocycle

$$\widetilde{\gamma}_t \equiv -tv_2^{p-3}v_3^{t-1}\widetilde{k}_0 \otimes t_1 \mod J_3 = (p, v_1, v_2^{p-1}) \tag{4.1}$$

in $\Omega^{3,(tp^2+(t-1)p+t-2)q}BP_*\{p^2-1\}$ (*cf.* [4, p. 239]). In this section, we consider a spectrum $V(2)_{p-1}$ in (2.6). Note that $BP_*(V(2)_{p-1}) = BP_*/J_3$.

Theorem 4.2. $\overline{\beta}_1^{p-2}\overline{\beta}_2\overline{\gamma}_{p+2} \neq 0 \in E_2^{2p+1,tq}(S^0)$ *for* $t = p^3 + 4p^2 + 2p + 1$.

Proof. Let $G \in C = \Omega^{2p+1,tq}BP_*$ be a cocycle representing the element $\overline{\beta}_1^{p-2}\overline{\beta}_2\overline{\gamma}_{p+2}$. Then, $G \equiv v_3^p G_2 \mod J_3$ for a cochain

$$G_2 = -2v_2^{p-3}v_3\widetilde{k}_0 \otimes t_1 \otimes (2v_2b_{1,0} + \widetilde{k}_0) \otimes b_{1,0}^{\otimes(p-2)}$$

in $\overline{C} = \Omega^{2p+1,(3p^2+p+1)q}BP_*\{p^2+2\}$ by (3.4) and (4.1). Note that G_2 is the cochain $\mathcal{D}$ of [4, p. 240] for $t = 2$, which is shown not to be a coboundary in $\overline{C}/J_3$. We claim that

(4.3) $\qquad G$ has no term with v_4 as a factor modulo J_3.

Indeed, if $G = v_3^pG_2 + v_4w + w' \mod J_3$ for $w, w' \in \Omega^*BP_*/(J_3 + (v_4))$, then, applying the differential d to the equality, we obtain $0 = v_3^p d(G_2) + d(v_4)\otimes w + v_4 d(w) + d(w')$. Since $d(G_2)$, $d(v_4)$ and $d(w')$ have no term with v_4, we deduce that $d(w) = 0$. Therefore, $[w] \in E_2^{2p,(3p^2+p)q}(V(2)_{p-1})$, which is zero by Lemma 2.19. It follows that there is a cochain $\overline{w}$ such that $w = d(\overline{w})$. So replace v_4w by $d(v_4)\otimes\overline{w}$ so that G has no term with factor v_4 modulo J_3.

Suppose that there is a cocycle $y \in \Omega^{2p,tq}BP_*$ such that $d(y) = G$ in C. Put $y = y_1 + v_4y_2 + v_3^py_3 + z$ for $y_i = \sum_{a,b} v_2^a v_3^b y_{i,a,b}$ $(i = 1,2,3)$ with $y_{i,a,b} \in \Omega^{2p,*}BP_*/I_5$ and $z \in J_3\Omega^{2p,*}BP_*$. By a similar argument showing (4.3), we replace v_4y_2 by a linear combination of terms without factor v_4. Thus we may put $y = y_1 + v_3^py_3 + z$. By (2.1), we see that $d(t_i) \in \Omega^2 BP_*/J_3\{p^2+2\}$ has the only one term $v_2b_{1,1}$ if $i = 3$, and $v_2b_{2,1}$ if $i = 4$ with factors v_2 and v_3. It follows that for $x \in \Omega^{2p,uq}BP_*/I_5$ with $u \le t$, $d(x) \in (\mathbb{Z}/p)\{1, v_2\} \otimes \Omega^{2p+1,uq}BP_*/I_5\{p^2+2\}$ by degree reason. Indeed, $v_2^2b_{1,1}^2 = 0 \in \Omega^{4,2e(3)q}BP_*/I_2\{p^2+2\}$ and $v_2^2b_{2,1}^2$ has an internal degree greater than tq. Since $d(v_3^b) = bv_2v_3^{b-1}t_1^{p^2}$ in $\Omega^{1,*}BP_*/J_3\{p^2+2\}$ by (2.1), we see that

$$d(y) = d(y_1) + v_3^p d(y_3) = v_3^pG_2 \in \Omega^{2p+1,tq}BP_*/J_3\{p^2+2\}.$$

Here, we notice that $d(z) \equiv 0 \mod J_3$, since J_3 is an invariant ideal. From the equality, we see that $d(y_1) = 0$ and $d(y_3) = G_2$ in $\Omega^{2p+1,(3p^2+p+1)q}BP_*/J_3\{p^2+2\}$. Thus, $G_2 (= \mathcal{D}$ in [4, p. 240]) is a coboundary in the complex. This contradicts to the conclusion of the proof of [4, Th. 4.1]. □

Corollary 4.4. $\beta_1^{p-2}\beta_2\gamma_{p+2} \neq 0 \in \pi_{(p^3+4p^2+2p)q-3}(S^0)$.

Proof. By virtue of Theorem 4.2, it suffices to show that there is no element $x \in E_2^{2,(t-1)q}(S^0)$ such that $d_{2p-1}(x) = \overline{\beta}_1^{p-2}\overline{\beta}_2\overline{\gamma}_{p+2}$ in the Adams-Novikov spectral sequence. In [7, Th. 2.6], it is shown that the E_2-term $E_2^{2,*}(S^0)$ is generated by the elements $\overline{\beta}_{sp^i/j,k+1}$ for integers $p \nmid s \ge 1$, $i, k \ge 0$, $j \ge 1$, subject to $j \le p^i$ if $s = 1$, $p^k \mid j \le a_{i-k}$ and $a_{i-k-1} < j$ if $p^{k+1} \mid j$, where $a_0 = 1$, $a_n = p^n + p^{n-1} - 1$ for $n \ge 1$. The internal degree of the element $\overline{\beta}_{sp^i/j,k+1}$ is $(sp^i(p+1) - j)q$, and we have an equation $t - 1 = sp^i(p+1) - j$ to find the element x. Note that $sp^i - j \ge 0$, and we have $2p^3 > sp^{i+1}$ and so $i \le 2$. Thus, we obtain the only solution $(i, j, s) = (1, p, p+3)$ of the equation. In this case, $k = 0$ by the relation $p^k \mid j \le a_{i-k}$. The element $\overline{\beta}_{(p+3)p/p} (= \overline{\beta}_{(p+3)p/p,1})$ is a permanent cycle by [8]. Thus, we have no such element x, and hence $\overline{\beta}_1^{p-2}\overline{\beta}_2\overline{\gamma}_{p+2}$ is not in the image of the differential d_{2p-1} of the spectral sequence. □

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