A NOTE ON THE PRODUCTS $\alpha_1 \beta_2 \gamma_t$ AND $\beta_1^{r+1} \beta_2 \gamma_t$ IN THE STABLE HOMOTOPY OF SPHERES

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Abstract. In the stable homotopy groups of spheres, we have Greek letter elements due to J. F. Adams [2], L. Smith [12] and H. Toda [13]. Here we study the non-triviality of products of the first alpha element, the first and the second beta elements and a gamma element in the homotopy groups.

1. Introduction

Let $S(p)$ denote the stable homotopy category of spectra localized at a prime number $p > 5$, and $S^0 \in S(p)$ be the sphere spectrum localized at $p$. Since $S^0$ is a generator of $S(p)$ in a sense, the homotopy groups $\pi_\ast(S^0)$ play an important role to understand the category $S(p)$. The homotopy groups $\pi_\ast(S^0)$ form a commutative graded algebra with multiplication given by composition. Unfortunately, the structure of $\pi_\ast(S^0)$ is little known. G. Nishida showed that every element in $\pi_t(S^0)$ for $t > 0$ is nilpotent. We have generators of the groups called Greek letter elements. In this paper, we study whether or not a product of the Greek letter elements $\alpha_1 \in \pi_{q-1}(S^0)$, $\beta_1 \in \pi_{pq-2}(S^0)$, $\beta_2 \in \pi_{2p+1}q-2(S^0)$ and $\gamma_t \in \pi_{(tp^2+(t-1)p+t-2)q-3}(S^0)$ for $t \geq 1$ is trivial. Hereafter, we put $q = 2p - 2$ as usual.

In [1], M. Aubry determined the homotopy groups $\pi_\ast(S^0)$ through total degree less than $(3p^2 + 4p)q$. In particular, we have the following:

Theorem 1.1 ([1]). $\alpha_1 \beta_2 \gamma_2$ and $\beta_1 \beta_2 \gamma_2$ for $r < p$ are non-trivial, and $\alpha_1 \beta_1 \beta_2 \gamma_2 = 0$.

X. Liu showed the theorems:

Theorem 1.2 ([5]). The products $\alpha_1 \beta_2 \gamma_s$ are non-trivial for $2 < s < p$.

Theorem 1.3 ([14]). The products $\alpha_1 \beta_1 \beta_2 \gamma_s$ are non-trivial for $2 < s < p$.

These two theorems are shown by use of the classical Adams spectral sequence. Thus, the subscript $s$ of $\gamma_s$ must be greater than two.

Consider the Adams-Novikov spectral sequence $\{E_r^\ast \ast(X)\}$ converging to the homotopy groups $\pi_\ast(X)$ of a spectrum $X$, and let

$\overline{\alpha}_1 \in E_2^{2,q}(S^0), \overline{\beta}_1 \in E_2^{2,pq}(S^0), \overline{\beta}_2 \in E_2^{2,(2p+1)q}(S^0)$ and

$\overline{\gamma}_t \in E_2^{3,(tp^2+(t-1)p+t-2)q}(S^0)$ \quad ($t \geq 1$)

be the elements detecting the Greek letter elements $\alpha_1$, $\beta_1$, $\beta_2$ and $\gamma_t$, respectively. Observing products of these elements in the $E_2$-term, we obtained the following theorems:

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Theorem 1.4 ([11, Th. 1.1]). The products $\alpha_1^t \gamma_{ap+t} \neq 0$ if $1 < t < t + u < p$ and $r \leq p - 2$.

Theorem 1.5 ([3, Th. 1.4]). Let $t$ be a positive integer with $p \nmid t(t^2 - 1)$. Then, $\beta_2 \gamma_t \neq 0 \in \pi_*(S^0)$.

C.-N. Lee showed that

Theorem 1.6 ([4, Th. 4.1, Th. 4.4]). Let $p \geq 7$. The products $\beta_2 \gamma_t$ and $\beta_1^{-1} \beta_2 \gamma_t$ are non-trivial if $0 < t < p$ and $r \leq p - 1$. The product $\alpha_1 \beta_2 \gamma_t$ is non-trivial if $2 \leq t < p$ and $r \leq p - 2$.

By using a result $\beta_1^{p-2} \beta_2 \gamma_2 \neq 0$ of Lee's, we deduce the non-triviality of the product $\beta_1^{p-2} \beta_2 \gamma_{p+2}$:

Theorem 1.7. Let $t$ be an integer with $1 < t < p$ or $t = p + 2$. Then, the products $\beta_1 \beta_2 \gamma_t$ are non-trivial for $0 \leq r \leq p - 2$.

Consider spectra $V(2)_k$ for $k \geq 1$ characterized by the Brown-Peterson homology $BP_*(V(2)_k) = BP_*/(p, v_1, v_2^k)$ (see (2.6)). The spectrum $V(2) = V(2)_1$ is the second Smith-Toda spectrum. It is well known that $\tau_1 = \tau_1 \tau_{p-1}^2$, and so $\tau_1 \tau_1 = 0$ as well as $\alpha_1 \gamma_1 = 0$. If $t = p, p + 1$, then $\tau_t = 0 \in E_2((p^2+(t-1)p+t-2)q) \pi_*(V(2))$ (see (3.5), cf. [4, Lemma 4.3]).

For products $\tau_1 \tau_2 \tau_t$ in the Adams-Novikov $E_2$-term for computing $\pi_*(V(2))$, we have

Theorem 1.8. $\pi_1 \tau_2 \tau_t = 0 \in E^6_2((p^2+(p+1)p+t-2)q) \pi_*(V(2))$ for $t \geq p$.

By use of the May and the Novikov spectral sequences together with Toda’s calculation [13] on the May $E_1$-term, we show the non-triviality of an element $\pi_1 \tau_2 \tau_{p+2} \neq 0 \in E^6_2((p^2+3p^2+4p+2)q) \pi_*(V(2)_3)$ in Lemma 2.20. From this, we extend non-triviality of products of Theorems 1.1 and 1.2 to the following:

Theorem 1.9. Let $t$ be an integer with $1 < t < p$ or $t = p + 2$. Then, $\alpha_1 \beta_2 \gamma_t \neq 0 \in \pi_*(S^0)$.

In the next section, we study the Adams-Novikov $E_2$-term by use of the May and the Novikov spectral sequences with Toda’s calculation [13] on the May $E_1$-term. We then show the non-triviality of $\alpha_1 \beta_2 \gamma_{p+2}$ in Theorem 1.9 and the triviality of the products in Theorem 1.8 in Section 3. The last section is devoted to show the non-triviality of the composite $\beta_1^{p-2} \beta_2 \gamma_{p+2}$ in Theorem 1.7.

2. The Adams-Novikov $E_2$-terms

We fix a prime number $p \geq 7$. Let $BP$ denote the Brown-Peterson spectrum at the prime $p$, and we have a Hopf algebroid

$$(BP_* \otimes BP_*(BP)) = (\mathbb{Z}[v_1, v_2, \ldots], BP_*[t_1, t_2, \ldots])$$

with structure maps: the left and the right units $\eta_L, \eta_R: BP_* \to BP_*(BP)$, the coproduct $\Delta: BP_*(BP) \to BP_*(BP) \otimes_{BP_*} BP_*(BP)$, the counit $\varepsilon: BP_*(BP) \to BP_*$ and the conjugation $c: BP_*(BP) \to BP_*(BP)$. Here, $v_i$ and $t_i$ are generators
of degree $2p - 2 = e(i)q$ for $e(i) = \frac{p - 1}{p - 1}$ and $q = 2p - 2$. We notice here the following action of the structure maps on the generators:

$$
\begin{align*}
\eta_R(v_n) &\equiv v_n + v_n - 1 t_1 p^{n-1} - v_n t_1 \mod I_{n-1} \ (n \geq 2), \\
\eta_R(v_0) &\equiv v_0 + v_0 t_1 p - t_1 \eta_R(v_2) + v_1 t_1 v_2 - v_1 t_2 \mod (p), \\
\eta_R(v_4) &\equiv v_4 + v_4 t_1^2 + v_2 t_2^2 - \eta_R(v_3^2) t_1 - v_3^2 t_2 \mod I_2, \\
\Delta(t_n) &\equiv \sum_{i=0}^n t_i \otimes t_{n-i}^p + v_n - 1 b_{n-1} \mod I_{n-1} \ (n \geq 1), \\
\Delta(t_4) &\equiv \sum_{i=0}^4 t_i \otimes t_4^{i-1} + v_3 b_{1,2} + v_2 b_{2,1} \mod I_2, \\
c(t_1) &\equiv -t_1, \ c(t_2) = t_1^2 - t_2 \quad \text{and} \\
\Delta(c(x)) &\equiv (c \otimes c) \Delta(x) \quad \text{for } x \in BP_*(BP).
\end{align*}
$$

(cf. [10, Ch. 4]). Here, $T : BP_*(BP) \otimes BP_*(BP) \to BP_*(BP) \otimes BP_*(BP)$ denotes the switching map given by $T(x \otimes y) = y \otimes x$, $I_{n-1}$ denotes the invariant ideal of $BP_*$ generated by $n - 1$ elements $v_0 = p$, $v_1$, ..., $v_{n-2}$ ($I_0 = 0$), $w_1(v_2) = \left( v_0 p^2 + v_1 t_1^2 - v_1 t_2 p - (v_0 + v_1 t_1 - v_1 t_1 t_2) / p, \right.$ and $b_{1,k}, b_{2,k}$ and $b_{3,k} \in \text{Ext}_BP_*(BP) \otimes BP_*(BP)$ for $k \geq 0$ are the elements fitting in the following equalities

$$
\begin{align*}
d(t_1^{k+1}) &\equiv pb_{1,k}, \\
d(t_2^{k+2}) &\equiv -t_1^{k+1} \otimes t_1^{k+2} - v_1^{k+1} b_{1,0} + pb_{2,k}, \\
d(t_3^{k+2}) &\equiv -v_1^{k+1} b_{1,0} - v_2^{k+1} b_{2,0} + pb_{3,k},
\end{align*}
$$
in which $d(x) = 1 \otimes x + x \otimes 1 - \Delta(x) \in BP_*(BP) \otimes BP_*(BP)$. By the definition (2.2) and the formulas on $\Delta(t_1)$ and $\Delta(t_2)$ in (2.1), we see that

$$
\begin{align*}
d(b_{2,i}) &\equiv b_{1,i} t_1^{i+2} - t_1^{i+1} b_{1,i+1} \quad \text{for } i \geq 0, \\
\Delta(b_{3,0}) &\equiv b_{1,0} t_2 - t_1 b_{2,1} + b_{2,0} t_1 - t_2 b_{1,2} \mod (p).
\end{align*}
$$

We have the Adams-Novikov spectral sequence:

$$
E_2^{s,t}(W) = \text{Ext}_BP_*(BP)(BP_*, BP_*(W)) \Longrightarrow \pi_{t-s}(W)
$$

for a spectrum $W$. In this paper, we use the cobar complex $\Omega^* \otimes BP_*(W)$ for studying elements of the $E_2$-term: $E_2^{s,t}(W) = H^{s,t}(BP_*(W))$ (cf. [7], [4]). Here,

$$
H^{s,t}(M) = \text{Ext}_BP_*(BP)(BP_*, M)
$$

for a $BP_*(BP)$-comodule $M$. Furthermore, we consider the $k$-th Smith-Toda spectrum $V(k)$ for $k = 0, 1, 2$ defined by the cofiber sequences

$$
\Sigma^0 \xrightarrow{p} \Sigma^0 \xrightarrow{\alpha} V(0) \xrightarrow{\beta} S^1, \quad \Sigma^0 V(0) \xrightarrow{\alpha} V(0) \xrightarrow{\beta} V(1) \xrightarrow{\delta} \Sigma^{q+1} V(0) \quad \text{and} \\
\Sigma^{(p+1)}(V(1)) \xrightarrow{\alpha} V(1) \xrightarrow{\beta} V(2) \xrightarrow{\delta} \Sigma^{(p+1)q+1} V(1)
$$

for the maps $p, \alpha$ and $\beta$, which induces a multiplication by $p$, $v_1$ and $v_2$ on the $BP_*$-homologies, respectively ([2], [12], cf. [10]). We also consider similar spectra $V(k)$ for $k \geq 2$ defined by the cofiber sequences

$$
\Sigma^{q+1} V(1) \xrightarrow{\delta} V(1) \xrightarrow{\beta} V(2) \xrightarrow{\delta} \Sigma^{q+1} V(1).
$$

We notice that $V(2)$ is a ring spectrum if $k \leq (p - 2) / 2$ ([9, Lemma 4.1], where it is denoted by $L_k$). Note that $BP_*(V(k)) = BP_*/I_{k+1}$, and $BP_*(V(2)) = BP_*/(p, v_1, v_2^2)$. 

\[\text{A NOTE ON THE PRODUCTS } \alpha_1 \beta_2 \gamma_1 \text{ AND } \beta_1^{*+1} \beta_2 \gamma_2 \]
Consider a Hopf algebra $\mathcal{T} = \mathbb{Z}/p[t_1, t_2, \ldots] = BP_*(BP)/(p, v_1, v_2, \ldots)$ with structure maps obtained from $BP_*(BP)$ under the projection $BP_*(BP) \to \mathcal{T}$. May [6] constructed spectral sequences:

$$E_1 = H^*(V(L)) \Rightarrow H^*(\mathcal{T}) \quad \text{and} \quad E_2 = P(b_{i,j}) \otimes H^*(U(L)) \Rightarrow H^*(V(L)).$$

Here, $L$ denotes the restricted Lie algebra associated to the Hopf algebra $\mathcal{T}$ and $U(L)$ and $V(L) = U(L)/(\xi(x) = x^p)$ are the enveloping algebras of $L$ ($\xi$ is the “$p$ operation”). The bidegree of the generator $b_{i,j}$ is $(2, p^i e^j(i)q)$, and $b_{i,j}$’s correspond to those given above for $i = 1, 2, 3$. The cohomology $H^*(U(L))$ is isomorphic to the cohomology of the exterior complex $E(t_{i,j} : i \geq 1, j \geq 0)$ over generators $t_{i,j}$ with bidegree $(1, p^i e^j(i)q)$ along with the differential given by

$$d(t_{i,j}) = \sum_{k=1}^{i-1} t_{i-k,j+k} t_{k,j}.$$ 

In [13], Toda determined $H^{s,t}(U(L))$ for $t - s \leq (p^3 + 3p^2 + 2p + 1)q - 4$, which is additively generated by the unit element 1 and the elements in the table:

<table>
<thead>
<tr>
<th>$h_0$</th>
<th>$h_1$</th>
<th>$g_0$</th>
<th>$k_0$</th>
<th>$k_0 h_0$</th>
<th>$h_2$</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>$p$</td>
<td>$p + 2$</td>
<td>$2p + 1$</td>
<td>$2p + 2$</td>
<td>$p^2$</td>
</tr>
<tr>
<td>$h_2 h_0$</td>
<td>$g_1$</td>
<td>$l_1$</td>
<td>$l_2$</td>
<td>$l_1 h_1$</td>
<td>$k_1$</td>
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<td>$p^2 + 2p$</td>
<td>$p^2 + 2p + 3$</td>
<td>$p^2 + 3p + 1$</td>
<td>$p^2 + 3p + 3$</td>
<td>$2p^2 + p$</td>
</tr>
<tr>
<td>$l_3$</td>
<td>$h_1 h_1$</td>
<td>$l_1 h_2$</td>
<td>$m_1$</td>
<td>$m_1 h_0$</td>
<td>$l_4$</td>
</tr>
<tr>
<td>$2p^2 + p + 2$</td>
<td>$2p^2 + 2p$</td>
<td>$2p^2 + 2p + 3$</td>
<td>$2p^2 + 4p + 2$</td>
<td>$2p^2 + 4p + 3$</td>
<td>$3p^2 + 2p + 1$</td>
</tr>
<tr>
<td>$l_4 h_0$</td>
<td>$l_4 h_1$</td>
<td>$l_4 h_2$</td>
<td>$l_4 h_3$</td>
<td>$l_4 h_0 h_0$</td>
<td>$l_4 h_3$</td>
</tr>
<tr>
<td>$l_4 h_3$</td>
<td>$l_4 h_4$</td>
<td>$l_4 h_5$</td>
<td>$l_4 h_6$</td>
<td>$l_4 h_7$</td>
<td>$g_2$</td>
</tr>
<tr>
<td>$3p^2 + 2p + 2$</td>
<td>$3p^2 + 3p + 1$</td>
<td>$3p^2 + 3p + 3$</td>
<td>$3p^2 + 4p + 2$</td>
<td>$3p^2 + 4p + 3$</td>
<td>$p^4$</td>
</tr>
<tr>
<td>$h_3 h_0$</td>
<td>$h_3 h_1$</td>
<td>$h_3 h_2$</td>
<td>$h_3 h_0 h_0$</td>
<td>$g_2$</td>
<td></td>
</tr>
<tr>
<td>$p^3 + 1$</td>
<td>$h_3 h_0$</td>
<td>$h_3 h_1$</td>
<td>$h_3 h_2$</td>
<td>$h_3 h_0 h_0$</td>
<td>$g_2$</td>
</tr>
<tr>
<td>$g_2 h_0$</td>
<td>$l_5$</td>
<td>$m_2$</td>
<td>$m_3$</td>
<td>$l_6$</td>
<td>$m_4$</td>
</tr>
<tr>
<td>$p^3 + 2p^2 + 1$</td>
<td>$p^3 + 2p^2 + 3p$</td>
<td>$p^3 + 2p^2 + 2p^2 + 3p + 4$</td>
<td>$p^3 + 3p^2 + p + 4p + 1$</td>
<td>$p^3 + 3p^2 + p + p^2 + 2$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.9

Here, the integer under each element is the degree of it divided by $q$, and

$$h_1 = [t_{1,1}], \quad g_i = [t_{1,i+1}, t_{2,1}], \quad k_i = [t_{1,i+1}, t_{2,1}], \quad (i \geq 0);$$

$$l_1 = [t_{3,0,t_2,0,t_1}], \quad l_2 = [t_{2,1}, t_{2,0}, t_{1,1}], \quad l_3 = [t_{3,0}, t_{2,1}, t_{1,1}], \quad l_4 = [t_{3,1}, t_{2,1}, t_{1,1}], \quad l_5 = [t_{3,2}, t_{2,1}, t_{1,1}], \quad l_6 = [t_{2,2}, t_{2,1}, t_{1,1}], \quad l_7 = [t_{3,1}, t_{2,2}, t_{1,1}];$$

$$m_1 = [t_{3,0}, t_{2,1}, t_{2,0}, t_{1,1}], \quad m_2 = [t_{3,0}, t_{2,1}, t_{2,0}, t_{1,1}], \quad m_3 = [t_{3,1}, t_{2,1}, t_{2,0}, t_{1,1}], \quad m_4 = [t_{3,2}, t_{2,1}, t_{2,0}, t_{1,1}].$$

(2.10)

**Lemma 2.11.** The cohomology $H^5(p^3 + 3p^2 + 3p + 1)q(T)$ is a subquotient of $\mathbb{Z}/p[l_4 h_3 h_1]$, and $H^5(p^3 + 3p^2 + 4p + 2)q(T) = 0$.

**Proof.** We consider the May spectral sequences (2.7). The module $(E(t_{i,j}))^{5,q}$ for $t = (p^3 + 3p^2 + ap + a - 2)$ with $a = 3$ or $a = 4$ is generated by the monomials of the form

$$t_{1,0}^{1.0} t_{1,1}^{1.1} t_{1,2}^{1.2} t_{1,3}^{1.3} t_{2,0}^{2.0} t_{2,1}^{2.1} t_{2,2}^{2.2} t_{2,3}^{2.3} t_{3,0}^{3.0} t_{3,1}^{3.1} t_{4,0}^{4.0}$$

$\cdots$
with \( \varepsilon_{i,j} \in \{0, 1\} \) satisfying equations

\[
\begin{align*}
(1) \quad 5 &= \varepsilon_{1,0} + \varepsilon_{1,1} + \varepsilon_{1,2} + \varepsilon_{1,3} + \varepsilon_{2,0} + \varepsilon_{2,1} + \varepsilon_{2,2} + \varepsilon_{3,0} + \varepsilon_{3,1} + \varepsilon_{4,0}, \\
(2) \quad 1 &= \varepsilon_{1,3} + \varepsilon_{2,2} + \varepsilon_{3,1} + \varepsilon_{4,0}, \\
(3) \quad 3 &= \varepsilon_{1,2} + \varepsilon_{2,1} + \varepsilon_{2,2} + \varepsilon_{3,0} + \varepsilon_{3,1} + \varepsilon_{4,0}, \\
(4) \quad a &= \varepsilon_{1,1} + \varepsilon_{2,0} + \varepsilon_{2,1} + \varepsilon_{3,0} + \varepsilon_{3,1} + \varepsilon_{4,0} \quad \text{and} \\
(5) \quad a - 2 &= \varepsilon_{1,0} + \varepsilon_{2,0} + \varepsilon_{3,0} + \varepsilon_{4,0}.
\end{align*}
\]

These equations implies

\[
\begin{align*}
(6) \quad 4 &= \varepsilon_{1,0} + \varepsilon_{1,1} + \varepsilon_{1,2} + \varepsilon_{2,0} + \varepsilon_{2,1} + \varepsilon_{3,0} \quad \text{by (1) and (2),} \\
(7) \quad 2 &= \varepsilon_{1,0} + \varepsilon_{1,1} + \varepsilon_{1,3} + \varepsilon_{2,0} \quad \text{by (1) and (3),} \\
(8) \quad 2 &= \varepsilon_{1,2} + \varepsilon_{2,1} + \varepsilon_{3,0} - \varepsilon_{1,3} \quad \text{by (2) and (3), and} \\
(9) \quad 2 &= \varepsilon_{1,1} + \varepsilon_{2,1} + \varepsilon_{3,1} - \varepsilon_{1,0} \quad \text{by (4) and (5).}
\end{align*}
\]

The case for \( \varepsilon_{3,1} = 0 \): In this case, we see that \( \varepsilon_{1,1} = \varepsilon_{2,1} = 1 \) and \( \varepsilon_{1,0} = 0 \) by (9). Then,

\[
2 = \varepsilon_{1,2} + \varepsilon_{2,0} + \varepsilon_{3,0} \quad \text{by (6) and} \quad \varepsilon_{1,3} + \varepsilon_{2,0} = 1 \quad \text{by (7).}
\]

- If \( \varepsilon_{1,3} = 1 \), then \( \varepsilon_{2,0} = 0 \), and so \( \varepsilon_{1,2} = \varepsilon_{3,0} = 1 \), and obtain a monomial
  \( t_{1,1}t_{2,1}t_{2,0}l_{1,3} \) at degree \( (p^3 + 3p^2 + 3p + 1)q \), which yields the element
  \( l_0h_1h_3 \).

- If \( \varepsilon_{1,3} = 0 \), then \( \varepsilon_{2,0} = 1 \), and so \( \varepsilon_{1,2} + \varepsilon_{3,0} = 1 \).

  - If \( \varepsilon_{1,2} = 1 \), then the monomial has a factor \( t_{1,1}t_{2,1}t_{2,0}l_{1,2} \) of degree
    \( (2p^2 + 3p + 1)q \), and so we obtain
      \[
      t_{1,1}t_{2,1}t_{2,0}l_{1,2}t_{2,2} \quad \text{at} \quad a = 3, \quad \text{and}
      \]
      \[
      t_{1,1}t_{2,1}t_{2,0}l_{1,2}t_{4,0} \quad \text{at} \quad a = 4.
      \]
      The first monomial gives us the element \( l_0g_2 = l_0g_0 \in H^{3,1}(U(L)) \).
      We name the second monomial \( x_1 \).

  - If \( \varepsilon_{1,2} = 0 \), then \( \varepsilon_{3,0} = 1 \), and the monomial has a factor \( t_{1,1}t_{2,1}t_{2,0}l_{3,3} \)
    of degree \( (2p^2 + 4p + 2)q \), and so the monomial is \( t_{1,1}t_{2,1}t_{2,0}l_{3,0}t_{2,2} \)
    at degree \( (p^3 + 3p^2 + 4p + 2)q \). We name it \( x_2 \).

The case for \( \varepsilon_{3,1} = 1 \): In this case, \( \varepsilon_{1,1} = \varepsilon_{2,2} = \varepsilon_{4,0} = 0 \) by (2). By (9),

\[
1 = \varepsilon_{1,1} + \varepsilon_{2,1} - \varepsilon_{1,0}.
\]

- If \( \varepsilon_{1,0} = 1 \), then \( \varepsilon_{1,1} = \varepsilon_{2,1} = 1 \), and the monomial has a factor
  \( t_{1,0}t_{1,1}t_{2,1}l_{3,1} \) of degree \( (p^3 + 2p^2 + 3p + 1) \). Therefore, we have monomials
  \( t_{1,0}t_{1,1}t_{2,1}t_{3,1}l_{3,3} \) at \( a = 3 \) and \( t_{1,0}t_{1,1}t_{2,1}l_{3,0}t_{3,1} \) at \( a = 4 \). The first
  monomial corresponds \( l_0h_2h_0 \). By Table 2.9, we see that \( l_0h_2 = 0 \) and the
  monomial yields nothing. We name the second one \( x_3 \).

- If \( \varepsilon_{1,0} = 0 \), then \( 1 = \varepsilon_{1,1} + \varepsilon_{2,1} \). This together with (6) implies \( 3 = \varepsilon_{1,2} + \varepsilon_{2,0} + \varepsilon_{3,0} \), and we obtain \( \varepsilon_{1,2} = \varepsilon_{2,0} = \varepsilon_{3,0} = 1 \). By (8), \( \varepsilon_{2,1} = 0 \), and so \( \varepsilon_{1,1} = 1 \). Therefore, we have \( t_{1,1}t_{1,2}l_{2,0}l_{3,0}l_{3,1} \) at degree
  \( (p^3 + 3p^2 + 4p + 2)q \). We name it \( x_4 \).

Now put

\[
\tilde{x}_1 = t_{1,1}t_{2,1}t_{2,0}l_{1,2}l_{3,1}l_{1,0} \quad \tilde{x}_2 = t_{1,1}t_{2,1}t_{2,0}l_{1,2}l_{3,1}l_{3,0}
\]

Then,

\[
d(x_1) = \tilde{x}_1 + \tilde{x}_2, \quad d(x_2) = -\tilde{x}_2, \quad d(x_3) = -\tilde{x}_1 \quad \text{and} \quad d(x_4) = -\tilde{x}_1 + \tilde{x}_2,
\]
and
\[ d(t_{1,1}t_{2,1}t_{3,0}t_{4,0}) = -x_1 - x_3 - x_2 \quad \text{and} \quad d(t_{2,1}t_{2,0}t_{3,0}t_{3,1}) = -x_2 + x_3 - x_4. \]

Thus, the elements \( x_i \) for \( i = 1, 2, 3, 4 \) yield no element of \( H^5(p^3 + 3p^2 + 4p + 2)(U(L)) \).

We also have
\[
\begin{align*}
\frac{d}{dt}(t_{1,1}t_{2,1}t_{3,0}t_{4,0}) - t_{2,0}t_{2,1}t_{3,1} & = -t_{1,1}t_{2,1}(t_{3,1} + t_{2,2}t_{2,0} + t_{1,3}t_{3,0}) - t_{1,1}t_{1,0}t_{2,1}t_{3,1} \\
+ t_{2,0}t_{2,1}t_{2,2}t_{1,1} & = -2t_{2}g_{2} + t_{4}h_{3}h_{1}.
\end{align*}
\]

\( H^5_{\cdot,q}(V(L)) \) for \( t = (p^3 + 3p^2 + ap - a - 2) \) with \( a = 3 \) or \( 4 \) also contains elements obtained from the \( E_{1} \)-term of the May spectral sequence (2.7):
\[
\begin{align*}
b_{1,0}H^3, t^{q}(U(L)) & \quad \text{for } t' = t - p = (p^3 + 3p^2 + (a - 1)p + a - 2), \text{ and} \\
b_{1,0}^2H^1, t^{q}(U(L)) & \quad \text{for } t'' = t - 2p = (p^3 + 3p^2 + (a - 2)p + a - 2).
\end{align*}
\]

The latter module is trivial. We have a monomial of the complex \((E(t_{i,j}))^3, t^{q}\): \( t_{2,1}t_{3,0}t_{4,0} \) \( (t' = p^3 + 3p^2 + 3p + 2) \), on which the differential acts by \( d(t_{2,1}t_{3,0}t_{4,0}) = t_{2,1}t_{2,1}t_{3,0} + \cdots \neq 0 \), and this monomial yields no element of \( H^3, t^{q}(U(L)) \). Thus there is no element in these modules.

From Table (2.9), we find no element of the form \( x_{b_{i,j}}b_{k,l} \) or \( x_{b_{i,j}} \) for \( x \in H^*(U(L)) \) in our degree. \( \square \)

For studying the Adams-Novikov \( E_2 \)-term, we consider the Novikov spectral sequences
\[
\begin{align*}
E_1 & = \text{Ext}_T(\mathbb{Z}/p, Q) \Rightarrow E_2^*(V(0)) \\
(\text{cf. } [1, \text{ Lemme in p. 61}]) \quad \text{and} \\
E_1 & = \mathbb{Z}/p[v_n] \otimes \text{Ext}_T(\mathbb{Z}/p, Q(n + 1)) \Rightarrow \text{Ext}_T(\mathbb{Z}/p, Q(n)) \\
(\text{cf. } [1, (1.4.3)]). \quad \text{Here,}
\end{align*}
\]
\[
\begin{align*}
Q & = \mathbb{Z}/p[v_1, v_2, \ldots ] \quad \text{and} \quad Q(n) = Q/(v_1, \ldots , v_{n-1})
\end{align*}
\]

are comodules with coactions given by
\[
\eta(v_n) = \sum_{i=0}^{n} v_i t_{n-i}^{p^i}.
\]

We note that
\[
\text{Ext}_T(\mathbb{Z}/p, Q(5)) = H^*(T)
\]
in our range.

Among the generators (2.10) of \( H^*(U(L)) \), the elements \( g_i \) and \( k_i \) for \( i \geq 0 \), \( t_2, t_4 \) and \( t_6 \) survive to the Adams-Novikov \( E_2 \)-term, \( E_2^*(V(2))_p \) by the Massey products
\[
\begin{align*}
g_i & = (h_i, h_i, h_{i+1}), \quad k_i = (h_i, h_{i+1}, h_{i+1}), \\
l_2 & = (h_0, h_1, g_1), \quad l_4 = -2(h_2, h_2, k_0) \quad \text{and} \quad l_6 = (h_1, h_2, g_2).
\end{align*}
\]

These satisfy
\[
\begin{align*}
g_i & = (h_{i+1}, h_i, h_i), \quad 2g_i = - (h_i, h_{i+1}, h_i) \quad \text{and} \quad 2k_i = - (h_{i+1}, h_i, h_{i+1}) \\
\text{for } i \geq 0.
\end{align*}
\]

By a juggling theorem of the Massey products, we also see that \( h_ig_i = 0 \), \( h_{i+1}g_i = h_ik_i \) and \( g_{i+2} = 0 \).
We moreover have elements of the $E_2^{*,*}(V(2)_p)$:

$$\begin{align*}
(2.18) \quad v_3 h_2 &= (v_2, h_2, h_2) \quad \text{and} \quad x b_{2,0} = \left< x, (h_1, h_2), \left( \frac{-b_1}{b_0} \right) \right>
\end{align*}$$

for an element $x \in E_2^{*,*}(V(2)_p)$ with $x h_1 = 0 = x h_2$. Hereafter, we write $b_i$ for the homology class of $b_{1,i}$ (see also (3.3)). For example, $x = h_1, h_2, g_2$ and $k_1 b_2$. Indeed, $k_1 h_2 b_1 = g_1 h_2 b_2 = g_1 h_3 b_1 = 0$.

**Lemma 2.19.** For the spectra $V(2)_k$ in (2.6), some of the Adams-Novikov $E_2$-terms are given as follows:

$$E_2^{3, (2p^2 + p)q}(V(2)_3) = \mathbb{Z}/p[h_2 b_{2,0}] \quad \text{and} \quad E_2^{2p, (3p^2 + p)q}(V(2)_{p-1}) = 0.$$ 

**Proof.** For $t \leq 2p^2 + 3p + 2$, $E_2^{*,*}(V(2)_3)$ is a subquotient of $\mathbb{Z}/p[v_2, v_3] \otimes H^*(T)$ by the spectral sequences (2.12) and (2.13), and $H^*(T)$ is a subquotient of $P(b_{1,i}) \otimes H^*(U(L))$ by the May spectral sequence.

We pick generators with given bidegrees out of the module $\mathbb{Z}/p[v_2, v_3] \otimes P(b_{1,i}) \otimes H^*(U(L))$ as in the following table, where $a, b \in \{0, 1, 2\}$ and $x \in H^*(U(L))$.

<table>
<thead>
<tr>
<th>bidegree</th>
<th>$a, b$</th>
<th>$\dim x$</th>
<th>$x$</th>
<th>generators</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(3, (2p^2 + p)q)$</td>
<td>$v_2^2 v_3^2 x$</td>
<td>$a = b = 0$</td>
<td>3</td>
<td>$h_2$</td>
</tr>
<tr>
<td></td>
<td>$v_2^2 v_3^2 x b_{1,j}$</td>
<td>$a = b = 0$</td>
<td>1</td>
<td>$h_2 b_{2,0}$</td>
</tr>
</tbody>
</table>

By (2.18), the element $h_2 b_{2,0}$ yields an element of the Adams-Novikov $E_2$-term. We easily find only one element $k_1$ of bidegree $(2, (2p^2 + p)q)$ in $\mathbb{Z}/p[v_2, v_3] \otimes P(b_{1,i}) \otimes H^*(U(L))$. This is an element of $E_2^{2, (2p^2 + p)q}(V(2)_3)$, and no differential hits $h_2 b_{2,0}$ in any above spectral sequences. Therefore, $h_2 b_{2,0}$ survives to the $E_3$-term $E_3^{3, (2p^2 + p)q}(V(2)_3)$.

To turn to the second. A monomial of bidegree $(2p, (3p^2 + p)q)$ of $\mathbb{Z}/p[v_2, v_3] \otimes P(b_{1,i}) \otimes H^*(U(L))$ has one of the forms $v_2^a v_3^b x b_{2,0}^{b_{1,0} - 2 - \frac{1}{2} \dim x}$, $v_2^a v_3^b x b_{2,0}^{b_{1,0} - 2 - \frac{1}{2} \dim x}$, $v_2^a v_3^b x b_{2,0}^{b_{1,0} - 2 - \frac{1}{2} \dim x}$, $v_2^a v_3^b x b_{2,0}^{b_{1,0} - 1 - \frac{1}{2} \dim x}$, $v_2^a v_3^b x b_{2,0}^{b_{1,0} - 1 - \frac{1}{2} \dim x}$, and $v_2^a v_3^b x b_{2,0}^{p - \frac{1}{2} \dim x}$.

The degrees of these elements are

<table>
<thead>
<tr>
<th>monomials</th>
<th>degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_2^a v_3^b x b_{2,0}^{b_{1,0} - 2 - \frac{1}{2} \dim x}$</td>
<td>$q \left( (p + 1)a + (p^2 + p + 1)b + \deg x + 3p^2 - \frac{2}{3} \dim x \right)$</td>
</tr>
<tr>
<td>$v_2^a v_3^b x b_{2,0}^{b_{1,0} - 2 - \frac{1}{2} \dim x}$</td>
<td>$q \left( (p + 1)a + (p^2 + p + 1)b + \deg x + 3p^2 - p - \frac{2}{3} \dim x \right)$</td>
</tr>
<tr>
<td>$v_2^a v_3^b x b_{2,0}^{b_{1,0} - 2 - \frac{1}{2} \dim x}$</td>
<td>$q \left( (p + 1)a + (p^2 + p + 1)b + \deg x + 2p - \frac{2}{3} \dim x \right)$</td>
</tr>
<tr>
<td>$v_2^a v_3^b x b_{2,0}^{b_{1,0} - 1 - \frac{1}{2} \dim x}$</td>
<td>$q \left( (p + 1)a + (p^2 + p + 1)b + \deg x + 2p^2 - \frac{2}{3} \dim x \right)$</td>
</tr>
<tr>
<td>$v_2^a v_3^b x b_{2,0}^{b_{1,0} - 1 - \frac{1}{2} \dim x}$</td>
<td>$q \left( (p + 1)a + (p^2 + p + 1)b + \deg x + 2p - \frac{2}{3} \dim x \right)$</td>
</tr>
<tr>
<td>$v_2^a v_3^b x b_{2,0}^{p - \frac{1}{2} \dim x}$</td>
<td>$q \left( (p + 1)a + (p^2 + p + 1)b + \deg x + p^2 - \frac{2}{3} \dim x \right)$</td>
</tr>
</tbody>
</table>

Since the degree is $(3p^2 + p)q$, we see that $\deg x/q \equiv -a - b \mod p$, and deduce that $a = b = 0$. Indeed, $\deg x/q \equiv d \mod p$ with $0 \leq d \leq 3$, $0 \leq a < p - 1$ and $0 \leq b \leq 2$.

Thus, $x = g_1 b_1$, and we have a candidate $g_1 b_2 b_0^{b_{1,0} - 2}$ for a generator. Note that $d_{2p-1}(g_1 b_2 b_0^{b_{1,0} - 2}) = g_1 h_2 b_0^{p - 1} = h_1 k_1 b_0^{p - 1}$ in the second May spectral sequence in (2.7). Since $h_1 k_1 \neq 0$ by Table 2.9, we have no generator at the degree.

**Lemma 2.20.** We have a non-zero element $v_2^2 v_3^2 b_0 b_2^2 \in E_2^{6, (3p^2 + 4p + 2)q}(V(2)_3)$.
Proof. Put \( t_0 = p^3 + 3p^2 + 4p + 2 \). We consider the element \( v_2^2 v_3^2 b_0 b_2^3 \in E^{6, t_0}_{V(2), 3} \) by the spectral sequences (2.7), (2.12) and (2.13). For this sake, we compute the Ext group \( E = \text{Ext}^{5, t_0}_{V}(Z/p, Q(2)) \) for the comodule \( Q(2) \) in (2.14). We study whether or not the element \( v_2^2 v_3^2 b_0 b_2^3 \) is in the image of a differential of the spectral sequences, and so it suffices to consider modules

\[
M(a, b, c) = \langle v_2^2 v_3^2 H^{5, t_0}(V(L)) \rangle^{5, t_0} \subset (P(v_2, v_3, v_4)/(v_3^3)) \otimes H^{5, t_0}(V(L))^{5, t_0}.
\]

We read off from Table 2.9 and Lemma 2.11, the module

\[
\begin{align*}
Z/p[v_4 q b_1] & \quad (a, b, c) = (0, 0, 1) \\
Z/p[v_3 v_4 h_2 b_0^3] & \quad (a, b, c) = (0, 1, 1) \\
Z/p[v_2 v_4 h_2 b_0 b_2, v_2 v_3 h_1 b_1 b_2] & \quad (a, b, c) = (1, 0, 1) \\
Z/p[v_1 v_2 b_1] & \quad (a, b, c) = (0, 1, 0) \\
Z/p[v_2 v_3 h_2 b_2, v_2 v_3 h_1 b_2, v_2 v_3 h_1 b_2, v_2 v_3 h_1 b_3] & \quad (a, b, c) = (1, 1, 0) \\
Z/p[v_2 v_3 h_2 b_2] & \quad (a, b, c) = (0, 2, 0) \\
Z/p[v_2 v_3 h_2 b_2] & \quad (a, b, c) = (1, 0, 0) \\
Z/p[v_2 v_3 h_2 b_2] & \quad (a, b, c) = (1, 2, 0) \\
Z/p[v_2 v_3 h_2 b_2] & \quad (a, b, c) = (2, 0, 0) \\
M(a, b, c) & \quad (a, b, c) = (2, 0, 0) \text{ otherwise.}
\end{align*}
\]

Here, we write \( A \subseteq B \) if \( A \) is a subquotient of \( B \). Let \( E(a, b, c) \) denote a submodule of \( E \) generated by elements detected by elements of \( M(a, b, c) \). We first verify which of the elements on the right hand side of the above relation yields an element of \( M(a, b, c) \), and then evaluate \( E(a, b, c) \) by the spectral sequences (2.13).

We consider the second spectral sequence (2.7). Note that the May filtration of the elements \( h_{i, j} \) and \( h_{i, j} \) are \( 2i - 1 \) and \( p(2i - 1) \), respectively. Then, the May differential \( d_{2p-1} : E_{2p-1}^{r, s} \to E_{2p-1}^{r+1, s-2p+1} \) of the spectral sequence acts as

\[
\begin{align*}
d_{2p-1}(b_{2i}) &= b_{1i} h_{i+1} - h_{i+1} b_{1i+1} & \quad \text{for } i \geq 0, \\
d_{2p-1}(b_{2i}) &= -h_{1i} b_{2i+1} + b_{2i} h_{3} & \quad \text{by (2.3).}
\end{align*}
\]

We start from the modules \( M(0, 1, 1) \), \( M(1, 0, 1) \), \( M(1, 1, 0) \) and \( M(2, p, 0) \). By (2.21), \( b_2^2 = h_1 b_0 b_1, b_2 b_0 b_2 = h_1 b_0 b_2, \) and \( b_2 b_0 b_2 = h_2 b_2^2 \) in \( H^r(V(L)) \), and

\[
\begin{align*}
d_{2p-1}(h_2^2) &= -2h_3(b_{10} h_2 - h_{11} b_{10}) b_{20} = 2h_3 h_{11} b_{12}, \\
d_{2p-1}(h_1 b_2, b_{30}) &= h_3 h_{11} b_{12}, \\
d_{2p-1}(h_1 b_{11}, b_{30}) &= -h_2 h_{11} b_{11} + h_{12} b_{11}, \\
d_{2p-1}(h_2 b_{11}, b_{30}) &= -h_2 h_{12} b_{11} + h_{13} b_{11}, \\
d_{2p-1}(h_1 b_{20}, b_{30}) &= (b_{10} h_2 - h_{11} b_{10}) b_{10} + b_{20} h_{11} b_{10} \\
&= h_{11} b_{11} b_{10} - h_{12} b_{11} b_{12} + h_{13} b_{11} b_{12}.
\end{align*}
\]

These differentials imply that the rank of the module \( M(1, 1, 0) \) is not greater than three. Therefore, \( M(0, 1, 1) \subseteq Z/p[v_4 q b_2 b_0^3] \), \( M(1, 0, 1) \subseteq Z/p[v_2 v_3 h_2 b_0 b_0 b_0] \), \( M(1, 1, 0) \subseteq v_2 v_3 (Z/p h_2 b_0 b_0 b_0, h_2 b_0 b_0, h_1 b_0 b_2 - h_{11} b_{10}, h_1 b_{12}) \) and \( M(2, p, 0) \subseteq
$\mathbb{Z}/p\{v_2^2 b_0 h_2 b_0 b_1\}$. Furthermore, we have $d_{4p-3}(b_2 b_1 h_2 b_3, 0) = -h_2 b_1 h_2(b_1 h_2 - h_2 b_1, 1, 2)$ and $d_{4p-3}(b_1 b_0 b_2 b_1, h_1 b_1, 0) = h_1 b_1 b_2 b_1, h_1 b_1 b_2 b_1, 0)$, we obtain $M(1, 1, 0) \subseteq \mathbb{Z}/p\{v_2 v_3 k_1 h_1 b_2\}$.

Consider the spectral sequence (2.13). The differentials of the spectral sequences are read off from the structure map (2.15). For example, $d_1(v_3) = v_3 h_3$ for $n = 3$ and $d_1(v_3) = v_2 h_2$ for $n = 2$. For $M(0, 1, 1)$, noticing that $v_3 h_2$ is represented by a cocycle $v_3 t_1^p + v_3 c(t_1^p)$ in the cobar complex $Q(2) \otimes \mathcal{T}$, we compute

$$d(v_3 t_1^p + v_3 c(t_1^p) + v_2 t_1^p t_2^p) = v_3 t_1^p \otimes t_1^p + v_2 t_2^p \otimes t_1^p + v_3 t_1^p \otimes c(t_1^p) - v_3 t_1^p \otimes t_1^p - v_2 t_2^p \otimes t_1^p - v_2 t_2^p \otimes t_1^p + t_2^p$$

in which the underlined terms with a subscript cancel each other out. The cocycle $2t_1^p \otimes t_1^p + t_2^p \otimes t_1^p$ appearing in the right hand side of the above computation represents $2g_2 \neq 0$ in $\text{Ext}_T(\mathbb{Z}/p, Q(3))$ (see (2.14) for $Q(3)$). It follows that $v_3 h_2$ does not survive to $\text{Ext}_T(\mathbb{Z}/p, Q(2))$ in (2.13). Thus, $E(0, 1, 1) = 0$.

For $M(1, 0, 1)$, we compute

$$h_3 h_2 b_2, 0 = h_3 \left\langle h_2, (h_1, h_2), \left\langle -b_1, b_0 \right\rangle \right\rangle$$

(2.22)

$$= \left\langle \langle h_3, h_2, h_1 \rangle, \langle h_3, h_2, h_2 \rangle \right\rangle \left\langle -b_1, b_0 \right\rangle = g_2 b_0$$

by the juggling theorem in the $E_{2p}$-term of the second spectral sequence in (2.7) by (2.18) and (2.17). We also note that $\langle h_3, h_2, h_1 \rangle = 0$ by considering $d(t_3^2)$. Therefore, $d_1(v_3 h_2 b_2, 0) = v_3 g_2 b_0$ in the spectral sequence (2.13) for $n = 3$, and $E(1, 0, 1) = 0$ follows.

In the spectral sequence (2.13) for $n = 2$, we compute

$$d_1(v_2^2 g_1 b_2) = 2v_2 v_3 h_2 g_1 b_2 = 2v_2 v_3 k_1 h_1 b_2$$

and

$$d_1(v_2^2 g_1 b_0 b_1) = v_2^2 v_3 h_2 b_0 b_1,$$

where we use the well known relation $g_1 h_2 = h_1 k_1$. Therefore, the triviality of $E(1, 1, 0)$ and $E(2, p, 0)$ follows.

Since $h_2 l_2 = 0 = h_3 l_2$ by Table 2.9, we see that

$$l_2 b_2, 1 = \left\langle l_2, (h_2, h_3), \left\langle -b_1, b_1 \right\rangle \right\rangle$$

in $H^*(V(L))$ in the same manner as (2.18). Note that $\langle h_2, l_2, h_2 \rangle = 2l_4 h_1$ and $\langle h_2, l_2, h_3 \rangle = 0$ in $H^*(V(L))$. Therefore, in the spectral sequence (2.13) for $n = 2$, we compute $d_1(v_3 l_2 b_2, 1) = -v_3 l_2 h_1 b_2 \neq 0$ and so $E(0, 1, 0) = 0$.

Since $d_{2p-1}(b_3, b_1, 0) = (-h_1 b_2, 1 + b_2, h_3) b_1, 0)$ and

$$d_{2p-1}(h_1 b_2, 1, b_1, 0) = -h_1 (b_1, h_3 - h_2 b_1, 2) b_1, 0) = -h_3 h_1 b_1, 1 b_1, 0),$$

we see that $M(0, 2, 0) \subseteq \mathbb{Z}/p\{v_2^2 h_1 b_2, 0 b_2\}$. In the spectral sequence (2.13) for $n = 2$,

$$d_1(v_3^2 h_1 b_2, 0) = 2v_2 v_3 h_2 \left\langle h_1, (h_1, h_2), \left\langle -b_1, b_0 \right\rangle \right\rangle$$

$$= 2v_2 v_3 \left\langle h_2, (h_1, h_2), \left\langle -b_1, b_0 \right\rangle \right\rangle = 2v_2 v_3 (g_1 b_1 - 2b_1 b_0)$$

by (2.17) and (2.18). It follows that $E(0, 2, 0) = 0$. 

In the spectral sequence in (2.7), $d_{2p-1}(k_1b_{3,0}) = k_1(-h_1b_{2,1} + b_{2,0}h_3) = -k_1h_1b_{2,1}$ and $k_1h_1b_{2,1} = 0 \in H^*(V(L))$. By (2.3), we compute the differential $d(t_1^p \otimes b_{2,0} \otimes b_{3,0})$ in the cobr complex for computing $H^*(V(L))$, and deduce that

$$d_{4p-3}(h_2b_{2,0}b_{3,0}) = h_2b_2h_1(b_1,0h_2) - h_{1,2}b_{1,2} = g_2b_2b_0h_1 - k_1b_1b_{2,0}$$

in the spectral sequence. Here, $xb_{2,0}$ for $x = g_2$, $k_1b_2$ are given in (2.18). Thus, $M(2,0,0) \subseteq \mathbb{Z}/p\{v_2^b\}$. We have $M(1,p,0) = 0$ and $M(0,p+1,0) = 0$, since $d_{2p-1}(h_0b_{2,0}) = -h_0(b_1,0,0h_2 - h_{1,1}) = h_2h_0b_{1,0}$.

Therefore, $E(1,p,0) = 0$ and $E(0,p+1,0) = 0$.

Therefore, Ext$_\mathbb{Z}/p\{v_4b_2b_1, v_4b_2h_3h_1, v_2^bh_0\}$.

We consider the element $v_4b_2$. By (2.16), $l_2 \in E_2^{*,*}(V(2)_2)$. Let $l_2$ denote a cocycle representing $l_2$ in the cobr complex for computing $E_2^{*,*}(V(2)_2)$. By Table 2.9 together with (2.16), we see that $h_0l_2 = 0$ and $h_3l_2 = 0$, and so we have cochains $y_i$ such that $d(y_i) = t_1^p \otimes l_2$ for $i = 0, 3$ in the cobr complex. Then,

$$d(v_4l_2 - v_3y_3 + v_3^p\gamma) = v_3t_1^p \otimes l_2 - v_3t_1 \otimes l_2 + v_2v_3^p \otimes l_2 - v_2t_1^p \otimes y_3$$

Since $t_1^p \otimes l_2 - t_1^p \otimes y_3$ represents an element of the Massey product $\langle h_2, h_3, l_2 \rangle$, which belongs to $H^{4, (p^2 + 3p + 3)/2}(U(L))$. Therefore, we deduce that $\langle h_2, h_3, l_2 \rangle = 0$ by Table 2.9, and so we have a cochain $z$ such that $d(z) = t_1^p \otimes l_2 - t_1^p \otimes y_3$. Now the element $v_4l_2b_1$ yields an element of $E_2^{*,*}(V(2)_3)$ represented by $(v_4l_2 - v_3y_3 + v_3^p\gamma) \otimes b_1$.

The other generators of the module are represented by the Massey products

$$-2v_4\langle h_2, h_3, l_2 \rangle h_3h_1 \quad \text{and} \quad v_2^p\langle h_1, h_2, g_2 \rangle b_0$$

in the Adams-Novikov $E_2$-term $E_2^{*,*}(V(2)_3)$ (cf. (2.16)). Therefore, the differentials of (2.12) on these generators act trivially, and $v_2^p\gamma b_0b_1^2$ is not in the image of any differentials of the spectral sequences. 

\section{On the Product $\alpha_1\beta_2\gamma_{p+2}$}

We recall the definition of the Greek letter elements. The Greek letter elements in the homotopy groups $\pi_*(S^0)$ are defined by composites

\begin{equation}
\alpha_s = j\alpha^s_i, \quad \beta_s = j j_1 j_2^s \iota_1 i \quad \text{and} \quad \gamma_s = j j_1 j_2 \gamma^s \iota_2 \iota_1 i
\end{equation}

for the maps in (2.5) and a map $\gamma: \Sigma^{(p^2 + 3p + 3)/2}V(2) \to V(2)$ inducing a multiplication by $v_3$ on $BP$-homologies given by Toda [13]. We notice that $(i \wedge V(0))\alpha_i \iota_1 \iota_1 i = v_3 \in BP_i$, $(\iota \wedge V(1))\beta_i \iota_1 i \in v_2^p \in BP_i$ and $(i \wedge V(2))\gamma \iota_2 \iota_1 i \iota_1 i = v_2^p \in BP_i$ for the unit map $i: S^0 \to BP$ of the ring spectrum $BP$. Then by the Geometric Boundary Theorem (cf. [10, Th. 2.3.4]), the Greek letter elements (3.1) are detected by those in the Adams-Novikov $E_2$-term defined by

\begin{equation}
\alpha_s = \delta_0(v_3^s) \in E_2^{*,q}(S^0), \quad \beta_s = \delta_0\delta_1(v_2^s) \in E_2^{*(p^2 + 3p + 3)/2}(S^0) \quad \text{and} \quad \gamma_s = \delta_0\delta_1\delta_2(v_1^s) \in E_2^{*(p^2 + 3p + 3)/2}(S^0).
\end{equation}
Lemma 3.6. Here $\delta_k: E_2^{*,*}(V(k)) \to E_2^{*,*+1}(V(k-1))$ denotes the connecting homomorphism associated to the cofiber sequences in (2.5) $(V(-1) = S^0)$. Traditionally we put

\begin{equation}
(3.3) \quad h_i = [\mu^i_1] \in E_2^{p,i,q}(S^0) \quad \text{and} \quad b_i = [b_{1,i}] \in E_2^{p+1,i,q}(S^0),
\end{equation}

where $[c]$ denotes the cohomology class of a cocycle $c \in \Omega^{*,*}BP_*$. We note that $h_i$ corresponds to $h_i$ in Table 2.9. Then, by definition, we have well known relations (cf. [4], [10]):

\begin{equation}
(3.4) \quad \Pi_1 = h_0, \quad \Pi_1 \equiv b_0 \quad \text{and} \quad \Pi_2 \equiv 2v_2b_0 + k_0 \mod I_2
\end{equation}
in the $E_2$-term. Furthermore, it is also shown in [4, Lemma 4.3] that

\begin{equation}
(3.5) \quad \gamma_t = 2\left(\frac{t}{2}\right)v_3^{-2}h_2b_{2,0} + 3\left(\frac{t}{3}\right)v_3^{-2}l_4 \mod I_3 = (p,v_1,v_2)
\end{equation}
in $E_2^{3,(t^2+(t-1)p+q+2)q}(S^0) = H_2^{3,(t^2+(t-1)p+q+2)q}(BP_*)$, where $h_2b_{2,0}$ and $l_4$ are given in (2.18) and (2.16). By Lemma 2.19, we have

Lemma 3.6. $\gamma_2 = 2h_2b_{2,0} \neq 0 \in E_2^{3,(2p^2+2p)q}(V(2)_3)$.

Lemma 3.7. The element $\tau_{p+1} \in E_2^{3,(p^2+3p^2+2p)q}(S^0)$ satisfies that $\tau_{p+1} \equiv v_3^p\gamma_2$ mod $(p,v_1,v_2)$.\hfill\Box

Proof. The relation $\tau_{p+2} \equiv v_3^p\gamma_2$ follows from computation:

\begin{align*}
\delta_2(v_3^p) & \equiv v_3^p\delta_2(v_3^2) \mod (v_3^6). \\
\delta_1\delta_2(v_3^p) & \equiv \delta_1(v_3^p\delta_2(v_3^2) + v_3^2x) \equiv v_3^p\delta_1\delta_2(v_3^2) \mod (v_3^3,v_2^2). \\
\delta_0\delta_1\delta_2(v_3^p) & \equiv \delta_0(v_3^p\delta_1\delta_2(v_3^2) + v_3^2y + v_3^2z) \equiv v_3^p\delta_0\delta_1\delta_2(v_3^2) \mod (p,v_1,v_2),
\end{align*}

for elements $x \in E_2^{1,+}(V(1))$, and $y, z \in E_2^{2,+}(V(0))$.\hfill\Box

Lemma 3.8. For the spectrum $V(2)_3$ in (2.6), we have

$h_0k_0\gamma_2 = 0 \in E_2^{6,(2p^2+3p+2)q}(V(2)_3)$.

Proof. By the juggling Theorem of the Massey products, (2.18) and Lemma 3.6, we compute

\begin{align*}
h_0k_0\gamma_2 & = g_0h_1\gamma_2 = 2g_0(\langle h_1, h_2, h_1 \rangle, \langle h_1, h_2, h_2 \rangle) \left(\begin{array}{c} -b_1 \\ b_0 \end{array}\right) \\
& = 4g_0b_1 + 2g_0k_1b_1 = 0
\end{align*}
in $E_2^{6,(2p^2+3p+2)q}(V(2)_3)$. Indeed, $\langle h_1, h_2, h_1 \rangle = -2g_1$ by (2.17), and $g_0g_1 = 0 = g_0k_1$. Therefore, the lemma follows.\hfill\Box

Lemma 3.9. In the Adams-Novikov $E_2$-term,

$\alpha_1\beta_2\gamma_2 = 4v_2^2b_0b_1^2 \in E_2^{6,(2p^2+3p+2)q}(V(2)_3)$.
Proof. By (3.4) and Lemma 3.8, we see that \( \pi_1\beta_2\gamma_2 = 2v_2\pi_1\beta_1\gamma_2 \), which is congruent to \( 4v_2h_0b_2h_2b_0,0 \) modulo \( (p,v_1,v_2^2) \) by Lemma 3.6. We compute
\[
\frac{1}{2}v_2\pi_1\beta_1\gamma_2 = v_2h_0b_0\left( h_2, (h_1, h_2), \left( \begin{array}{c} -b_1 \\ b_0 \end{array} \right) \right) \\
= h_0b_0\left( v_2, (h_1, h_2) \right) \left( \begin{array}{c} -b_1 \\ b_0 \end{array} \right) \\
= h_0b_0\left( \langle v_2, h_2, (h_1, h_2) \rangle \right) \left( \begin{array}{c} -b_1 \\ b_0 \end{array} \right) \\
= h_0b_0\left( -\langle v_2, h_2, h_1 \rangle b_1 + \langle v_2, h_2, h_2 \rangle b_0 \right) \\
= -v_2(h_2, h_1, h_0)b_1b_0 + \langle v_2, h_2, h_2 \rangle b_0b_0b_0 \\
= v_2^3b_0b_1^2 + v_3h_2h_0b_0^2.
\]
Here, the differential \( d(c(t_3)) \) (see (2.1)) gives us a relation \( \langle h_2, h_1, h_0 \rangle \equiv v_2h_1 \mod I_2 \) in the \( E_2 \)-term. We further see that \( h_2h_0b_0^2 = 0 \in H^* (V(L)) \), since \( d_{2p-1}(h_0b_1b_2,0) = h_0h_2b_1^2 \) in the May spectral sequence. □

**Theorem 3.10.** \( \pi_1\beta_2\gamma_{p+2} \neq 0 \in E_2^{6,(p^3+3p^2+4p+2)q}(S^0). \)

Proof. By Lemma 3.7, we have \( \gamma_{p+2} = v_3^p\gamma_2 \in E_2^{3,(p^3+3p^2+2p)q}(V(2)_3) \), and so
\[
\pi_1\beta_2\gamma_{p+2} = v_3^p\beta_1\gamma_{p+2} = 4v_2^p\beta_0b_1^2 \in E_2^{5,(p^3+3p^2+4p+2)q}(V(2)_3)
\]
by Lemma 3.9. Now the theorem follows from Lemma 2.20. □

Proof of Theorem 1.8. For \( t = p \) and \( p+1 \), \( \gamma_t = 0 \) by (3.5), and so the proposition holds in these cases. Suppose now \( t \geq p+2 \). Note that \( \beta_2 = [k_0] = k_0 \) and \( \gamma_t = 2(t_3^t/3)_2h_2b_0,0 + 3(t_3^t/3)_3^t - 3 t_4^t \) for \( t \geq 2 \) in \( E_2^t(V(2)) \) by (3.4) and (3.5) (cf. \[4, p. 234\], \[4, Lemma 4.3\]). Here, \( BP_*(V(2)) = BP_*/I_2 \) and \( I_2 \) denotes the generator given in \[13, p. 55\]. This implies that \( \gamma_t = v_3^p\gamma_{t-p} \) for \( t \geq p+2 \) in \( E_2^t(V(2)) \), and we also see \( v_0^p h_0 = v_3h_3 \) in \( E_3^t(V(2)) \) by \( d(v_4) \), where \( h_i \in E_2^{1,p^0q}(V(2)) \) is an element represented by a cocycle \( t_i^t \). Therefore, \( \pi_1\beta_2\gamma_t \) is represented by \( v_3^{t-p-2}h_3k_0(2(t_3^t/3)_2v_3h_2b_0,0 + 3(t_3^t/3)_3^t - 3 t_4^t) \). Here, we see that \( h_3k_0h_2b_0,0 = k_0g_2b_1,0 \) by (2.22). We also see that \( h_3k_0k_4 = h_3h_2m_1 \) for the generators in Toda’s calculation \[13, p. 55\]. Since both of \( k_0g_2 \) and \( h_3h_2 \) are zero by Toda’s calculation (see Table 2.9), these imply the triviality of \( \pi_1\beta_2\gamma_t \) for \( t \geq p+2 \). □

4. Non-triviality of \( \beta_1^{p-2}\beta_2\gamma_{p+2} \)

We begin with a recollection of some results from \[4\]: \( \Omega^{*-*}BP_\{a\} \) denotes a quotient complex of the cobar complex \( \Omega^{*-*}BP_\ast \) by a subcomplex generated by monomials \( m \in t^{E_1} \circ \cdots \circ t^{E_n} \) with \( \sum_{i=1}^n E_i > (a,0,\ldots) \). Here, \( t^E \) for a sequence \( E = (e_1,e_2,\ldots) \) denotes the monomial \( t_1^{e_1}t_2^{e_2} \cdots \in BP_\ast(BP_\ast) \), and the set of sequences admits the lexicographical ordering (cf. \[4, p.235\]).

Then, the gamma elements \( \gamma_t \) for \( t \geq 2 \) in the Adams-Novikov \( E_2 \)-term are represented by a cocycle
\[
\gamma_t \equiv -tv_2^{p-3}v_3^{t-1}k_0 \otimes t_1 \mod J_3 = (p,v_1,v_2^{p-1})
\]
in \( \Omega^{(tp^2+(t-1)p+t-2)}BP_\{p^2-1\} \) (cf. \[4, p. 239\]). In this section, we consider a spectrum \( V(2)_{p-1} \) in (2.6). Note that \( BP_\ast(V(2)_{p-1}) = BP_\ast/J_3 \).
Theorem 4.2. $β_1^{p-2}β_2^2π_{p+2}^0 \neq 0 \in E_2^{2p+1,1q}(S^0)$ for $t = p^3 + 4p^2 + 2p + 1$.

Proof. Let $G \in C = \Omega^{2p+1,1q}BP_*$ be a cocycle representing the element $β_1^{p-2}β_2^2π_{p+2}^0$. Then, $G \equiv v_4^pG_2 \bmod J_3$ for a cochain

$$G_2 = -2v_2^{-p-3}v_3\bar{k}_0 \otimes t_1 \otimes (2v_2b_{1,0} + \bar{k}_0) \otimes b_{1,0}^{(p-2)}$$

in $\overline{C} = \Omega^{2p+1,1q}BP_*(p^2 + 2)$ by (3.4) and (4.1). Note that $G_2$ is the cocchain $D$ of [4, p. 240] for $t = 2$, which is shown not to be a coboundary in $\overline{C}/J_3$. We claim that $G$ has no term with $v_4$ as a factor modulo $J_3$.

Indeed, if $G = v_4^pG_2 + v_4w + w'$ mod $J_3$ for $w, w' \in \Omega^*BP_*/(J_3 + (v_2))$, then, applying the differential $d$ to the equality, we obtain $0 = v_4^pd(G_2) + d(v_4) \otimes w + v_4d(w) + d(w')$. Since $d(G_2)$, $d(v_4)$ and $d(w')$ have no term with $v_4$, we deduce that $d(w) = 0$. Therefore, $[w] \in E_2^{2p,3(p^2+p+1)}(V(2)p-1)$, which is zero by Lemma 2.19. It follows that there is a cochain $\bar{w}$ such that $w = d(\bar{w})$. So replace $v_4w$ by $d(v_4) \otimes \bar{w}$ so that $G$ has no term with factor $v_4$ modulo $J_3$.

Suppose that there is a cocycle $y \in \Omega^{2p,1q}BP_*$ such that $d(y) = G$ in $C$. Put $y = y_1 + v_4y_2 + v_0y_3 + z$ for $y_1 = \sum_{a,b}v_3^av_3^by_{a,b} \in \Omega^{2p,*}BP_*/I_5$ and $z \in J_3\Omega^{2p,1q}BP_*$. By a similar argument showing (4.3), we replace $v_4y_2$ by a linear combination of terms without factor $v_4$. Thus we may put $y = y_1 + v_4y_3 + z$. By (2.1), we see that $\{t_i\} \in \Omega^2BP_*/J_3(p^2 + 2)$ has the only term $v_2b_{1,1}$ if $i = 3$, and $v_2b_{2,1}$ if $i = 4$ with factors $v_2$ and $v_3$. It follows that for $x \in \Omega^{2p,1q}BP_*/I_5$ with $u \leq t_i$, $d(x) \in (\mathbb{Z}/p)[1, v_2] \otimes \Omega^{2p+1,1q}BP_*/I_5\{p^2 + 2\}$ by degree reason. Indeed, $v_2^2b_{1,1} = 0 \in \Omega^{2,2e(3)p}BP_*/I_2\{p^2 + 2\}$ and $v_2^2b_{2,1}^0$ has an internal degree greater than $tq$. Since $d(v_4^p) = bv_2v_3^{b-1}t_1^2$ in $\Omega^{1,*}BP_*/J_3\{p^2 + 2\}$ by (2.1), we see that

$$d(y) = d(y_1) + v_4^pd(y_3) = v_4^pG_2 \in \Omega^{2p+1,1q}BP_*/J_3\{p^2 + 2\}.$$

Here, we notice that $d(z) \equiv 0 \bmod J_3$, since $J_3$ is an invariant ideal. From the equality, we see that $d(y_1) = 0$ and $d(y_3) = G_2 \in \Omega^{2p+1,1q}BP_*/J_3\{p^2 + 2\}$. Thus, $G_2(= D$ in [4, p. 240]) is a coboundary in the complex. This contradicts to the conclusion of the proof of [4, Th. 4.1].

Corollary 4.4. $β_1^{p-2}β_2^2γ_{p+2}^0 \neq 0 \in π_{p^3+4p^2+2p+3}(S^0)$.

Proof. By virtue of Theorem 4.2, it suffices to show that there is no element $x \in E_2^{2,2(p+1)}(S^0)$ such that $d_{2p-1}(x) = β_1^{p-2}β_2^2π_{p+2}^0$ in the Adams-Novikov spectral sequence. In [7, Th. 2.6], it is shown that the $E_2$-term $E_2^{2,2}(S^0)$ is generated by the elements $β_{sp^2/j+k+1}$ for integers $p \geq 1$, $i, k \geq 0$, $j \geq 1$, subject to $j \leq p^i$ if $s = 1$, $p^i | j \leq a_{i-k}$ and $a_{i-k-1} < j$ if $p^k | j$, where $a_0 = 1, a_n = p^{an} + p^{an-1} - 1$ for $n \geq 1$. The internal degree of the element $β_{sp^2/j+k+1}$ is $(sp^2 + p - j)q$, and we have an equation $t - 1 = sp^2(p + 1) - j$ to find the element $x$. Note that $sp^2 - j \geq 0$, and we have $2p^3 > sp^2 + 1$ and so $i \leq 2$. Thus, we obtain the only solution $(i, j, s) = (1, p, p + 3)$ of the equation. In this case, $k = 0$ by the relation $p^k | j \leq a_{i-k}$. The element $β_{(p+3)p/p}(= β_{(p+3)p/p})$ is a permanent cycle by [8]. Thus, we have no such element $x$, and hence $β_1^{p-2}β_2^2π_{p+2}^0$ is not in the image of the differential $d_{2p-1}$ of the spectral sequence. □
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