ON THE PICARD GROUP GRADED HOMOTOPY GROUPS OF
A FINITE TYPE TWO $K(2)$-LOCAL SPECTRUM AT THE
PRIME THREE

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Abstract. Consider Hopkins’ Picard group of the stable homotopy category
of $E(2)$-local spectra at the prime three, consisting of homotopy classes of
invertible spectra $[1]$. Then, it is isomorphic to the direct sum of an infinite
cyclic group and two cyclic groups of order three. We consider the Smith-Toda
spectrum $V(1)$ and the cofiber $V_2$ of the square $a^2$ of the Adams map, which is
a ring spectrum. In this paper, we introduce imaginary elements which make
computation clearer, and determine the module structures of the Picard group
graded homotopy groups $\pi_\ast(V(1))$ and $\pi_\ast(V_2)$.

1. Introduction

We work on the stable homotopy category $S(3)$ of spectra localized at the prime
three. Consider the Brown-Peterson spectrum $BP$ with coefficient algebra
$\mathbb{Z}(3)[v_1, v_2, \ldots]$ on the generators $v_i$ of degree $2 \times 3^i - 2$ for $i \geq 1$. Then, the second
Johnson-Wilson spectrum $E(2)$ is the spectrum representing the Landweber
exact functor $E(2)_\ast(X) = \mathcal{H}_2 \otimes_{BP} BP_\ast(X)$ for $E(2)_\ast = \mathbb{Z}(3)[v_1, v_2, v_2^{-1}]$ on
$X \in S(3)$. Let $L_2$ denote the full subcategory of $S(3)$ consisting of spectra localized
with respect to $E(2)$ in the sense of Bousfield. Then, we have the Bousfield local-
ization functor $L_2 :: S(3) \rightarrow L_2$, which is a retraction. A spectrum $X \in L_2$ is called
invertible if there is a spectrum $Y$ such that $X \wedge Y = L_2 S^0$ for the sphere spectrum
$S^0$. Hopkins’ Picard group $\text{Pic}(L_2)$ is defined to be a group consisting of the
homotopy equivalence classes of invertible spectra with multiplication defined by the
smash product. For an element $\lambda \in \text{Pic}(L_2)$, $S^\lambda$ denotes an invertible spectrum
that represents $\lambda$. Note that $E(2)_\ast(S^\lambda) = E(2)_\ast$, shown by Hovey and Sadofsky
[2]. In [1], Goerss, Henn, Mahowald and Rezk showed that $\text{Pic}(L_2)$ is isomorphic
to $\mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$. The generator of the summand $\mathbb{Z}$ is represented by $S^1 = \Sigma L_2 S^0$.
Let $\omega_1, \omega_2$ denote a generator of the $i$-th summand of $\mathbb{Z}/3 \oplus \mathbb{Z}/3 \subset \text{Pic}(L_2)$.
The Picard group graded homotopy groups $\pi_\ast(X)$ of a spectrum $X$ is

$$\pi_\ast(X) = \bigoplus_{\lambda \in \text{Pic}(L_2)} [S^\lambda, L_2 X]$$

Note that $S^{\omega_1 + b \omega_2 + c \omega_3 + d}$ for $a \in \mathbb{Z}$ and $b, c \in \mathbb{Z}/3$ is represented by the invertible
spectrum $S^{\omega_1}(S^{\omega_2})^b \wedge (S^{\omega_3})^c$.

Let $M$ denote the mod 3 Moore spectrum fitting in the cofiber sequence

$$S^0 \rightarrowtail S^0 \rightarrowtail M \twoheadrightarrow S^1.$$
For an integer $e \in \{1, 2\}$, we have spectra $V_e$ given by the cofiber sequence

\[(1.2) \quad \Sigma^{4e} M \xrightarrow{\alpha} M \xrightarrow{\iota} V_e \xrightarrow{\eta} \Sigma^{4e+1} M,\]

for the Adams map $\alpha$ satisfying $E(2)_*(\alpha) = v_1$. Then,

\[(1.3) \quad E(2)_*(V_e) = E(2)_*/(3, v_1^e).\]

Note that $E(2)_*(V_1) = K(2)_*$, the coefficient algebra of the second Morava $K$-theory. The spectrum $V_1$ is the first Smith-Toda spectrum $V(1)$. We note that Toda [10] showed that $V_1$ is not a ring spectrum, while Oka [6] showed that $V_2$ is a ring spectrum. We tried to determine homotopy groups of $L_2V_1 = L_2V(1)$, $V_1 \wedge S^{\omega_1}$ and $V_1 \wedge S^{\omega_2}$ ([8], [4], [3]). Unfortunately, there are some missing relations on the differential $d_9$ in [3], and the result is not correct. In this paper, we correct the result (see Remark 2.23), and furthermore, determine the additive structure of the homotopy groups of $L_2V_2$, $V_2 \wedge S^{\omega_1}$ and $V_2 \wedge S^{\omega_2}$. Our main tool is the $E(2)$-based Adams spectral sequence

\[E_2^{s,t}(X) = \text{Ext}_{E(2)_*(E(2))}^{s,t}(E(2)_*, E(2)_*(X)) \Rightarrow \pi_{-s}(L_2X)\]

for a spectrum $X$. The generators of the $E_2$-terms behave very complicated in the spectral sequences. To make the behavior clearer, we introduce some imaginary generators. In order to compute $E_r$-terms, we consider differential algebras $C_e$ for $e \in \{1, 2\}$, whose cohomologies are easily determined, so that the $E_\infty$-terms for $V_e$ are obtained from the cohomologies.

In the next section, we state our main theorem, the homotopy groups $\pi_*(V_e \wedge S^{\omega_1})$ for $e \in \mathbb{Z}/3$, after introducing the elements. We determine the $E_2$-terms $E_2^{s,t+\omega_2}(V_e)$ in section three, and the Adams-Novikov differentials $d_5$ and $d_9$ for $\pi_{s+\omega_2}(V_e)$ in section four. Sections five and six are devoted to compute the cohomologies of the differential algebras $C_1g^l$ and $C_2g^l$ for $l \in \mathbb{Z}/3$, respectively. Here, $g$ denotes a generator of $E(2)_*(S^{\omega_2})$. In the last section, we deduce our main theorems Theorems 2.22 and 2.24 from the results of the cohomologies of $C_1g^l$ and $C_2g^l$.

2. Statement of results

By the $3 \times 3$ lemma, the cofiber sequences in (1.2) give rise to another cofiber sequence

\[(2.1) \quad \Sigma^4 V_1 \xrightarrow{\pi} V_2 \xrightarrow{\iota} V_1 \xrightarrow{\eta} \Sigma^5 V_1.\]

On the generator $\omega_1 \in \text{Pic}(\mathbb{L}_2)$, we have the following

**Theorem 2.2** ([4, Th. A]). There is a homotopy equivalence $v_3^2 : \Sigma^{48} V_1 \simeq V_1 \wedge S^{\omega_1}$.

Since $\pi_{-5}(L_2V_1) = 0$ by [8, Th. 10.6] (see (4.10)), this theorem implies that $\pi_{48}(V_1 \wedge S^{\omega_1}) = 0$. It follows that $(\tilde{\eta} \wedge 1) v_3^2 \tilde{v}_1 i = 0$ for $v_3^2$ in Theorem 2.2, and so $v_3^2 \tilde{v}_1 i \in \pi_{48}(V_1 \wedge S^{\omega_1})$ is pulled back to $\pi_{48}(V_2 \wedge S^{\omega_1})$ under $(\tilde{\eta} \wedge 1)_*$. Notice that $V_2$ is a ring spectrum, and we obtain the following

**Proposition 2.3.** There is a homotopy equivalence $v_3^2 : \Sigma^{48} V_2 \simeq V_2 \wedge S^{\omega_1}$.

Consider the $E(2)$-based Adams spectral sequence

\[E_2^{s,t}(X) = \text{Ext}_{E(2)_*(E(2))}^{s,t}(E(2)_*, E(2)_*(X)) \Rightarrow \pi_{-s}(L_2X)\]
for a spectrum \( X \). The \( E_2 \)-term is given by the cohomology of the cobar complex \( \Omega^* E(2)_*(X) \) of the \( E(2)_*(E(2)) \)-comodules. Here,

\[
E(2)_*(E(2)) = E(2)_*[t_1, t_2, \ldots] \otimes_{BP_*} E(2)_*
\]

with \( |t_i| = 2(3^i - 1) \). Note that

\[
E(2)_*(S^{\omega_i}) = E(2)_* \{ g_i \}
\]

for \( i \in \{1, 2\} \) and generators \( g_i \in E(2)_0(S^{\omega_i}) \) (see [2, Th. 2.4]).

**Proposition 2.4.** Let \( e \in \{1, 2\} \). The Picard graded homotopy groups \( \pi_{s+t_1\omega_1+t_2\omega_2}(L_2 V_e) \) for \( s \in \mathbb{Z} \) and \( l_1, l_2 \in \mathbb{Z}/3 \) is isomorphic to \( \pi_{s+48l_1+t_2\omega_2}(L_2 V_e) \).

We concentrate the determination of the homotopy groups \( \pi_{s+t_2\omega_2}(L_2 V_e) \) for \( s \in \mathbb{Z} \) and \( l \in \mathbb{Z}/3 \) and \( e \in \{1, 2\} \), and abbreviate \( \omega_2 \) and \( g_2 \) to \( \omega \) and \( g \), respectively.

For the homotopy equivalences \( v_2^2 \) in Theorem 2.2 and Proposition 2.3, consider the composite map \( B_e \colon \Sigma^{144} V_e \xrightarrow{v_2^2} \Sigma^{96} V_e \wedge S^{\omega} \xrightarrow{v_2^2 \wedge 1} \Sigma^{48} V_e \wedge S^{\omega} \wedge S^{\omega} \xrightarrow{v_2^2 \wedge 1} V_e \wedge S^{\omega} \wedge S^{\omega} \wedge S^{\omega} = V_e \), in which \( S^{\omega} \wedge S^{\omega} \wedge S^{\omega} = L_2 S^0 \) since \( \omega_2 = 0 \).

**Proposition 2.5.** There exist self maps \( B_e \colon \Sigma^{144} V_e \xrightarrow{v_2^2} V_e \) for \( e \in \{1, 2\} \) such that \( E(2)_*(B_e) = v_2^2 : E(2)_*(V_e) \rightarrow E(2)_*(V_e) \).

The maps \( B_e \) induce the isomorphisms \( (B_e)_* : \pi_{s+t_2\omega_2}(L_2 V_e) \rightarrow \pi_{s+t_2\omega_2}(L_2 V_e) \) of the homotopy groups as well as the isomorphisms \( v_2^2 : E_{r+1}^{s+t_2\omega}(V_e) \rightarrow E_{r+1}^{s+t_2\omega}(V_e) \) of the Adams-Novikov \( E_\ast \)-terms, and so it suffices to determine \( E_{r+1}^{s+t_2\omega}(V_e) \otimes_{K(3)} \mathbb{Z}/3 \) for \( r \geq 2 \) for the homotopy groups \( \pi_{s+t_2\omega}(L_2 S^0) \). Here,

\[
K^{(k)} = \mathbb{Z}/3[v_2^{-3k}, v_2^{-3k}]
\]

for \( k \in \{0, 1, 2\} \). Note that \( K^{(0)} = K(2)_* \). Moreover, \( \mathbb{Z}/3 \) is considered to be a \( K^{(2)} = \mathbb{Z}/3[v_2^0, v_2^{-3}] \)-module by sending \( v_2^0 \) to 1. Hereafter, we abuse notation, and a \( K^{(2)} \)-module \( M \) denotes

\[
M \otimes_{K^{(2)}} \mathbb{Z}/3.
\]

So degrees run over \( \mathbb{Z}/144 \), and \( K^{(2)} \) is considered to be \( \mathbb{Z}/3 \). We also consider the algebra

\[
K^{(0)} \otimes (F^b \oplus F^h \oplus F^{h\varphi} \oplus F^{h\varphi}) \otimes L(\zeta_2)
\]

for a generator \( b \) corresponding to \( b_0 \in E_2^{2,12}(V_e) \), which detects \( i_e i_\beta \in \pi_{10}(V_e) \) for the well known generator \( \beta_1 \in \pi_{10}(S^0) \).

**Proposition 2.6.** ([8, Th. 5.8]) The \( E_2 \)-term \( E_{r+1}^{s,t_2\omega}(V_1) \) is isomorphic to a free \( P^{(0)} \)-module

\[
K^{(0)} \otimes (F^b \oplus F^h \oplus F^{h\varphi} \oplus F^{h\varphi}) \otimes L(\zeta_2)
\]

for

\[
F^b = P^{(2)} \{ 1, b_1 \}, \quad F^h = P^{(2)} \{ h_1, h_0 \},
\]

\[
F^{h\varphi} = P^{(2)} \{ \varphi_0, \varphi_1 \} \quad \text{and} \quad F^{h\varphi} = P^{(2)} \{ \xi, \xi, b_1 \}.
\]

Here, \( \zeta = E_2^{1,0}(V_1) \), \( h_1 = E_2^{1,12}(V_1) \) and

\[
\begin{align*}
\tilde{h}_0 &= v_2^2 h_0 \in E_2^{1,84}(V_1), \quad \tilde{b}_1 &= v_2^2 b_1 \in E_2^{1,84}(V_1), \\
\tilde{\xi} &= -v_2^2 \xi \in E_2^{2,120}(V_1), \quad \tilde{\varphi}_0 &= v_2^2 \varphi_0 \in E_2^{2,120}(V_1) \quad \text{and} \quad \tilde{\varphi}_1 = -v_2^2 \varphi_1 \in E_2^{3,120}(V_1)
\end{align*}
\]

for the generators \( h_0, b_1, \xi, \varphi_0 \) and \( \varphi_1 \) in [8]. The generators satisfy the relations:
(2.10) ([8, Prop. 5.9])
\[ h_0 h_1 = 0, \quad h_0 \xi = 0, \quad h_1 \xi = 0, \]
\[ h_0 b_0 = h_1 b_1, \quad h_1 b_0 = -h_0 b_1, \]
\[ b_1 \xi = h_0 \bar{\psi}_1 = -h_1 \bar{\psi}_0, \quad b_1 \xi = h_0 \bar{\psi}_0 = h_1 \bar{\psi}_1, \]
\[ v_0^0 b_0^2 = -b_1, \quad b_0 \bar{\psi}_1 = b_1 \bar{\psi}_0 \quad \text{and} \quad b_0 \bar{\psi}_0 = -b_1 \bar{\psi}_1, \]
as well as
\[ (2.11) \quad h_0^2 = 0, \quad h_1^2 = 0, \quad \xi^2 = 0, \quad \bar{\psi}_0^2 = 0, \quad \bar{\psi}_1^2 = 0 \quad \text{and} \quad \zeta^2 = 0. \]

We introduce imaginary generators \( u \) and \( \varphi \) such that
\[ u^2 = -v_0^3 = -1, \quad \bar{\psi_0} = b \varphi \quad \text{and} \quad \bar{\psi}_1 = u b \varphi, \]
and put \( h = h_1 \) and \( \zeta = \zeta_2 \). We further identify the elements as follows:
\[ (2.12) \quad h_0 = uh, \quad b_1 = ub, \quad \xi = uh \varphi. \]

Here, the bidegrees of the generators are
\[ \| v_1 \| = (0, 4), \quad \| v_2 \| = (0, 16), \quad \| u \| = (0, 72), \quad \| h \| = (1, 12), \]
\[ \| \varphi \| = (1, 36), \quad \| \xi \| = (1, 0) \quad \text{and} \quad \| b \| = (2, 12). \]

In the table, we notice that
\[ (2.13) \quad h \varphi \notin E_2^{2,48}(V_1) \quad \text{and} \quad h b \varphi \in E_2^{4,60}(V_1). \]

The modules in (2.9) are rewritten as
\[ (2.14) \quad F^h = K^{(2)} \oplus b P_u^{(2)}, \quad F^h = h P_u^{(2)}, \quad F^b \varphi = b \varphi P_u^{(2)} \quad \text{and} \quad F^h \varphi = u h \varphi K^{(2)} \oplus h \varphi b P_u^{(2)} = u h \varphi F^b \]
for
\[ (2.15) \quad K_u^{(k)} = \mathbb{Z}/3[v_0^k, v_2^k] / (u^2 + 1) \quad \text{and} \quad P_u^{(k)} = K_u^{(k)}[b], \]
where \( k \in \{0, 1, 2\} \), and so
\[ (2.16) \quad E_2^*(V(1)) \cong \left( K^{(0)}[1, u h, u h \varphi] \oplus b P_u^{(0)} \otimes \Lambda(h, \varphi) \right) \otimes \Lambda(\zeta). \]

We notice that the relations (2.10) follow from the two relations
\[ u^2 = -1 \quad \text{and} \quad h^2 = 0. \]
Furthermore, we consider the element
\[ (2.17) \quad \zeta = u \varphi \zeta \quad (\in E_2^{2,108}(V_1)), \]
and modules
\[ (2.18) \quad K = \mathbb{Z}/3\{1, v_2, v_2^5\} \quad \text{and} \quad K' = \mathbb{Z}/3\{1, v_2^5\}, \]
The homotopy groups
Theorem 2.24.\(P(2)) = \mathbb{Z}/3[b]/(b), P(x) = P(k) \oplus uP(k), \]
\(P(k, l) = P(k) \oplus v_2^2 P(l), P(k, b) = P(k) \oplus v_2^2 bP(l)\) and
\(P(k, b, l) = P(k) \oplus v_2^2 bP(l) \oplus v_2^2 P(m)\)

for \(i \in \{1, 2\}, k, l, m \in \{-\} \cup \{n \in \mathbb{Z} | n \geq 0\}, \) where
\(P(-) = P(2)\) and \(P(0) = 0.\)

We also note that
\[ub^t = (ub)b^{t-1} = \tilde{b}_1b^{t-1}\] for \(t \geq 1.\)

By use of these notation, we determine the homotopy groups:

**Theorem 2.22.** The homotopy groups \(\pi_{*+l}(L_2V_1)\) for \(l \in \mathbb{Z}/3\) are given by:
\[
\pi_*(L_2V_1) = K \otimes \Lambda(\zeta) \otimes \left[ (P(5) \oplus uP(4) \oplus v_2 h (P(2, 2) \oplus uP(3, 3))) \right]
\concat \varphi(b (P(4) \oplus uP(5))) \oplus v_2 h (bP(2, 2) \oplus uP(3, 3)))] \]
\[\oplus \left[ v_2^2 (P(3) \oplus uP(3)) \oplus v_2 h (P(2, b) \oplus uP(3, b^2)) \right]
\concat \varphi(b (P(3) \oplus uP(3))) \oplus v_2 h (P(1, 3) \oplus uP(2))\]
\[\oplus \left( (P(5) \oplus uP(4)) \oplus v_2 h (P(2, 2) \oplus uP(3, 3)) \right) \] \text{and} \(K_{2^{\pm 1}}.\)

**Remark 2.23.** From the structure, we find missing differentials in the paper [3]:
\(d_9(v_2^{-2} h_1 g_9) \equiv v_2^{j-4} v_2 h_1 g_9\) \(j \equiv 2, 6, 7 \) \((9),\)
\(d_9(v_2^6 g_9) \equiv v_2^{j+6} v_2 h_1 g_9\) \(j \equiv 0, 1, 5 \) \((9),\)
\(d_9(v_2^6 g_9) \equiv v_2^{j+6} v_2 h_1 g_9\) \(j \equiv 0, 1, 5 \) \((9),\)
up to sign. Here, the notations are those used in [3].

**Theorem 2.24.** The homotopy groups \(\pi_{*+l}(L_2V_2)\) for \(l \in \mathbb{Z}/3\) are given by:
\[
\pi_*(L_2V_2) = (\mathcal{M} \oplus \varphi \mathcal{M}^{\varphi}) \otimes \Lambda(\zeta) \otimes S_2
\]
for
\[\mathcal{M} = v_1 v_2^6 (P(3, 3) \oplus uP(2, 2)) \otimes K' \otimes (P(5) \oplus uP(4)) \otimes \Lambda(v_1 v_2)\]
\[\oplus h (P(4) \oplus uP(5)) \otimes K' \otimes v_2 h (P(2, 2) \oplus uP(3, 3)) \otimes \Lambda(v_1 v_2),\]
\[\mathcal{M}^{\varphi} = v_1 v_2^6 b (P(2, 2) \oplus uP(3, 3)) \otimes K' \otimes b (P(4) \oplus uP(5)) \otimes \Lambda(v_1 v_2)\]
\[\oplus h (bP(4) \oplus uP(5)) \otimes K' \otimes v_2 h (bP(2, 2) \oplus uP(3, 3)) \otimes \Lambda(v_1 v_2),\] and
\[S_2 = uv_1 v_2 h K^{(1)} \otimes K \otimes \Lambda(\varphi, \zeta); \text{ and } \]
\[\pi_{*+l}(L_2V_2) = \left[ (\mathcal{M} \oplus \varphi \mathcal{M}^{\varphi}) \otimes \zeta \mathcal{M} \oplus \varphi \mathcal{M}^{\varphi} \otimes S_2 \right] g^{+1}\]
for
\[\mathcal{M} = v_1 v_2^6 (P(3, b^2 1) \oplus uP(2, b)) \otimes K' \otimes b^2 P_a(3) \otimes \Lambda(v_1 v_2),\]
\[\oplus h b^2 (P(2) \oplus uP(3)) \otimes K' \otimes v_2 h (P(2) \oplus uP(3, b^2 1)) \otimes \Lambda(v_1 v_2),\]
\[\mathcal{M}^{\varphi} = v_1 v_2^6 b (P(1, 3) \oplus uP(1, 2)) \otimes K' \otimes bP_a(3) \otimes \Lambda(v_1 v_2)\]
\[\oplus h (P(3, 1) \oplus uP(3)) \otimes K' \otimes v_2 h (P(1, 3, 1) \oplus uP(1, 2)) \otimes \Lambda(v_1 v_2).\]
We notice that these are isomorphism of modules, and so the modules are not expressed uniquely. For example, in the summands of $\pi_{p+q}(L_2 V_2)$,
\[
g\left([hh^2 P(2) \oplus h \kappa P(3, 1)] \oplus K' \oplus (v_2 h P(2) \oplus v_2 h \kappa P(1, 3, 1)) \oplus \Lambda(v_1 v_2)\right) \\
= (hh P(3) \oplus h \kappa P(3)) \oplus K' \oplus (v_2 h P(2, b1) \oplus v_2 h \kappa P(1, 3)) \oplus \Lambda(v_1 v_2)].
\]
Indeed, these are isomorphic to
\[
(hh^2 g P(2) \oplus h \kappa g P(3) \oplus h (bg) P(1)) \oplus K' \\
\oplus (v_2 h g P(2) \oplus v_2 h \kappa g P(1, 3) \oplus v_2^3 h (bg) P(1)) \oplus \Lambda(v_1 v_2)
\]
for the element $\langle bg \rangle = bg + v_3^2 g$ in (5.11).

3. The Adams-Novikov $E_2$-terms for $\pi_*(V_2)$

By (2.18), we rewrite the $E_2$-term as follows:
\[
E_2^{s,t}(V_2) = E^{(1)} \otimes K \otimes \Lambda(\zeta)
\]
for $E^{(1)} = K^{(1)} \oplus (F^h \oplus F^h \oplus F^{b\phi} \oplus F^{h \phi})$.

Consider the exact sequence
\[
E_2^{s,t-4}(V_1) \xrightarrow{v_1} E_2^{s,t}(V_2) \xrightarrow{\tilde{t}_s} E_2^{s,t}(V_1) \xrightarrow{\delta} E_2^{s+1,t-4}(V_1)
\]
associated to the cofiber sequence (2.1). Recall Landweber’s formula $\eta_R(v_2) \equiv v_2 + v_1 t_1^2 - v_1^3 t_1 \bmod (3)$ in $BP_*(BP)$. Then, we see that
\[
\delta(v_2^s) = sv_2^{s-1} h.
\]
Indeed, $h = [t_1^3] \in E_2^{1,12}(V_1)$. Hereafter, $[c] \in E_2^{s,t}(V_2)$ for a cocycle $c \in \Omega^{s,t}E_2^{*,*}(V_2)$ denotes the homology class of $c$. Under the exact sequence (3.2), (3.3) implies
\[
v_1 v_2^s h = 0 \in E_2^{1,s}(V_2) \quad \text{unless } s \equiv 2 (3).
\]
We also recall (1.3) that
\[
E_2^{s,t}(V_2) = K(2)_s \quad \text{and} \quad E_2^{s,t}(V_2) = E_2^{s,t}(3, v_1^3).
\]
For a cocycle $c \in \Omega_{s-t}^{3,4} K(2)_s$, we have a cocycle $c^9 \in \Omega_{s-t}^{3,36} E_2^{s,t}(3, v_1^3)$. Furthermore, we see that
\[
\tilde{t}_s([c^9]) = [v_2^{3s} c] \in E_2^{s,36t}(V_1),
\]
since $t_1^3 = v_2^{3s-1} t_1 \in \Omega^{1,s} K(2)_s$.

**Lemma 3.5.** The connecting homomorphism $\delta$ acts trivially on the submodule $E_2^{1,s}(V_1)$.

**Proof.** It suffices to show that, for each element $x \in E_2^{1,s}(V_2)$, we have an element $(x)^{\sim} \in E_2^{1,s}(V_2)$ such that $\tilde{t}_s((x)^{\sim}) = x$. For the generators of $E_2^{1,s}(V_2)$, we may put
\[
(b)^{\sim} = [b_{1,0}], \quad (ab)^{\sim} = [v_2 b_{1,1}], \quad (h)^{\sim} = [t_1^3], \quad (v_1 h)^{\sim} = [v_2^3 v_1^3],
\]
\[
(ab \phi)^{\sim} = [v_2^3 v_1^3 X^{3}], \quad (b_2 \phi)^{\sim} = [v_2^3 Y_0^{3}] \quad \text{and} \quad (ub \phi)^{\sim} = [v_2^3 Y_1^{3}].
\]
Here, $b_{1,k} = (t_1 \otimes t_1^2 + t_1^2 \otimes t_1)^{3k}$, and $X \in \Omega_{s-t}^{2,s} K(2)_s$, $Y_0$ and $Y_1 \in \Omega_{s-t}^{3,s} K(2)_s$ denote cocycles representing $\overline{\xi} = u b \phi$, $\overline{\psi}_0 = b_2 \phi$ and $\overline{\psi}_1 = u b \phi$, respectively. \qed
The exact sequence (3.2) together with an isomorphism (3.1) gives rise to the the exact sequences

\[ v_2^3 E^{(1)} \xrightarrow{\delta} \tilde{E}_1^{(1)} \xrightarrow{\tau} E^{(1)} \xrightarrow{\delta} v_2^3 E^{(1)} \]  
(3.7)

and we obtain

\[ E_2^{*,*}(V_2) = \left( \tilde{E}_1^{(1)} \oplus \tilde{E}_0^{(1)} \right) \otimes \Lambda(\zeta). \]  
(3.8)

The homomorphism \( \tilde{\tau} \) induces an isomorphism

\[ \mathbb{Z}/3\{(v_2^3h)^-\} = E_2^{1,16s+12}(V_2) \xrightarrow{\tau} E_2^{1,16s+12}(V_1) = \mathbb{Z}/3\{v_2^3h\} \]

for \( v_2^3 \in K \) (see the chart below (2.14)). The representatives for \((v_2^3h)^-\) are given by

\[ (v_2^3h)^- = [v_2^3t^3_1 - sv_2^{1+4}t^3_1]. \]  
(3.9)

It follows that

**Lemma 3.10.** In \( E_2^{*,*}(V_2) \), the generators satisfy the relations:

\[ h(v_2^3h)^- = v_1v_2^{-2}ub, \quad h(v_2^3h)^- = -v_1v_2^{-2}ub \quad \text{and} \quad (v_2^3h)^- (v_2^3h)^- = v_1v_2^{-1}ub. \]

In other words, \((v_2^3h)^- (v_2^3h)^- = (t - s)v_1v_2^{-t+4}ub.\)

**Proof.** This follows from computation

\[ h(v_2^3) = [t_1^3 \otimes v_2t_1^3 - v_1t_1 \otimes t_1^3] = [v_2t_1^3 \otimes t_1^3 + v_1t_1^3 \otimes t_1^3 - v_1t_1 \otimes t_1^3] = v_1v_2^{-3}ub, \]

\[ h(v_2^3h) = [t_1 \otimes v_2^3t_1^3 + v_1v_2^3t_1 \otimes t_1^3] = [v_2^3t_1^3 \otimes t_1^3 + v_1v_2^3t_1 \otimes t_1^3 + v_1v_2^3t_1 \otimes t_1^3] = v_1v_2^{-3}ub, \]

\[ (v_2^3h)^- (v_2^3h)^- = [v_2^3t_1^3 \otimes v_2^3t_1^3 + v_1v_2^3t_1 \otimes t_1^3 + v_1v_2^3t_1 \otimes t_1^3] = v_1v_2^{-1}ub. \]  
\( \square \)

We note that the multiplication by \( b \) (resp. \( ub \)) defines the monomorphism \( b: E_2^{*,*}(V_2) \rightarrow E_2^{1+t+2s+12}(V_2) \) (resp. \( ub: E_2^{*,*}(V_2) \rightarrow E_2^{1+t+2s+16}(V_2) \)).

**Lemma 3.11.** We have an element \((v_2^3uh)^- \in E_2^{*,*}(V_2)\) satisfying

\[ (v_2^3uh)^- = b = (v_2^3h)^- \quad \text{for } v_2^3 \in K. \]

**Proof.** Since \( \delta(v_2^3uh) = 0 \), we have an element \((v_2^3uh)^- \in E_2^{*,*}(V_2)\) such that \( \tilde{\tau}((v_2^3uh)^-) = v_2^3uh \). Then, \( \tilde{\tau}((v_2^3uh)^-)b = v_2^3ubb = \tilde{\tau}((v_2^3h)^-ub) \). Thus, \((v_2^3uh)^-b - (v_2^3h)^-ub\) is an image of \( v_1 \). By degree reason, \((v_2^3uh)^-b - (v_2^3h)^-ub = kuv_2^{s-4}b\) for some \( k \in \mathbb{Z}/3 \). Thus the lemma follows by setting \((v_2^3uh)^- = (v_2^3uh)^- = kuv_2^{s-4}b\).

\( \square \)

We also have

\[ (v_2^3uh\varphi)^- = [v_2^{3+s}X^9 - sv_1v_2^{s-4}Z^9] \in E_2^{*,*}(V_2) \]

for a cochain \( Z \in \Omega^2K(2) \), such that \( d(Z) = t_1^3 \otimes X \). Since \( v_2\psi_0 \in (h_1, h_1, \xi) \in E_2^{*,*}(V_1) \), we may put

\[ (b\varphi)^- = [v_2^3t_1^3 \otimes X^9 + t_1^3 \otimes Z^9] \in E_2^{*,*}(V_2). \]
We note that $v_2 Y_0 = t_1^4 \otimes X + t_1^3 \otimes Z$ for $Y_0$ in the proof of Lemma 3.5.

**Lemma 3.13.** In $E_2^s(V_2)$, the generators satisfy the relations:

$$(v_2^s h)^-(v_2^s u h \varphi)^- = (t-s)v_1 v_2^{s+t} b \varphi \quad \text{and} \quad (v_2^s h)^-(b \varphi)^- = (v_2^s u h \varphi)^- u b$$

for $s, t \in \{1, 2\}$.

**Proof.** The first relation follows from

$$(v_2^s h)^-(v_2^s u h \varphi)^- = \left[ (v_2^s t_1^4 - sw_1 v_2^{s-1} t_1^6) \otimes (v_2^s t_1^4 X^9 - t_1^4 Z^9) \right]$$

$$- \left[ s v_1 v_2^{s+4} t_1^6 \otimes X^9 - t_1^4 Z^9 \right]$$

$$\implies -d(v_2^{s+t-3}Z^9) = -(s+1)v_1 v_2^{s+t-4} t_1^4 \otimes X^9 - t_1^4 Z^9 \right)$$

Here, the underlined terms with subscript (1) cancel each other out, and the coefficient of the sum of the wave under lined terms is $t - s$.

Similarly, we verify the second relation by computing

$$(v_2^s h)^-(b \varphi)^- = \left[ (v_2^s t_1^4 - sw_1 v_2^{s-1} t_1^6) \otimes (v_2^s t_1^4 X^9 + t_1^4 Z^9) \right]$$

$$- s v_1 v_2^{s+4} t_1^6 \otimes X^9 + t_1^4 Z^9 \right]$$

Indeed,

$$-d(v_2^{s-1}Z^9) = -s v_1 v_2^{s-1} t_1^4 \otimes t_1^4 Z^9 - s v_1 v_2^{s-1} t_1^4 \otimes (v_2^s t_1^4 X^9)$$

$$-d(v_1 v_2^{s+4} t_1^1 \otimes X^9) \right)$$

By (3.3) and (3.7), we see that

$$\text{Im} (\delta; v_2^s E^{(1)} \rightarrow v_2^{s-1} E^{(1)}) = v_2^{s-1} K^{(1)} \otimes (b F^h \otimes F \varphi)$$

$$\text{Ker} (\delta; v_2^s E^{(1)} \rightarrow v_2^{s-1} E^{(1)}) = v_2^{s-1} K^{(1)} \otimes (F^h \otimes F \varphi)$$

for $s \in \{1, 5\}$, where $F \varphi = h b P_a^{(1)}$ such that $K^{(1)} \otimes F^h \varphi = u h \varphi K^{(1)} \otimes F \varphi$. From this, we obtain the following

**Lemma 3.14.** The submodules $\tilde{E}_s^{(1)}$ for $s \in \{0, 1, 5\}$ are:

$$\tilde{E}_0^{(1)} = E^{(1)} \otimes \Lambda(v_1 v_2^s)$$

$$\tilde{E}_s^{(1)} = \left( F \varphi^s \otimes \tilde{F}^s \varphi \right) \oplus v_1 v_2^{s-1} K^{(1)} \otimes (F^h \otimes F \varphi) \oplus u h K^{(1)} \otimes \Lambda(\varphi)$$
for $s \in \{1, 5\}$. Here,
\[
\tilde{F}_s^h = P^{(1)}\{ (v_2^s h)^-, (w_2^s h)^- \} \quad \text{and} \quad \tilde{F}_s^{h\varphi} = P^{(1)}\{ (v_2^s uh\varphi)^-, (v_2^s uh\varphi)^-uh \}.
\]

Hereafter, we abbreviate $(x)^-$ to $x$. Then, we may identify $\tilde{F}_s^h = v_2^s K^{(1)} \otimes F^h$ and $\tilde{F}_s^{h\varphi} = v_2^s K^{(1)} \otimes F^{h\varphi}$.

**Corollary 3.15.** $E_2^{s*,*}(V_2)$ is isomorphic to the tensor product of $K^{(1)}$, $\Lambda(\zeta)$ and the direct sum of
\[
(F^b \oplus F^{h\varphi} \oplus F^h \oplus F^{h\varphi}) \otimes \Lambda(v_1 v_2^s)
\]
and
\[
v_2^s K' \otimes \left(F^b \oplus F^{h\varphi} \oplus v_1 v_2^s \left(F^b \oplus F^{h\varphi} \oplus uhK^{(2)} \otimes \Lambda(\varphi) \right)\right).
\]
The generators satisfy $h^2 = 0$. Therefore, the relations in (2.10) also hold in $E_2^{s*,*}(V_2)$.

We note that
\[
E_2^{s*,*}(V_2) = K^{(1)} \otimes \Lambda(\zeta) \otimes \left( (F^b \oplus F^{h\varphi} \oplus v_1 (F^b \oplus F^{h\varphi}) \otimes K \right)
\]
\[
\oplus \left( (F^b \oplus F^{h\varphi}) \otimes K \oplus v_1 v_2^s (F^b \oplus F^{h\varphi}) \right) \oplus v_1 v_2 uK^{(2)} \otimes \Lambda(\varphi) \otimes K'.
\]
By Lemmas 3.10 and 3.13, we have
\[
(v_2^s h)(v_2^s h\varphi) = (t - s)v_1 v_2^{s+t-4} uh\varphi = (v_2^s h)(v_2^s h)\varphi.
\]

4. **The Adams-Novikov differentials on $E_2^{s*,*+l\omega}(V_e)$ for $e \in \{1, 2\}$ and $l \in \mathbb{Z}/3$**

Let $b_1 \in \pi_{10}(S^0)$ be the well known generator. Note that it is detected by $b = b_0 \in E_2^{2,12}(S^0)$. Consider a spectrum $W$ fitting in the cofiber sequence
\[
S^{10} \xrightarrow{b_1} S^0 \xrightarrow{f} W \xrightarrow{g} S^{11}.
\]
Then, $E_2^{2,11}(W) = E_2^{2,0} \oplus E_2^{2,11,0}$ for a generator $b \in E_2^{2,11}(W)$ such that $s_2(b) = 1 \in E_2^{2,0}$.

Hereafter, we abbreviate the generators $\omega_2$ of Pic($L_2$) and $g_2$ of $E_2^{2,0}(S^{\omega_2})$ to $\omega$ and $g$, respectively. We set
\[
V_e^{(l)} = V_e \wedge S^{l\omega} \quad \text{for } e \in \{1, 2\} \text{ and } l \in \mathbb{Z}/3.
\]
Then, $E_2^{s*,*+l\omega}(V_e) = E_2^{s*,*}(V_e^{(l)})$ for $e \in \{1, 2\}$. Note that $E_2^{s*,*}(V_e^{(l)}) = E_2^{s*,*}(V_e)$ for $l \in \mathbb{Z}/3$, and $b_1$ induces a monomorphism $b: E_2^{s*,*}(V_e^{(l)}) \to E_2^{s*,*+l\omega}(V_e^{(l)})$ by (2.9) and Corollary 3.15. For the next lemma, we recall an exact couple defining the Adams-Novikov spectral sequence:

\[
\begin{array}{cccccccc}
* & \overset{k_1}{\longrightarrow} & E \wedge X & \overset{k_2}{\longrightarrow} & E_2 \wedge X & \overset{k_3}{\longrightarrow} & \cdots & \overset{k_2}{\longrightarrow} & E_3 \wedge X & \overset{k_3}{\longrightarrow} & \cdots \\
\downarrow & & \downarrow j_0 & & \downarrow i_1 & & & & \downarrow j_1 & & \cdots \\
E \wedge X & & E \wedge E \wedge X & & E \wedge E \wedge E \wedge X & & \cdots
\end{array}
\]
for a spectrum $X$. Here, $E = E(2)$, and $S^0 \xrightarrow{i} E \xrightarrow{j} E$ is a cofiber sequence.

**Lemma 4.2.** The Adams-Novikov $E_3$-term $E_3^{s*,*}(V_e^{(l)} \wedge W)$ is trivial for $e \in \{1, 2\}$, $l \in \mathbb{Z}/3$ and $s \geq 6$. 
The cofiber sequence (4.1) induces a short exact sequence
\begin{equation}
0 \to E_2^{s,t}(V_c^{(l)}) \xrightarrow{d} E_2^{s,t}(V_c^{(l)} \wedge W) \xrightarrow{\pi_s} E_2^{s,-t+11}(V_c^{(l)}) \to 0.
\end{equation}

Consider the generator \(g' \in E(2)_0(V_c^{(l)})\), and let \(i(l) \in \pi_2(\mathbb{E}_3 \wedge V_c^{(l)})\) be an element such that \(k_1b_2(i(l)) = g'\). Let \(b' \in \pi_2(E \wedge \mathbb{E}_3^{i+1} \wedge V_c^{(l)})\) be an element representing \(b\). Since \((\mathbb{E}_3 \wedge V_c^{(l)})(V') = 0\), the element \(\iota_k(b)\) in the \(E_2\)-term \(E_2^{2,12}(V_c^{(l)} \wedge W)\) is in the image of a differential \(d_r\) of the spectral sequence. By degree reason, we have \(d_2(bg') = b \in E_2^{2,12}(V_c^{(l)} \wedge W)\). Therefore, the induced connecting homomorphism from (4.3) of the \(d_2\)-differential modules is the multiplication by \(b\) and so we obtain an exact sequence of the Adams-Novikov-\(E_2\)-terms
\begin{equation}
E_3^{s,t}(V_c^{(l)}) \xrightarrow{b} E_3^{s+2,t+12}(V_c^{(l)}) \xrightarrow{\alpha} E_3^{s+2,t+12}(V_c^{(l)} \wedge W) \xrightarrow{\pi_s} E_3^{s+1,t}(V_c^{(l)}).
\end{equation}

Here, note that \(E_3^{s,t}(V_c^{(l)}) = E_2^{s,t}(V_c^{(l)})\) by degree reason.

Consider a commutative diagram
\[
\begin{array}{cccccccc}
E_2^{s-1, t} & \xrightarrow{d} & E_2^{s-t, t-4} & \xrightarrow{v_1} & E_2^{s-t, (V_c^{(l)})} & \xrightarrow{\tau_s} & E_2^{s, t} & \xrightarrow{d} & E_2^{s+1, t-4} \\
\downarrow b & & \downarrow b & & \downarrow b & & \downarrow b & & \downarrow b \\
E_2^{s+1, t+12} & \xrightarrow{d} & E_2^{s+2, t+12} & \xrightarrow{v_1} & E_2^{s+2, t+12} & \xrightarrow{\tau_s} & E_2^{s+2, t+12} & \xrightarrow{d} & E_2^{s+3, t+8}
\end{array}
\]

associated to the cofiber sequence (2.1), where \(E_2^{s,t}\) denotes \(E_2^{s,t}(V_c^{(l)})\). By (2.9), we see that \(b: E_2^{s,t}(V_c^{(l)}) \to E_2^{s+1,t+12}(V_c^{(l)})\) is an isomorphism if \(s \geq 4\), and a monomorphism with \(\operatorname{Coker} b = K(0)\{hb\zeta\}\) if \(s = 3\) (see (2.15)). The Five Lemma shows that \(b: E_2^{s,t}(V_c^{(l)}) \to E_2^{s+1,t+12}(V_c^{(l)})\) is an isomorphism if \(s \geq 5\) and an epimorphism if \(s = 4\). Therefore, the lemma follows from the exact sequence (4.4).

**Lemma 4.5.** In \(E_r^{*,*}(V_c^{(l)})\) for \(e \in \{1, 2\}\) and \(l \in \mathbb{Z}/3\), if \(d_r(xb) = yb\) for elements \(x, y \in E_r^{*,*}(V_c^{(l)})\), then \(d_r(x) = y\). Similarly, a relation \(d_r(xub) = yub\) also implies \(d_r(x) = y\).

**Proof.** Since \(E_3^{s,t}(V_c^{(l)}) = 0\) unless \(4 \leq t\), we see that \(E_3^{s,t}(V_c^{(l)}) = E_3^{s,t}(V_c^{(l)})\). By (2.9) and (3.8), we see that \(b\) in (4.4) is a monomorphism on the \(E_2\)-terms. Therefore, the lemma holds for \(r = 5\).

Suppose inductively that the lemma holds for \(s\) with \(5 < s < r\). Suppose also \(d_s(xb) = yb \in E_r^{k,m}(V_c^{(l)})\) and put \(d_r(x) = y'\). Then \(by = by' \in E_r^{k,m}(V_c^{(l)})\), and so we have an integer \(s < r\) and an element \(z \in E_s^{k-s,m-s+1}(V_c^{(l)})\) such that \(d_s(z) = b(y - y')\). Note that \(r - s \geq 4\). Since \(k \geq r + 2\), we see that \(k - s \geq r + 2 - s \geq 6\). Therefore, \(\iota_k(z) = 0\) in (4.4) by Lemma 4.2 and we have \(zb = z\). It follows that \(d_r(2b) = d_r(z) = b(y - y')\), and by the inductive hypothesis we have \(d_r(z) = y - y'\) and \(d_r(x) = y\) as desired.

Since \(ub\) is a permanent cycle (see 4.13), multiplying the relation \(d_r(xub) = yub\) by \(ub\) implies \(d_r(x(ub)^2) = y(ub)^2\). Therefore, \(d_r(xb^2) = yb^2\), and we obtain \(d_r(x) = y\).

**Corollary 4.6.** In \(E_2^{*,*}(V_c^{(l)})\) for \(e \in \{1, 2\}\) and \(l \in \mathbb{Z}/3\), if \(xb\) (resp. \(xub\) is a permanent cycle, then so is \(x\).
By [5] and [1], the differential $d_5: E_2^{*,*}(S^\infty) \to E_3^{*,*,4}(S^\infty)$ acts on $g$ by
\begin{equation}
(4.7) \quad d_5(g) = \omega g \equiv v_2uhb\varphi g \in E_2^{*,*}(V_\mathcal{E} \wedge S^\infty) \quad \text{for} \quad e \in \{1, 2\}.
\end{equation}

By [8, Props 8.4, 9.9, 9.10], we deduce that
\begin{equation}
(4.8) \quad d_5(v_2^{3t+s}g^l) = -tv_2^{3t+s-2}hb^2g^l + tv_2^{3t+s}u(v_2h)b\varphi g^l \in E_2^{*,*}(V_1 \wedge S^{l\omega}),
\end{equation}
for $l \in \mathbb{Z}/3$ and $s \in \{0, 1, 5\}$, and
\begin{equation*}
\quad d_5(v_2^{3t+s}xg^l) = d_5(v_2^{3t+s}g^l)x \in E_2^{*,*}(V_1 \wedge S^{l\omega})
\end{equation*}
for $x \in \{b, h, uh, ub, ub\varphi, b\varphi, ub\varphi, hb\varphi, \zeta\} = \{b, \tilde{b}_0, h_1, \tilde{b}_1, \tilde{\xi}, v_0, \tilde{v}_1, \tilde{b}_1\tilde{\xi}, \zeta_2\}$. In particular,
\begin{equation*}
\quad d_5(v_2^{3t+s}g^l) = 0 \in E_2^{*,*}(V_1 \wedge S^{l\omega})
\end{equation*}
by (4.8) together with (2.11). We also have
\begin{equation}
(4.9) \quad \text{For} \quad s \in \{0, 1, 5\}, \quad \text{we have an integer} \quad \sigma(s) \in \{1, 2\} \quad \text{such that}
\end{equation}
\begin{equation*}
\quad d_9(v_2^{3t+s}h) = \sigma(s)v_2\varphi ub^5 \in E_9^{*,*}(V_1) \quad (ub^5 = \tilde{b}_1b^4).
\end{equation*}

The integer $\sigma(s)$ is not determined in [8]. We determine it to be two in Lemma 4.15.

\begin{equation}
(4.10) \quad \text{(8, Th. 10.6)} \quad \text{The} \quad E_{10} \text{-term for} \quad V_1 \quad \text{is isomorphic to the tensor product of} \quad \Lambda(\zeta), \quad K \quad \text{and}
\end{equation}
\begin{equation*}
\quad P^{(2)}/(b^4\{v_2, b\varphi\}) \oplus P^{(2)}/(b^5\{1, ub\varphi\}) \oplus (P^{(2)}/(b^5\{v_2h, v_2hb\varphi\}) \oplus P^{(2)}/(b^7\{v_2uh, v_2uh\varphi\})) \oplus \mathbb{Z}/3\{1, v_2^3\}.
\end{equation*}

See (2.20) for $K$.

In particular, we have:
\begin{equation}
(4.11) \quad \text{Every element of} \quad K \subset E_2^{*,*}(V_1) \quad \text{and} \quad v_1K \subset E_2^{*,*}(V_2) \quad \text{is a permanent cycle in the spectral sequences.}
\end{equation}

\begin{equation}
(4.12) \quad \text{The elements} \quad v_2^5h \in E_2^{1*,*}(V_1) \quad \text{for} \quad s \in \{0, 1, 2, 4, 5, 6\} \quad \text{and} \quad v_1v_2^3h \in E_2^{1*,*}(V_1)
\end{equation}
\begin{equation*}
\quad \text{for} \quad s \in \{2, 5\} \quad \text{are permanent cycles in the spectral sequences. (see (3.4).)}
\end{equation*}

The following is well known (cf. [7]):
\begin{equation}
(4.13) \quad \text{For} \quad e \in \{1, 2\}, \quad \text{the elements} \quad h \quad \text{and} \quad v_2h \in E_2^{1*,*}(V_e) \quad \text{and} \quad b \quad \text{and} \quad ub \in E_2^{2*,*}(V_e)
\end{equation}
\begin{equation*}
\quad \text{are permanent cycles detecting} \quad i_e\beta'_1 \quad \text{and} \quad i_e\beta'_2 \quad \text{in} \quad \pi_*(V_e) \quad \text{and} \quad i_e\beta_1 \quad \text{and} \quad i_e\beta_2/j_3 \quad \text{in} \quad \pi_*(V_e), \quad \text{respectively. Here,} \quad i \quad \text{and} \quad i_e \quad \text{are the maps in (1.1) and (1.2), the element}
\end{equation*}
\begin{equation*}
\quad \beta_1 \quad \text{is the one in (4.1),} \quad \beta_2 \in \pi_{20}(S^4) \quad \text{is the generator, and} \quad \beta'_s \in \pi_{15s-5}(M)) \quad \text{for} \quad s \in \{1, 2\} \quad \text{denotes an element such that} \quad j_3\beta'_s = \beta_s \quad \text{for the map} \quad j \quad \text{in (1.1).}
\end{equation*}

Among the Adams-Novikov differentials for $V_e^{(l)}$ for $e \in \{1, 2\}$ and $l \in \mathbb{Z}/3$, the following relation is also well known (cf. [9]):
\begin{equation}
(4.14) \quad \text{Consider the exact sequence of the} \quad E_2 \text{-terms}
\end{equation}
\begin{equation}
E_2^{*,*}(V_1 \wedge S^{l\omega}) \xrightarrow{e} E_2^{*,*}(V_1 \wedge S^{l\omega}) \xrightarrow{u} E_2^{*,*}(V_2 \wedge S^{l\omega}) \xrightarrow{f} E_2^{*,*}(V_1 \wedge S^{l\omega}) \xrightarrow{h} E_2^{*,*}(V_1 \wedge S^{l\omega}),
\end{equation}
and let $E \xrightarrow{f} F \xrightarrow{g} G \xrightarrow{h} E$ be a part of the exact sequence. Then, we have a relation described below:
Lemma 4.15. Let $s \in \{0, 1, 5\}$ and $t \in \mathbb{Z}/3$. Then, the integers $\sigma(s)$ for $s \in \{0, 1, 5\}$ in (4.9) are all two. Furthermore, in $E_2^{s,s}(V_2)$,

\begin{align*}
d_5(v_2^{3s}) &= -tv_1v_2^{3s-3}(v_2h)b^2, \\
d_5(v_2^{3s+h}) &= t(1-s)v_1v_2^{3s+3-6}ub^3, \\
d_5(v_1v_2^{3s}) &= \begin{cases} \\
-tv_1v_2^{3s-1}hb^2 & s = 1 \\
0 & s \in \{0, 5\} \\
\end{cases} \quad \text{and} \\
d_5(v_1v_2^{3s+2}h) &= 0.
\end{align*}

**Proof.** We read off $E_2^{5,48}(V_1) = \mathbb{Z}/3\{v_2^{-3}ab^2\}$ by (2.9), and may put $d_5(v_2^0) = -v_2hb^2 + kv_1v_2^{-3}ab^2 \in E_2^{5,52}(V_2)$ for $k \in \mathbb{Z}/3$ by (4.8). Since the differential $d_5$ is a derivation, we have

\begin{equation}
(4.16) \quad d_5(v_2^{3s}) = -tv_1v_2^{3s-3}(v_2h)b^2 + tkv_1v_2^{3s+3}ub^2\zeta, \quad \text{and} \\
d_5(v_2^{3s+h}) = -tv_1v_2^{3s-3}(v_2h)(v_2^0h)b^2 + tkv_1v_2^{3s+3+3}uabh^2\zeta + v_2^{3}d_5(v_2^{5}h).
\end{equation}

It follows that $d_5(v_2^{3s+1}) = 0$ by Lemma 3.10, (3.4) and (4.13). Thus, we have $d_5(v_2^{3s+h})$ for $s = 1$ in the lemma.

Suppose that $s \in \{0, 5\}$. Put

\begin{align*}
a &= (s-1)\sigma(s-4)v_2^{-5}ab^5, \\
c &= \sigma(s-4)v_2^{-4}ab^5, \\
x &= (s-1)\sigma(s-4)v_2^{-3}ab^3, \\
y &= v_2^{3+s}h, \\
z &= \mathbb{Z}_s(y), \\
w &= v_1x,
\end{align*}

and we have $d_5(z) = c$ by (4.9), $\delta(c) = a$ by (3.4) and $d_5(x) = a$ by (4.8). Therefore, we have $d_5(y) = w$ by (4.14), that is,

\begin{equation}
(4.17) \quad d_5(v_2^{3s+h}) = (s-1)\sigma(s-4)v_1v_2^{-3}ab^3.
\end{equation}

Similarly, put

\begin{align*}
a &= v_1c, \\
c &= (1-s)\sigma(s)v_2^ab^5, \\
x &= v_2^{6+s}hb^2, \\
y &= -v_2^{3+s}, \\
z &= (1-s)v_2^{3+s}h, \\
w &= \mathbb{Z}_s(x),
\end{align*}

and we have $d_5(y) = w$ by (4.8), $\delta(y) = z$ by (3.3) and $d_5(z) = c$ by (4.9). Thus, we have $d_5(x) = a$. By Lemma 4.5,

\begin{equation}
(4.18) \quad d_5(v_2^{6+s}h) = (1-s)\sigma(s)v_1v_2^{3}ab^3.
\end{equation}

Since $(v_2h)(v_2^0h) = (s-1)v_1v_2^{-3}ab$ by Lemma 3.10, the second relation of (4.16) is:

\begin{align*}
d_5(v_2^{3s+h}) &= \begin{cases} \\
tv_1v_2^{3s-6}ab^3 & s = 0 \\
-tv_1v_2^{3s-1}ab^3 + tkv_1v_2^{3s+8}uabh^2\zeta + v_2^{3}d_5(v_2^{5}h) & s = 5
\end{cases}
\end{align*}

by (3.4) and (4.13). Compare it with (4.17) and (4.18), we obtain

\begin{align*}
\sigma(5) &= -1 = \sigma(0); \\
v_2^{0}d_5(v_2^{5}h) &= (1 + \sigma(1))v_1v_2^{3}ab^3 - kv_1v_2^{3}abh^2\zeta \\
v_2^{0}d_5(v_2^{5}h) &= kv_1v_2^{3}abh^2\zeta.
\end{align*}
The last two relations show \( \sigma(1) = -1 \) and \( k = 0 \), and then \( d_5(v_2^1) = 0 \). Thus the top two relations of the lemma follow from (4.16).

The third relation of the lemma follows from the first one together with (3.4) and (4.11). Multiplying the permanent cycle \( v_1 \) in (4.11) to the second relation of the lemma implies the last one.

\[ \square \]

**Lemma 4.19.** The elements \( uh, uh\varphi = \overline{\xi} \), \( v_2^1uh\varphi = v_2^1\overline{\xi} \), \( h\varphi = \overline{\varphi}_0 \), \( v_2^0h\varphi = v_2^0\overline{\varphi}_0 \), \( ub\varphi = \overline{\varphi}_1 \), \( v_2^0ub\varphi = v_2^0\overline{\varphi}_1 \) and \( \zeta = \zeta_2 \) of \( E_2^{*,*}(V_2) \) are permanent cycles.

**Proof.** Let \( V_3 \) denote the cofiber of \( \alpha^3 : \Sigma^{12}M \to M \), and consider the cofiber sequence \( \Sigma^4V_2 \xrightarrow{\pi} V_3 \xrightarrow{\tau} V_1 \xrightarrow{\gamma} \Sigma^5V_2 \) obtained similarly to (2.1). Let \( \delta_2 : E_2^{*,*}(V_1) \to E_3^{*,*,8}(V_2) \) denote the associated connecting homomorphism. In the complex \( \Omega^*E(2)_*(V_3) \), we compute \( d(v_2^1t_1^1 + v_1v_2^1t_1^1) = -v_1v_2^1t_1^1 \otimes t_1^1 + v_2^1v_2^1t_1^1 \otimes t_1^1 + v_2^1v_2^1t_1^1 \otimes t_1^1 + v_1v_2^1t_1^1 \otimes t_1^1 = v_2^1v_2^1b_{1,1} \). It follows that \( \delta_2(v_2^1h\zeta) = uh\zeta \), and so \( uh\zeta \) is a permanent cycle by the Geometric Boundary Theorem, since \( v_2^1h\zeta \in E_2^{*,*}(V_1) \) is a permanent cycle by (4.10). Therefore, \( \zeta \) is a permanent cycle by (4.13) and Corollary 4.6. Since \( (uh)\tilde{h} = h(ub) \) by Corollary 3.15 ((2.10)) and \( h \) is a permanent cycle by (4.13), the element \( uh \) is a permanent cycle.

We also compute \( \delta_2(v_2^3x_3^3t_3^3\overline{x}^3) = v_2^3v_2^3\overline{\varphi}_0 \) by [9, Lemma 4.4], which is \( \delta_2(v_2^3u_3^3uh\varphi) = v_2^3v_2^3h\varphi \) in our notation. Since \( v_2^3uh\varphi \) and \( v_2^3uh\varphi \) are permanent cycles of \( E_3^{*,*}(V_2) \) by (4.10), their \( \delta_2 \)-images \( v_2^3b\varphi \) and \( b\varphi \) are permanent cycles of \( E_4^{*,*}(V_2) \) by the Geometric Boundary Theorem. By (4.13) and Corollary 3.15 ((2.10)), we have \( uh(v_2^3b\varphi) = b(v_2^3uh\varphi) \) and \( ub(v_2^3b\varphi) = b(v_2^3ub\varphi) \) in \( E_4^{*,*}(V_2) \) for \( s \in \{0, 6\} \). Noticing that \( uh \) and \( ub \) are permanent cycles, these show that \( uh\varphi, v_2^0uh\varphi, ub\varphi \) and \( v_2^0ub\varphi \) are all permanent cycles by Corollary 4.6.

Here, consider an element

\[ (4.20) \quad g^l = b_2^l g + tv_2^1x_vh\varphi g \in E_2^{1,24}(V_e \wedge S^\omega) \quad \text{for} \ l \in \mathbb{Z}/3 \text{ and } e \in \{1, 2\}. \]

We notice that the element \( v_2^3ub\varphi g \) is not divisible by \( b \) in the \( E_2 \)-term.

\[ \square \]

**Lemma 4.21.** Let \( s \in \{0, 1, 5\} \). In \( E_5^{*,*}(V_1 \wedge S^\omega) \), we have

\[
\begin{align*}
d_5(v_2^3t_2^4 + (v_2^3h)g) &= \begin{cases} 
0 & t = 0 \\
-v_2^3b^3\varphi g & t = 1 \\
-v_2^3ub^3g & t = 2
\end{cases}.
\end{align*}
\]

In particular, \( g (= g^1) \) is a permanent cycle.

**Proof.** We notice that

\[ d_5(eg) = d_5(xg) + (-1)^{|x|}x(v_2^3h)ub\varphi g \in E_2^{*,*+k\omega}(V_e) \]

for \( e \in \{1, 2\} \) by (4.7). Suppose that \( s \in \{0, 5\} \) and put

\[ a = v_1c, \quad c = (t-1)tv_2^3t_2^3u^5b^2g - tv_2^3t_2^3b^3\varphi g, \quad y = (s-1)v_2^3t_2^3g, \quad w = t_s(x), \quad x = (t-1)((t-1)tv_2^3t_2^3h^2b^2g - v_2^3t_2^3ub\varphi g), \quad z = v_2^3t_2^3h^2g. \]

Then, \( d_5(x) = a \in E_2^{10,*}(V_2 \wedge S^\omega) \) by Lemmas 4.15, 4.19 and 3.10, \( d_5(y) = w \in E_2^{2,*}(V_1 \wedge S^\omega) \) by (4.8), and \( \delta(y) = z \) by (3.3). By (4.14), we have \( d_5(z) = c \). For
the case for $s = 1$, we set
\[
\begin{align*}
  a &= \delta(c), \\
  c &= (t - 1)tv_2^{3t-5}ub^5 - tv_2^{3t-2}b^4\varphi g, \\
  y &= v_2^{t+2}hg \\
  w &= v_1x, \\
  x &= (1 - t)v_2^{3t-4}ub^3g - v_2^{3t-1}b^2\varphi g, \\
  z &= \overline{t}_s(y).
\end{align*}
\]
Then, $d_5(x) = a \in E_2^{10,*}(V_1 \wedge S^2)$ by (4.8) and Lemmas 4.19, and $d_5(y) = w \in E_2^{5,*}(V_2 \wedge S^2)$ by Lemmas 4.15 and 3.10. By (4.14), we also have $d_3(z) = c$ in this case.

\begin{corollary}
In the spectral sequence $E_2^{*,*}(V_1 \wedge S^2)$, $v_2^2b\varphi g$ and $v_2^2ub\varphi g$ are permanent cycles for $s \in \{0, 1, 5\}$.
\end{corollary}

\begin{proof}
Since we have a pairing $V_1 \wedge V_2 \to V_1$, we have $d_3(v_2^7u^s\varepsilon b\varphi g) = -v_2^5u^{1-\varepsilon}b\varphi g$ in $E_2^{*,*}(V_1)$ for $\varepsilon \in \{0, 1\}$ by Lemmas 4.19 and 4.21. This shows that $v_2^5u^{1-\varepsilon}b\varphi g$ is a permanent cycle, and hence the corollary follows from Corollary 4.6.
\end{proof}

By Lemma 4.15, among the elements of $(v_1K(0) \oplus K(1)) \otimes F^b$ and $(v_1v_2^5K(1) \oplus K(0)) \otimes F^b$ in the $E_2$-term $E_2^{*,*}(V_2)$, the following elements survive to $E_3$-term
\[
\begin{align*}
  &v_1v_2^{3t+s}, &\text{for } s \in \{0, 5\}, \\
  &v_1v_2, &v_1v_2^{3t+2}h, &h, &v_2^{3t+1}h &\text{and } v_2^5h
\end{align*}
\]
for $t \in \mathbb{Z}/3$.

\begin{lemma}
In $E_2^{*,*}(V_2)$, we have
\[
\begin{align*}
  d_3(v_1v_2^j) &= hh, &d_3(v_1v_2^3) &= -v_2^5hh, \\
  d_3(v_1v_2^5h) &= -v_1v_2ub^5 &d_3(v_2^5h) &= -ub^5.
\end{align*}
\end{lemma}

The following generators are permanent cycles:
\[
\begin{align*}
  v_1v_2^j &\text{ for } j \in \{0, 1, 2, 5, 6\}, \\
  v_1v_2^j h &\text{ for } j \in \{2, 5\}, \text{ and } \\
  v_2^5h &\text{ for } j \in \{0, 1, 4, 5\}.
\end{align*}
\]

\begin{proof}
We begin with verifying the permanent cycles. The elements $v_1v_2^j$ for $j \in \{0, 1, 5\}$ and $v_1v_2^5h$ for $j \in \{2, 5\}$ are permanent cycles by (4.11) and (4.12). The second relation in Lemma 4.15 with $(t, s) = (1, 0)$ and $(1, 5)$ shows that $v_1v_2^{3t}ub^3$ and $v_1v_2^{3t}ub^3$ are permanent cycles. Corollary 4.6 implies that $v_1v_2^j$ for $j \in \{2, 6\}$ are permanent. Similarly, the first relation in Lemma 4.15 with $t = 1$ and $s = 2$ implies that $v_2^tub^2$ and $v_2^tub^2$ are permanent, and so $v_2^j$ for $j \in \{1, 4\}$ is a permanent cycle by Corollary 4.6. By the same argument, the top two relations of this lemma imply that $v_2^5h$ for $j \in \{0, 5\}$ is permanent.

To turn to the top two relations. For $s \in \{0, 5\}$, put
\[
\begin{align*}
  a &= \overline{t}_s(c) \\
  c &= (s - 1)t(t + 1)v_2^{3t+s-3}bb^2 \\
  w &= -v_2^{3t+s-2}bb^2 \\
  x &= -(s - 1)t(v_2^{3t+s-1}b^2 \\
  y &= v_2^{3t+s} \\
  z &= v_1v_2^{3t+s}.
\end{align*}
\]
Then, these satisfy the relations in (4.14) other than $d_3(z) = c$ by (4.8) and (3.3). Hence, $d_3(z) = c$.

\[
\begin{align*}
\text{(4.24)} &\quad d_3(v_1v_2^{3t+s}) = (s - 1)t(t + 1)v_2^{3t+s-3}bb^4 \in E_2^{*,*}(V_2).
\end{align*}
\]
This with $t = 1$ shows the first two equalities.

Multiply by $h$ to the second equality, and Lemma 3.10 implies
\[
d_3(v_1v_2^8h) = -(v_2^5h)hh^4 = -v_1v_2ub^5,
\]
which is the third one. Since \( t_4(v_2^2 h) = v_2^2 h \in E_9^{1,0}(V_1) \) and \( d_9(v_2^2 h) = -ub^5 \in E_9^{1,0}(V_1) \) by (4.9) and Lemma 4.15, we see that \( d_9(v_2^2 h) = -ub^5 + kv_1v_2^7 h_\varphi b^2 = -ub^5 - d_9(kv_1v_2^b h_\varphi) \) for \( k \in \mathbb{Z}/3 \) by (4.8). Thus, the fourth \( d_9 \)-differential follows.

Now, the next lemma follows from Lemma 4.15 (see also Lemma 3.10).

**Lemma 4.25.** Let \( s \in \{0, 1, 5\} \) and \( t, l \in \mathbb{Z}/3 \). Then, in \( E_2^{s,*}(V_2 \land S^\omega) \),

\[
\begin{align*}
&d_5(v_2^g g^t) = -tv_2^{3t-3}(v_2 h)b^2g^t + lv_2^3(v_2 h)ub_2\varphi g^t, \\
&d_5(v_2^{3t+s}hg^t) = t(1-s)v_1v_2^{3t+s-3}b^2\varphi g^t + l(1-s)v_1v_2^{3t+s-3}bb^2zg^t, \\
&d_5(v_1v_2^{3t+s}g^t) = \begin{cases} 
-2v_1v_2^{3t-1}bb^2g^t + lv_1v_2^{3t+2}ub_2\varphi g^t & s = 1 \\
0 & s \in \{0, 5\}
\end{cases}
\]

and

\[
d_5(v_1v_2^{3t+2}g^t) = 0.
\]

By Lemma 4.25, among the elements of \((v_1K^{(0)} \oplus K^{(1)}) \otimes F^b \oplus (v_1v_2^2K^{(1)} \oplus K^{(0)}) \otimes F^h)g \) in the \( E_2 \)-term \( E_2^{s,*}(V_2 \land S^\omega) \), the following elements survive to \( E_9 \)-term \( v_1v_2^{3t+s}g \) for \( s \in \{0, 5\} \), \( v_1v_2^{3t+2}hg \) and \( v_1v_2^{3t+1}hg \) for \( t \in \mathbb{Z}/3 \).

The relation with \((t, s) = (2, 0)\) in Lemma 4.21 is \( d_9(v_2^3hg) = -ub^3 \varphi \in E_9^{10,132}(V_1 \land S^\omega) \). We see that \( v_1E_2^{16,128}(V_1) = v_1b^3E_2^{4,92}(V_1) = \mathbb{Z}/3 \langle v_1v_2^3bb^2 \varphi \rangle \subset E_2^{16,132}(V_2) \) by (2.9). The generator is zero in the \( E_9 \)-term by \( d_5(v_1v_2^3ub_2\varphi g) = v_1v_2^3b^4\varphi g \), which follows from the last relation in Lemma 4.25 multiplied by the permanent cycle \( ub_2\varphi \) (Lemma 4.19). Thus, the relation in \( E_2^{s,*}(V_1) \) is pulled back to the one in \( E_2^{s,*}(V_2) \):

\[
d_9(v_2^3hg) = -ub^3 \varphi \in E_9^{10,132}(V_2 \land S^\omega).
\]

It follows from Corollary 4.6 that

\[
g = b^2g + v_1v_2^3ub_2\varphi g \in E_9^{13,24}(V_2 \land S^\omega) \quad \text{is a permanent cycle for the element} \quad g = g^1 \text{ in (4.20).}
\]

**Lemma 4.28.** In \( E_0^{s,*}(V_2 \land S^\omega) \), we have

\[
\begin{align*}
d_9(v_1v_2^{3t+s}g) &= \begin{cases} 
(s-1)v_2^3ub_2^3\varphi g & t = 0 \\
(1-s)v_1v_2^3bb^2g & t = 1 \quad \text{for } s \in \{0, 5\}, \text{ and} \\
0 & t = 2
\end{cases} \\
d_9(v_2^{3t+1}hg) &= \begin{cases} 
-b^3\varphi g & t = 1 \\
-ub^3g & t = 2
\end{cases}
\end{align*}
\]

**Proof.** For a permanent cycle \( x \) of \( E_2^{s,*}(V_2) \) with \( d_9(xg) = 0 \), we have \( d_9(xg) = 0 \in E_9^{s,*}(V_2 \land S^\omega) \), and so

\[
d_9(xb^2g) = -d_9(xv_2^3\varphi g) = -d_9(xv_2^3g)ub_2\varphi g \in E_9^{s,*}(V_2 \land S^\omega).
\]

Put \( x_4^{(0)} = v_1v_2^{3t+s} \) and \( x_4^{(1)} = v_2^{3t+4}h \). By Lemma 4.25, \( d_5(x_4^{(0)}g) = 0 \) for \( x \in \{0, 1\} \), and so \( x_4^{(1)}g \in E_9^{s,*}(V_2 \land S^\omega) \). Furthermore, Lemma 4.23 shows that \( x_4^{(1)}g \) for \( x \in \{0, 1\} \) is a permanent cycle unless \( t = 1 \). Therefore, by (4.29), we compute

\[
\begin{align*}
d_9(x_4^{(0)}b^2g) &= -d_9(x_4^{(0)}g)ub_2\varphi g, \quad \text{and} \\
d_9(x_4^{(1)}b^2g) &= -d_9(x_4^{(1)}g)ub_2\varphi g = d_9(x_4^{(1)}g)(ub_2\varphi g)^2 = 0.
\end{align*}
\]
Thus, the relations for \( t = 0 \) follow from those for \( t = 1 \), and the relations for \( t = 2 \) follow from Corollary 4.6.

Now we consider the differential \( d_9 \) on \( x^{(c)}_1 \). Lemma 4.25 together with Lemma 4.19 also shows that

\[
(4.30) \quad v_2 u b^2 \varphi \zeta g, \quad v_1 v_2^2 b^2 \varphi \zeta g \text{ and } v_1 v_2^3 b^2 \varphi \zeta g
\]

are zero in \( E_{9}^{\ast}(V_2 \wedge S^{\ast}) \). Therefore,

\[
d_9(x^{(0)}_1 b^2 g) = d_9(x^{(0)}_1 (b^3 g + v_2^2 u b^2 \varphi \zeta g)) = d_9(v_1 v_2^3 b^2 g) = (1 - s) v_2^3 u b^2 g, \quad \text{and}
\]

\[
d_9(x^{(1)}_1 b^2 g) = d_9(x^{(1)}_1 (b^2 g + v_2^3 u b \varphi \zeta g)) = d_9(v_2^3 b^2 g) = -u b^2 g
\]

for \( s \in K' \). By Corollary 4.6, we obtain the relations for \( d_9(x^{(c)}_1) \). \( \square \)

5. The Cohomology of a Differential Algebra \( C_1 \)

Consider algebras \( K^{(k)} \), \( K_{u}^{(k)} \), \( P^{(k)} \) and \( P_u^{(k)} \) in (2.6) and (2.17) and

\[
A^{(k)}_1 = P_u^{(k)} \otimes A(v_2 h)
\]

for \( k \in \{0, 1, 2\} \). Recall that these algebras are considered to be the tensor products with \( \mathbb{Z}/3 \) over \( K^{(2)} \) (see (2.7)). In this section, we consider the module

\[
C_1(g^l) = \left( A^{(0)} \otimes A(\varphi, \zeta) \right) g^l
\]

for \( l \in \mathbb{Z}/3 \), which contains \( E_2^{\ast}(V_1)g^l = E_2^{\ast}(V_1 \wedge S^{l}) \). We use the relation

\[
g^l g^m = g^{l+m} \quad \text{for } l, m \in \mathbb{Z}/3.
\]

In order to consider a differential algebra, we consider the subalgebra

\[
C_1^{(1)} = A(1^{(1)}) \otimes A(\varphi, \zeta) \subset C_1.
\]

We begin with introducing a differential algebra structure on \( C_1^{(1)}[g]/(g^3) \) so that the inclusion \( E_2^{\ast}(V)[g]/(g^3) \to C_1[g]/(g^3) \) is the one of differential \( C_1^{(1)} \)-modules with differential \( \partial_5 \):

\[
\begin{align*}
\partial_5(x) &= 0 \quad \text{for } x \in \{1, u, b, v_2 h, \varphi, \zeta\}, \\
\partial_5(v_2^3 t) &= -t v_2^{3-t} (v_2 h) b^2 \quad \text{for } t \in \mathbb{Z}/3, \text{ and} \\
\partial_5(g) &= \omega g
\end{align*}
\]

on the generators, where

\[
\omega = av_2 u b \varphi \zeta = v_2 b h \zeta \in A(1^{(2)}) \quad (\zeta = u \varphi \zeta).
\]

We make \( C_1 = C_1^{(1)} \otimes K \) a differential module by setting

\[
\partial_5(v_2^3) = 0 \quad \text{and } \partial_5(v_2^2) = 0 \quad \text{for } v_2^3 \in K,
\]

and we obtain

\[
H^{\ast}(C_1 g^l, \partial_5) = H^{\ast}(C_1^{(1)} g^l, \partial_5) \otimes K.
\]

In addition to (2.21), we consider \( P_u^{(2)} \)-algebras

\[
(5.8) \quad P_u(b^s k) = b^s P_u(k), \quad P_u(b^s k, b^s l) = b^s P_u(k) \oplus v_2^3 b^s P_u(l) \quad \text{and}
\]

\[
P_u(b^s k, b^s l, b^s m) = b^s P_u(k) \oplus v_2^3 b^s P_u(l) \oplus v_2^3 b^s P_u(m)
\]

for \( k, l, m, e_i \in \{-1\} \cup \{n \in \mathbb{Z} \mid n > 0\} \), and we set \( b^{-} = 0 \). We notice that

\[
P_u^{(1)} = P_u(-, -, -).
\]
Since \( \partial_h \) acts as \( P_u(-,-,-) \to v_2 h P_u(b^2-h^2,-) \subset v_2 h P_u(-,-,-) \), we immediately obtain the following lemma from the second equality of (5.5):

**Lemma 5.9.** The cohomology \( H^*(A_1^{(1)}, \partial_h) \) is isomorphic to

\[
A_1^{(1)} = P_u^{(2)} \oplus v_2 h P_u(2,2,-)
\]

as an algebra.

Put

\[
B_1^{(1)} = A_1^{(1)} \otimes \Lambda(\varsigma) \quad \text{for} \quad \varsigma = u \varphi \zeta.
\]

Consider an element

\[
\langle bg^l \rangle = bg^l + lv_2^3 \varsigma g^l,
\]

and we see that this is a \( \partial_h \)-cocycle. Note that the element \( g \) in (4.20) equals \( b \langle bg \rangle \)
but that

\[
\partial_h(v_2^3 g) = -v_2 h b^2 g + v_2^3(v_2 h) b \varsigma g \neq v_2 h b \langle bg \rangle
\]

by (5.5).

**Lemma 5.12.** The cohomology \( H^*(B_1^{(1)}; g^{\pm 1}, \partial_h) \) is isomorphic to

\[
\mathbb{B}_1^{(1)} g^{\pm 1} = \langle bg^{\pm 1} \rangle P_u^{(2)} \oplus \left( v_2 h P_u(2,2,-) \oplus \varsigma \left( P_u^{(2)} \oplus v_2 h P_u(1,2,-) \right) \right) g^{\pm 1}.
\]

Then, Lemmas 5.9 and 5.12 imply the following:

**Corollary 5.13.** The cohomology \( H^*(C_1^{(1)} g^l, \partial_h) \) for \( l \in \mathbb{Z}/3 \) is isomorphic to

\[
C_1^{(1)} g^l = \begin{cases} A_1^{(1)} \otimes \Lambda(\varphi, \zeta) & l = 0 \\ \left( \mathbb{B}_1^{(1)} + A_1^{(1)} \{ \varphi, \zeta \} \right) g^l & l = \pm 1 \end{cases}
\]

and \( H^*(C_1 g^l, \partial_h) \) is isomorphic to \( \mathbb{C}_1^{(1)} g^l \otimes K \).

Now, we introduce \( \mathbb{C}_1 g^l \) for \( l \in \mathbb{Z}/3 \) a differential module structure with differential \( \partial_h \) given by

\[
\partial_h(v_2^{3l+2} h g^l) = \begin{cases} 0 & t = 0 \\ -l v_2^2 b^2 \varsigma g^l & t = 1 \\ -u v_2^2 b^4 \langle bg^l \rangle & t = 2 \end{cases}
\]

for \( t \in \mathbb{Z}/3 \) and \( s \in \{0, 1, 5\} \). In particular, we assume that

\[
\partial_h(\langle bg^l \rangle) = 0 = \partial_h(\varsigma g^l) \quad \text{for} \quad l \in \mathbb{Z}/3.
\]

By definition, we immediately obtain the following:

**Lemma 5.16.**

\[
H^*(\mathbb{A}_1^{(1)}, \partial_h) = A_1^{(1)} \quad \text{and} \quad H^*(\mathbb{B}_1^{(1)} g^l, \partial_h) = A_1^{(1)} g^l \oplus \mathbb{A}_1^{(1)} g^l
\]

for \( l \in \{1,2\} \). Here,

\[
A_1^{(1)} = P_u(5) \oplus v_2 h P_u(2,2),
\]

\[
\mathbb{A}_1^{(1)} = b P_u(4) \oplus v_2 h P_u(2, b^1) \quad \text{and} \quad \mathbb{A}_1^{(1)} = P_u(4) \oplus v_2 h P_u(1,2).
\]

Since \( H^*(\mathbb{C}_1 g^l, \partial_h) = H^*(\mathbb{C}_1^{(1)} g^l, \partial_h) \otimes K \), we obtain
Corollary 5.17. The cohomology $H^*(C_1g^l, \partial_9)$ for $l \in \mathbb{Z}/3$ is isomorphic to

$$\mathcal{C}_1G^l = \begin{cases} A_1^{(1)} \otimes K \otimes \Lambda(\varphi, \zeta) & l = 0 \\ \left( A_1^{(1)} \otimes \mathcal{A}(1^{(1)}) \right) \oplus A_1^{(1)} \otimes \mathbb{Z}/3\{\varphi, \zeta\} \oplus Kn^{l+1} & l = \pm 1. \end{cases}$$

Corollary 5.18. On $H^{**}(C_1g^l, \partial_9)$, there is no more non-trivial differential $\partial_9$ other than those in (5.14). Furthermore, no more differential $\partial_r$ for $r \geq 10$ can be defined on the cohomologies on them.

Proof. Since the submodule with the homology dimension of $\mathcal{C}_1g^l$ greater than ten is trivial, $\partial_r$ is trivial for each $r \geq 10$. For $r = 9$, $\partial_9$ originates $H^{**}(C_1g, \partial_9)$ for $s \in \{0, 1\}$, on which the differential $\partial_9$ is defined. □

6. THE COHOMOLOGY OF THE DIFFERENTIAL ALGEBRA $C$

We consider an algebra $E = \mathbb{Z}/3[v_1, v_2, v_2^{-1}]/(v_2^3)$ and $E$-algebras

$$Q_u = v_1P_u^{(0)} \otimes P_u^{(1)}, \quad \text{and} \quad Q_u^{h} = v_1v_2^phP_u^{(1)} \otimes hP_u^{(0)},$$

in which $h$ is an element with bidegree $\|h\| = (1, 12)$, and the $E$-action and the multiplication on $Q_u^{h}$ satisfies

$$v_1v_2^ph = 0 \quad \text{unless} \quad s = 2 (3),$$

$$xy = 0 \quad \text{for} \quad x \in v_1v_2^phP_u^{(1)} \otimes y \in Q_u^{h}, \quad \text{and}$$

$$(v_2^ph)(v_2^sh) = (t - s)v_1v_2^{s+t-4}ub.$$

We notice that $Q_u^{h}$ has a $Q_u$-module structure by (6.2). In this section, we consider the algebra

$$A = Q_u \oplus Q_u^{h}, \quad C = A \otimes \Lambda(\varphi, \zeta) \quad \text{and} \quad C^g = C[g]/(g^3 - 1)$$

for generators $\varphi$, $\zeta$ (cf. above (2.12)) and $g$ with $g^3 = 1$. We introduce differentials $\partial_9 : C \rightarrow C^g$ and $\partial_9 : H^*(C, \partial_9) \rightarrow H^*(C^g, \partial_9)$ so that $H^*(H^*(C, \partial_9), \partial_9)$ is closely related to $E_{15}^{15}(V_2)$. We moreover assume that $\partial_r$ is a derivation. For the generators $u, \varphi, \zeta, v_1v_2, v_2h, b$ and $g$, we set

$$\partial_r(u) = 0, \quad \partial_r(\varphi) = 0, \quad \partial_r(\zeta) = 0, \quad \partial_r(v_1v_2) = 0, \quad \partial_r(b) = 0, \quad \partial_r(v_2^ph) = 0 \quad \text{and} \quad \partial_r(g) = \omega g = v_2hbg.$$ 

for $r \in \{5, 9\}$, $s \in \{0, 1, 5\}$, and $\omega$ and $\zeta$ of (5.6). We define the differential $\partial_9$ by

$$\partial_9(v_2^{3t}) = -tv_2^{3t-2}hb^2 \quad \text{for} \quad v_2^{3t} \in K^{(1)}, \quad (6.6)$$

We notice that the relations in Lemma 4.25 hold after replacing $d_5$ with $\partial_9$ by (6.2) and (6.3). We define differential $\partial_9$ on the algebra $C_g = H^*(C, \partial_9)$ by

$$\partial_9(v_1v_2^{3t+s}g^l) = \begin{cases} (s - 1)v_2^{3t+3s}g^l & t = 0 \\ (1 - s)v_2^{3t+3s} \langle bg^l \rangle & t = 1 \quad \text{for} \quad s \in \{0, 5\}, \\ 0 & t = 2. \end{cases}$$

$$\partial_9(v_2^{3t+1}hg^l) = \begin{cases} 0 & t = 0, \\ -tub^4b^4g^l & t = 1, \quad \text{and} \quad -ub^4 \langle bg^l \rangle & t = 2. \end{cases} \quad (6.7)$$
for $l \in \mathbb{Z}/3$ and $\langle bg \rangle$ in (5.11). We also assume that the relations in (5.15) hold in $C_g$. We further notice that

$$Q_u = v_1v_2^3P_u^{(1)} \otimes K' \oplus P_u^{(1)} \otimes \Lambda(v_1v_2)$$

and

$$Q^b_u = hP_u^{(1)} \otimes K' \oplus v_2hP_u^{(1)} \otimes \Lambda(v_1v_2).$$

By (6.5), (6.6) and (6.7), we easily obtain the following:

**Lemma 6.8.** The cohomology $H^*(\mathcal{A}, \partial_{h})$ is isomorphic to

$$\mathcal{A} = \left( v_1v_2^6P_u(3, 3, 3) \otimes K' \oplus P_u(5) \otimes \Lambda(v_1v_2) \right) \oplus \left( hP_u(2) \otimes K' \oplus v_2hP_u(2, 2, -) \otimes \Lambda(v_1v_2) \right).$$

The cohomology $H^*(\mathcal{A}, \partial_{h})$ is isomorphic to

$$\mathcal{A} = \left( v_1v_2^6P_u(3, 3, 3) \otimes K' \oplus P_u(5) \otimes \Lambda(v_1v_2) \right) \oplus \left( hP_u(2) \otimes K' \oplus v_2hP_u(2, 2, -) \otimes \Lambda(v_1v_2) \right).$$

Consider a differential subalgebra of $C$

$$B = A \otimes \Lambda(\zeta),$$

Then, in the same manner as the proof of Lemma 6.8, we verify the following lemma easily by (6.2), (6.3), (6.5), (6.6) and (6.7) (cf. Lemma 4.25):

**Lemma 6.9.** The cohomology $H^*(\mathcal{B}g^{\pm 1}, \partial_{h})$ is isomorphic to

$$\mathcal{B}g^{\pm 1} = (\mathcal{A} \oplus \mathcal{K}) g^{\pm 1},$$

where

$$\mathcal{A} = \left( v_1v_2^6P_u(3, 3, -, -) \otimes K' \oplus P_u(5) \otimes \Lambda(v_1v_2) \right) \oplus \left( hP_u(2) \otimes K' \oplus v_2hP_u(2, 2, -) \otimes \Lambda(v_1v_2) \right)$$

and

$$\mathcal{K} = \left( v_1v_2^6P_u(3, 3, -, -) \otimes K' \oplus P_u(5) \otimes \Lambda(v_1v_2) \right) \oplus \left( hP_u(2) \otimes K' \oplus v_2hP_u(2, 2, -) \otimes \Lambda(v_1v_2) \right).$$

The cohomology $H^*(\mathcal{B}g^{\pm 1}, \partial_{h})$ is isomorphic to

$$\mathcal{B}g^{\pm 1} = (\mathcal{A} \oplus \mathcal{K}) g^{\pm 1},$$

where

$$\mathcal{A} = \left( v_1v_2^6P_u(3, 3, -) \otimes K' \oplus P_u(5) \otimes \Lambda(v_1v_2) \right) \oplus \left( hP_u(2) \otimes K' \oplus v_2hP_u(2, 2, -) \otimes \Lambda(v_1v_2) \right)$$

and

$$\mathcal{K} = \left( v_1v_2^6P_u(3, 3, -) \otimes K' \oplus P_u(5) \otimes \Lambda(v_1v_2) \right) \oplus \left( hP_u(2) \otimes K' \oplus v_2hP_u(2, 2, -) \otimes \Lambda(v_1v_2) \right).$$

**Remark 6.10.** In $\mathcal{A}g^{\pm 1}$, the elements $v_1b^k g^{\pm 1}$, $b^k g^{\pm 1}$, $b^k h g^{\pm 1}$ and $v_2^4 h g^{\pm 1}$ are the classes of $v_1b^k g^{\pm 1} + v_1v_2^2b^k-1 g^{\pm 1} = v_1b^k (bg^{\pm 1})$, $b^k g^{\pm 1} + v_2^4 b^k-1 g^{\pm 1} = b^{k-1} (bg^{\pm 1})$, $b^k h g^{\pm 1} + v_2^2b^k-1 h g^{\pm 1} = b h^{k-1} (bg^{\pm 1})$ and $v_2^4 h b - v_2^4 h c g^{\pm 1} = v_2^4 h (bg^{\pm 1})$, respectively.

**Corollary 6.11.** The cohomology $H^*(\mathcal{H}^*(Cg^l), \partial_{h})$ for $l \in \mathbb{Z}/3$ is isomorphic to

$$\mathcal{H}g^l = \begin{cases} 
\mathcal{A} \otimes \Lambda(\varphi, \zeta) & l = 0 \\
(\mathcal{B} \oplus \mathcal{A}\{\varphi, \zeta\}) g^l & l = \pm 1.
\end{cases}$$
Corollary 6.12. The other differentials $\partial_r: \mathbb{C}^r \to \mathbb{C}^{r+1}$ for $r \geq 9$ are all trivial.

Proof. By Corollary 6.11, the submodules of $\mathcal{C}^l$ for $l \in \mathbb{Z}/3$ with the homology dimension greater than nine are:

\[ \mathcal{C}^{10, s} = v_1v_2b^sK^{(2)}_m \oplus b^sK^{(2)}_m \text{ and} \]

\[ \mathcal{C}^{10, s}g^l = 0 \text{ for } s = 10 \text{ and } l = \pm 1 \text{ or } s \geq 11. \]

Therefore, $\partial_r = 0$ for $r \geq 10$. The differential $\partial_9$ is defined only on the element of $\mathcal{C}^{10, s}g^l$ for $s \in \{0, 1\}$ and $l \in \mathbb{Z}/3$, and no more differential can be defined.

7. The $E_\infty$-terms from the cohomologies of $C_1$ and $C$

In this section, we show a lemma by which the $E_\infty$-terms $E_\infty^* (V_c)g^l$ for $l \in \mathbb{Z}/3$ are deduced from $\mathcal{C}^l g^l$ for $c \in \{1, 2\}$. Hereafter, $C_2 = C$, $C_2 = \mathcal{C}$ and $C_2 = \mathcal{C}$. Let $R_c$ and $S_c$ denote modules fitting in the diagram

\[
\begin{array}{ccc}
\mathcal{S}_c g^l & \xrightarrow{i} & \mathcal{B}_c g^l \\
\downarrow & & \downarrow \mathcal{J} \\
\mathcal{C}_c g^l & \xrightarrow{i} & \mathcal{R}_c g^l
\end{array}
\]

in which the row and the column are exact. Then, $\mathcal{J}$ and $\mathcal{P}$ are monomorphisms. Indeed, if $\mathcal{P}(x) = 0$, then we have an element $bc \in b\mathcal{C}_c g^l$ such that $bc = \mathcal{P}(x)$. $bc = \mathcal{J}(bc) = \mathcal{J}(x) = 0$ and so $\mathcal{P}(x) = 0$. Since $\mathcal{P}$ is a monomorphism, $x = 0$ as desired. Here, $S_1 g^l = 0, S_2 g^l = \mathcal{S}_2 \otimes \Lambda(\zeta)g^l$ and $R_c g^l = \mathcal{R}_c \otimes \Lambda(\zeta)g^l$ for

\[
\begin{align*}
\mathcal{S}_2 &= u_1v_2hK^{(1)} \otimes \Lambda(\varphi) \otimes K' \\
\mathcal{R}_1 &= K^{(0)} \{1, h, uh, uh\varphi\} \\
\mathcal{R}_2 &= K^{(0)} \{v_1, h\} \otimes (K^{(1)} \otimes \Lambda(\varphi) \otimes \mathcal{S}_2) \\
&= (v_2^2P(1, 1, 1) \otimes K^{(1)} \otimes \Lambda(\varphi) \otimes \mathcal{S}_2) \\
&= (v_2^2P(1, 1, 1) \otimes K^{(1)} \otimes \Lambda(\varphi) \otimes \mathcal{S}_2) \\
&= (v_2^2P(1, 1, 1) \otimes K^{(1)} \otimes \Lambda(\varphi) \otimes \mathcal{S}_2)
\end{align*}
\]

Indeed, we deduce $\mathcal{S}_2$ and $\mathcal{R}_2$ from (3.16), (6.4) and isomorphisms

\[
\begin{align*}
bQ_u \otimes K^{(1)} \otimes v_1K^{(0)} &= (K^{(1)} \otimes v_1K^{(0)}) \otimes F^h, \\
bQ_u \varphi &= (K^{(1)} \otimes v_1K^{(0)}) \otimes F^h, \\
bQ_u \varphi &= (K^{(1)} \otimes v_1K^{(0)}) \otimes F^h, \\
bQ_u \varphi &= (K^{(1)} \otimes v_1K^{(0)}) \otimes F^h.
\end{align*}
\]

obtained by (2.16) and (6.1).

We see that

\[
p(\bigoplus_{s \geq 4} E_\infty^*(V_c)g^l) = 0.
\]

Lemma 7.5. Every element of $S_3 g^1 \subset E_\infty^*(V_2)g^l$ is a permanent cycle.
Proof. Since \( v_1v_2z^2hbg^l = 0 \in E_2^{s,0}(V_g) \) (by \((3.3)\)) unless \( s \equiv 2 \mod 3 \), we see that \( bS_2g^l = 0 \), and so the lemma follows from Corollary 4.6. \( \square \)

Put\n
\[
(7.6) \quad b_*C_e^l = H^*(bc_5g^l, \partial_5) \quad \text{and} \quad b_*C_5^l = H^*(b_*C_e^l, \partial_5).
\]

We notice that the generator \( b \) induces isomorphisms \( C_5g^l \rightarrow b_*C_5g^l \) and \( C_5g^l \rightarrow b_*C_5g^l \). Since \( p \) in \((7.1)\) is an epimorphism, for each \( x \in R_e \), we have an element \( \bar{x} \in E_2^{s,0}(V_e) \) such that \( p(\bar{x}) = x \).

**Lemma 7.7.** There is an isomorphism\n
\[
E_1^{s,0+1}(V_e) \cong b_*C_e^l/D_e^l \oplus Z_e^l \quad \text{for } l \in \mathbb{Z}/3
\]

of modules. Here,

\[
D_e^l = \{ [xg^l] \in b_*C_e^l \mid xg^l = d_5(\tilde{w}g^l) \text{ or } [xg^l] = d_5(\tilde{w}g^l) \text{ for } w \in R_e \} \quad \text{and}
\]

\[
Z_e^l = \{ xg^l \in R_e \mid d_5(\bar{x}g^l) = 0 \text{ and } d_5(\tilde{w}g^l) = 0 \}.
\]

**Proof.** Note that the differentials \( d_5 \) and \( d_9 \) act on \( R_e g^l \) trivially by \((7.2)\) (and \((7.4)\)). Indeed, it has no element of cohomology dimension greater than two. The short exact sequence in \((7.1)\) induces the long exact sequence

\[
R_e g^l \xrightarrow{\delta_5} b_*C_e^l \xrightarrow{inc} E_0^{s,0}(V_e)g^l \xrightarrow{p_*} R_e g^l
\]

of \( d_5 \)-cohomologies. Hereafter, \( inc \) denotes an homomorphism induced from the inclusion. This gives rise to the short exact sequence

\[
0 \rightarrow b_*C_e^l/(\text{Im } \delta_5) \xrightarrow{inc} E_0^{s,0}(V_e)g^l \xrightarrow{p_*} \text{Ker } \delta_5 \rightarrow 0.
\]

Here, \( \delta_5(x) = d_5(\bar{x}) \in E_2^{s,0}(V_e) \), and so \( \text{Im } \delta_5 = \{ [x] \mid x = d_5(w), w \in R_e \} \). For \( d_9 \)-cohomologies, we obtain a long exact sequence

\[
\text{Ker } \delta_5 \xrightarrow{\delta_5} H^*(b_*C_e^l/(\text{Im } \delta_5), \partial_5) \xrightarrow{inc} E_0^{s,0}(V_e)g^l \xrightarrow{p_*} \text{Ker } \delta_5 \xrightarrow{\delta_5} \cdots,
\]

which splits into a short exact sequence

\[
0 \rightarrow H^*(b_*C_e^l/(\text{Im } \delta_5), \partial_5)/(\text{Im } \delta_5) \xrightarrow{inc} E_0^{s,0}(V_e)g^l \xrightarrow{p_*} \text{Ker } \delta_0 \rightarrow 0.
\]

Now we deduce the lemma by verifying that \( H^*(b_*C_e^l/(\text{Im } \delta_5), \partial_0)/(\text{Im } \delta_0) = b_*C_e^l/D_e^l \) and \( \text{Ker } \delta_0 = Z_e^l \).

Since \( V_e \) is an \( M \)-module spectrum, the homotopy groups \( \pi_\ast(L_2V_e) \) are \( \mathbb{Z}/3 \)-modules, and hence \( \pi_{\ast-s}(L_2V_e) \cong \bigoplus E_1^{s,0}(V_e) \). So it suffices to determine the structures of \( E_1^{s,0}(V_1) \).

**Proof of Theorem 2.22.** The structure of \( E_1^{s,0}(V_1) \) follows from \((4.10)\).

For \( E_1^{s,0+1}(V_1) \), we obtain\n
\[
Z_1^{s+1} = [v_1hP_1(1) \oplus w_2h\varphi P(1,1) \oplus \zeta(P(1) \oplus v_2hP_1(1,1))] \otimes Kg^{s+1} \quad \text{and}
\]

\[
D_1^{s+1} = [v_1hbbP_1(1) \oplus v_2hbb^2P(1,1) \oplus b^8P_1(1) \oplus b^8\varphi P(1) \oplus \zeta(v_2hbbP_1(1,1))] \otimes Kg^{s+1}
\]

from \( H_1 \) in \((7.2)\) by \((4.8)\) and Lemma 4.21 (cf. \((5.5)\) and \((5.14)\)). We notice that the last summand of \( Z_1^{s+1} \) is given by the permanent cycles of \((5.14)\) by setting...
\[ v_2^j \eta c g^{±1} = (u_2 h_c \pm v_2 h b) g^{±1}. \] Therefore, by Corollary 5.17, the module \( b_± c_± 1 / D_± 1 \) is isomorphic to the tensor product of \( K g^{±1} \) and

\[
  b^2 P_2(3) \oplus v_2 h b P(1) \oplus u_2 h b P(2, b) \oplus \zeta (b P_2(3) \oplus v_3 h b P(2) \oplus u_2 h b P(1, 2)) \oplus \varphi (b P(4) \oplus u b P(5) \oplus v_2 h b P(2, 2)) \oplus \zeta (b P_2(4) \oplus v_2 h b P(1, 1) \oplus u_2 h b P(2, 2)),
\]

and the structure of the \( E_{10} \)-terms follow from Lemma 7.7. We add the summand \( v_3 h b P(1) \otimes K g^{±1} \) to the \( E_{10} \)-term instead of the last summand \( v_2 h c P(1) \otimes K g^{±1} \) of \( Z_{10}^{±1} \), since both of the generators of the modules represent the generator \( v_2 h (b g^{±1}) \).

\[ \square \]

**Proof of Theorem 2.24.** By (4.13), Lemmas 4.15, 4.19, 4.23 and 7.5, we read off from (7.2):

\[
  Z^0 = (Z_2 \oplus u_2 \overline{Z}_2 \oplus \overline{S}_2) \otimes \Lambda (\zeta) \quad \text{and} \quad D^0 = (\overline{D}_2 \oplus \varphi D^0 \otimes \Lambda (\zeta),
\]

for

\[
  Z_2 = v_1 v_2^2 P(1, 1) \otimes K' \otimes P(1) \otimes \Lambda (v_1 v_2) \oplus h P_2(1, 1) \otimes K' \otimes v_2 h P(1, 1) \otimes \Lambda (v_1 v_2),
\]

\[
  \overline{Z}_2 = h P(1) \otimes K' \oplus v_2 h P(1, 1) \otimes \Lambda (v_1 v_2),
\]

\[
  D_2 = h b^2 P(1) \otimes K' \oplus v_2 h b^2 P(1, 1) \otimes \Lambda (v_1 v_2)
\]

\[
  \oplus (v_1 v_2 h b^3 P_2(1, 1) \oplus K' \oplus b^3 P_2(1) \otimes \Lambda (v_1 v_2) \otimes \Lambda (v_1 v_2))
\]

\[
  \oplus \zeta D_2 \oplus \varphi D_2 \oplus \zeta D_2 \oplus \varphi D_2.
\]

By Lemmas 4.25, 4.28 and 7.5,

\[
  Z_{±1} = v_1 v_2^2 P(1) \otimes K' \oplus v_2 h P_2(1) \otimes \Lambda (v_1 v_2) \oplus \zeta \overline{Z}_2 \oplus u_2 \overline{Z}_2
\]

\[
  \oplus (h P(1, 1) \otimes K' \oplus v_2 h P(1, 1, 1) \otimes \Lambda (v_1 v_2)) \oplus \overline{S}_2
\]

and

\[
  D_{±1} = (h b^3 P(1) \oplus h b^5 P(1)) \otimes K' \oplus (v_2 h b^2 P(1, 1) \oplus v_2 h b P(1) \oplus \Lambda (v_1 v_2)
\]

\[
  \oplus (v_1 v_2^3 b^3 P_2(1, 1) \oplus v_1 v_2 b^5 P_2(1) \oplus K' \oplus b^3 P_2(1) \oplus b^5 P_2(1) \otimes \Lambda (v_1 v_2))
\]

\[
  \oplus \zeta \overline{D}_2 \oplus \varphi D_2 \oplus \zeta \overline{D}_2 \oplus \varphi D_2.
\]

Here, every element of \( \zeta \overline{Z}_2 g^{±1} \) for

\[
  \overline{Z}_2 = v_2^j h P(1) \otimes K' \oplus v_2^j h P(1) \otimes \Lambda (v_1 v_2)
\]

is a permanent cycle. Indeed, \( v_2^{j+s} h c g^{±1} \) for \( s \in \{0, 1, 5\} \) denotes a permanent cycle \( (v_2^{j+s} h c \pm v_2^{j+s} h b) g^{±1} \). Furthermore, for

\[
  Z^0 = v_1 v_2^2 P(1) \otimes K' \oplus v_2 h P_2(1) \otimes \Lambda (v_1 v_2),
\]

we have

\[
  Z_{±1} = Z^1 \oplus \zeta \overline{Z}_2 \oplus u_2 \overline{Z}_2 \oplus \zeta \left( \overline{Z}_2 \otimes \overline{Z}_2 \right) \oplus S_2.
\]

Put

\[
  D_{±1}^{1, \varphi} = (v_1 v_2^3 b^3 P_2(1) \oplus v_1 v_2 h b^3 P(1) \otimes \Lambda (v_1 v_2)
\]

\[
  \oplus h b^5 P(1) \otimes K' \oplus v_2 h b P(1) \otimes \Lambda (v_1 v_2)),
\]

and we see that

\[
  D_{±1} = D_2 \oplus \zeta D_2 \otimes \Lambda (\zeta) \oplus \varphi D_2.
\]

Then, we notice that

\[
  b_\lambda \cap_0 \cap_0 = \left( \frac{\partial \lambda \cap_0 \cap_0}{\partial \lambda \cap_0 \cap_0} \right) \otimes \Lambda (\zeta), \quad \text{and}
\]

\[
  b_\lambda \cap_0 \cap_0 \cap_0 \cap_0 \cap_0 = \left( \frac{\partial \lambda \cap_0 \cap_0 \cap_0 \cap_0 \cap_0}{\partial \lambda \cap_0 \cap_0 \cap_0 \cap_0 \cap_0} \right) \otimes \Lambda (\zeta) \oplus \varphi \left( \frac{\partial \lambda \cap_0 \cap_0 \cap_0 \cap_0 \cap_0}{\partial \lambda \cap_0 \cap_0 \cap_0 \cap_0 \cap_0} \right).
\]
by Corollary 6.11. Furthermore, we read off the summands:

\[ b_*A/D_2 = (v_1v_2^6bP_n(2,2) \otimes K' \oplus bP_n(4) \otimes \Lambda(v_1v_2)) \]
\[ \oplus (hb(P(3) \oplus uP(4)) \otimes K' \oplus v_2hb(P(1,1) \oplus uP(2,2)) \otimes \Lambda(v_1v_2)), \]

\[ b_*A/D_2^p = (v_1v_2^6b(P(2,2) \oplus uP(3,3)) \otimes K' \oplus b(P(4) \oplus uP(5)) \otimes \Lambda(v_1v_2)) \]
\[ \oplus (hbP_n(4) \otimes K' \oplus v_2hbP_n(2,2) \otimes \Lambda(v_1v_2)), \]

\[ b_*A/D_2 = (v_1v_2^6bP_n(2,2) \otimes K' \oplus bP_n(3) \otimes \Lambda(v_1v_2)) \]
\[ \oplus (hbP_n(4) \otimes K' \oplus v_2hbP_n(2,2) \otimes \Lambda(v_1v_2)), \] and

\[ b_*A/D_2^{+1,\phi} = v_1v_2^6b(P(1,3) \oplus uP(1,2)) \otimes K' \oplus bP_n(3) \otimes \Lambda(v_1v_2) \]
\[ \oplus (hbP_n(4) \otimes uP(3)) \otimes K' \oplus v_2hb(P(1) \oplus uP(2,2)) \otimes \Lambda(v_1v_2)). \]

Put that \( M = b_*A/D_2 \oplus \mathbb{Z}_2, \ M' = b_*A/D_2^p \oplus u\mathbb{Z}_2^p, \ M = b_*A/D_2 \oplus \mathbb{Z}^p \)
and \( M' = b_*A/D_2^{+1,\phi} \oplus (\mathbb{Z}_2^p \oplus \mathbb{Z}^p) \), and we obtain the \( E_{10} \)-terms from Lemma 7.7, and
the homotopy groups of the \( M \)-module spectrum \( V_2 \) are isomorphic to the corresponding \( E_{10} \)-terms. \( \square \)

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