GENERALIZED MOORE SPECTRA AND HOPKINS' PICARD GROUPS FOR A SMALLER CHROMATIC LEVEL

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Abstract. Let $\mathcal{L}_n$ for a positive integer n denote the stable homotopy category of $v_n^{-1}BP$-local spectra at a prime number p. Then, M. Hopkins defines the Picard group of $\mathcal{L}_n$ as a collection of isomorphism classes of invertible spectra, whose exotic summand $\mathrm{Pic}^0(\mathcal{L}_n)$ is studied by several authors. In this paper, we study the summand for n with $n^2 \le 2p+2$. For $n^2 \le 2p-2$, it consists of invertible spectra whose $K(n)$-localization is the $K(n)$-local sphere. In particular, X is an exotic invertible spectrum of $\mathcal{L}_n$ if and only if $X \wedge MJ$ is isomorphic to a $v_n^{-1}BP$-localization of the generalized Moore spectrum MJ for an invarinat regular ideal J of length n. For n with $2p-2 < n^2 \le 2p+2$, we consider the cases for $(p,n) = (5,3)$ and $(7,4)$. In these cases, we characterize them by the Smith-Toda spectra $V(n-1)$. For this sake, we show that $L_3V(2)$ at the prime five and $L_4V(3)$ at the prime seven are ring spectra.

1. Introduction

Let $\mathcal{S}_p$ be the stable homotopy category of p-local spectra for an odd prime number p. Consider the Brown-Peterson spectrum BP characterized by the homotopy groups $\pi_*(BP) = BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ over generators v_k with degree $|v_k| = 2(p^k-1)$. We work in the stable homotopy category $\mathcal{L}_n$ for $n \ge 0$ consisting of $v_n^{-1}BP$-local spectra. A spectrum $X \in \mathcal{L}_n$ is called *invertible* if there is a spectrum Y such that $X \wedge Y \simeq L_nS^0$. Here, $L_n \colon \mathcal{S}_p \to \mathcal{L}_n$ denotes the Bousfield localization functor. Mike Hopkins introduced the Picard group $\mathrm{Pic}(\mathcal{L}_n)$ consisting of the isomorphism classes of invertible spectra (*cf.* [14]). Mark Hovey and Hal Sadofsky [4] showed that $\mathrm{Pic}(\mathcal{L}_n)$ is an abelian group with the decomposition

$$\mathrm{Pic}(\mathcal{L}_n) \cong \mathbb{Z} \oplus \mathrm{Pic}^0(\mathcal{L}_n), \tag{1.1}$$

and

$$\mathrm{Pic}^0(\mathcal{L}_n) = 0 \quad \text{if } n^2+n \le q \tag{1.2}$$

for the integer

$$q = 2p-2.$$

For each $n \ge 1$, consider an invariant ideal

$$J = (p^{e_0}, v_1^{e_1}, \dots, v_{n-1}^{e_{n-1}}) \subset BP_* \tag{1.3}$$

for positive integers e_i (see (2.2)). We call a spectrum MJ with $BP_*(MJ) = BP_*/J$ a *type n generalized Moore spectrum.*

(1.4) (Hopkins and Smith [2]) *For each invariant ideal J of the form (1.3), there exists a type n generalized Moore spectrum MJ' for an invariant ideal J' of the form (1.3) with $J' \subset J$.*

Let MJ for an invariant ideal J in (1.3) be a type n generalized Moore spectrum, and put

$$S_J = \{[X] \mid X \in \mathrm{thick}\left\langle L_n S^0 \right\rangle,\ X \wedge MJ \simeq L_n MJ\},$$

where $[X]$ denotes the isomorphism class of X.

Theorem 1.5. *S_J is a subgroup of* $\mathrm{Pic}^0(\mathcal{L}_n)$.

Let $E(n)$ be the n-th Johnson-Wilson spectrum. Then, the category $\mathcal{L}_n$ also consists of $E(n)$-local spectra. The spectrum gives rise to the Hopf algebroid

$$\begin{aligned}&(E(n)_*, E(n)_*(E(n)))\\ &\qquad = (\mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}], E(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E(n)_*)\end{aligned}$$

induced from the Hopf algebroid $BP_*(BP) = BP_*[t_1, t_2, \dots]$ over t_k with $|t_k| = 2(p^k - 1)$. We notice that

(1.6) ([6, Th. 1.1]) $X \in \mathrm{Pic}^0(\mathcal{L}_n)$ *if and only if* $E(n)_*(X) \cong E(n)_*$ *as an* $E(n)_*(E(n))$-*comodule.*

We put

(1.7) $\quad H^{s,t}M = \mathrm{Ext}^{s,t}_{E(n)_*(E(n))}(E(n)_*, M)$ for an $E(n)_*(E(n))$-comodule M.

Then, we have the $E(n)$-based Adams spectral sequence

$$E_2^{s,t}(X) = H^{s,t}E(n)_*(X) \Longrightarrow \pi_{t-s}(L_n X). \tag{1.8}$$

The isomorphism in (1.6) induces an isomorphism

$$E_2^{s,t}(X) \cong E_2^{s,t}(S^0) \quad \text{for } X \in \mathrm{Pic}^0(\mathcal{L}_n). \tag{1.9}$$

From now on, we assume that the integer n satisfies

$$n \geq 3 \quad \text{and} \quad n^2 + n \leq 2q. \tag{1.10}$$

We notice that $n < p - 1$ under (1.10), and that by [10, (10.10)],

$$E_2^{kq+1,kq}(S^0) = 0 \quad \text{for } k > 1. \tag{1.11}$$

In this case, we have a monomorphism $\varphi\colon \mathrm{Pic}^0(\mathcal{L}_n) \to E_2^{q+1,q}(S^0)$ by [6, Th. 1.2] defined by $\varphi(X) = w$ for w in the differential

$$d_{q+1}(1_X) = w1_X \in E_2^{q+1,q}(X), \tag{1.12}$$

where 1_X is the generator of $E_2^{0,0}(X) \cong E_2^{0,0}(S^0) \cong \mathbb{Z}_{(p)}$. The monomorphism is actually an isomorphism:

(1.13) ([12, Cor. 1.9]) *Under the condition (1.10),* $\mathrm{Pic}^0(\mathcal{L}_n) \cong E_2^{q+1,q}(S^0)$.

Let MJ be a type n generalized Moore spectrum for an invariant ideal J of (1.3), and

$$i_J\colon S^0 \to MJ, \tag{1.14}$$

the inclusion to the bottom cell. Consider the induced homomorphism

$$(i_J)_*\colon E_2^{q+1,q}(S^0) \to E_2^{q+1,q}(MJ). \tag{1.15}$$

Then, Theorem 1.5 and (1.13) imply the following:

Proposition 1.16. *Suppose that an integer n satisfies (1.10). Then,*

$$\mathrm{Ker}\ (i_J)_* \cong S_J$$

for $(i_J)_*$ in (1.15) if L_nMJ is a ring spectrum.

This together with (1.13) implies the following theorem:

Theorem 1.17. *Let n be an integer satifying (1.10), and suppose that there exists a type n generalized Moore spectrum MJ for an ideal J in (1.3). If L_nMJ is a ring spectrum with $E_2^{q+1,q}(MJ) = 0$, then $\mathrm{Pic}^0(\mathcal{L}_n) = E_2^{q+1,q}(S^0) = \mathrm{Ker}\ (i_J)_* = S_J$.*

In this paper, we call M a *ring spectrum* if there exist maps $\mu_M\colon M \wedge M \to M$ and $i_M\colon S^0 \to M$ such that the composite $M = S^0 \wedge M \xrightarrow{i_M \wedge M} M \wedge M \xrightarrow{\mu_M} M$ is homotopic to the identity. For a spectrum MJ' in (1.4), Devinatz further showed

(1.18) (Devinatz [1]) *We may take MJ' in (1.4) to be a ring spectrum.*

The celebrated theorems (1.4) and (1.18) enable us to consider an inverse system of type n generalized Moore ring spectra

$$MJ^{(1)} \xleftarrow{\pi^1} MJ^{(2)} \xleftarrow{\pi^2} \cdots \xleftarrow{\pi^{k-1}} MJ^{(k)} \xleftarrow{\pi^k} MJ^{(k+1)} \xleftarrow{\pi^{k+1}} \cdots,$$

in which $J^{(k)} \supset J^{(k+1)}$ are invariant ideals such that $\bigcap_{k\ge 1} J^{(k)} = 0$, and $BP_*(\pi^k)\colon BP_*(MJ^{(k+1)}) \to BP_*(MJ^{(k)})$ are the canonical projections. Furthermore, we assume that the Spanier-Whitehead dual $D(MJ^{(k)})$ of $MJ^{(k)}$ is isomorphic to $\Sigma^a MJ^{(k)}$ for some $a \in \mathbb{Z}$. We fix such an inverse system, and denote the set of the invariant ideals by

(1.19) $\qquad \mathcal{I}_n = \{J^{(k)} \subset BP_* \mid J^{(k)} \text{ is the invariant ideal in the above system}\}.$

Let $L_E\colon \mathcal{S} \to \mathcal{S}$ denote the Bousfield localization functor with respect to a spectrum E. We notice that $L_n = L_{v_n^{-1}BP}$. Furthermore, $L_{F(n)}$ denotes L_F for a type n finite spectrum F, which is well defined since $L_F = L_{F'}$ for type n finite spectra F and F'.

(1.20)(Hovey [3, Th. 2.1]) $\qquad L_{F(n)}X \simeq \operatorname*{holim}_{J\in\mathcal{I}_n} X \wedge MJ$

Consider a spectrum $v_n^{-1}BP \wedge MJ$ for J of (1.3). Then, the Bousfield class $\langle v_n^{-1}BP \wedge MJ\rangle$ of $v_n^{-1}BP \wedge MJ$ equals the Bousfield class $\langle K(n)\rangle$ of the n-th Morava K-theory $K(n)$. Since $L_{F(n)\wedge E}X = L_{F(n)}L_EX$ by [3, Cor. 2.2], we obtain $L_{F(n)}L_nX \simeq L_{K(n)}X$. In particular, taking $X = L_nS^0$ in (1.20), we have

$$L_{K(n)}S^0 \simeq \operatorname*{holim}_{J\in\mathcal{I}_n} L_nMJ.$$

Consider the kernel of the homomorphism $\mathrm{Pic}^0(\mathcal{L}_n) \to \kappa_n \subset \mathrm{Pic}(\mathcal{L}_{K(n)})$ (*cf.* [4, Cor. 2.5]) induced from the localization $L_{K(n)}\colon \mathcal{L}_n \to \mathcal{L}_{K(n)}$:

(1.21) $\qquad S_{(n)} = \{[X] \in \mathrm{Pic}^0(\mathcal{L}_n) \mid L_{K(n)}S^0 \simeq L_{K(n)}X\}.$

Proposition 1.22. $\bigcap_{J\in\mathcal{I}_n} S_J = S_{(n)}$.

The decomposition (1.1) implies that for any invertible spectrum X in $\mathcal{L}_n$, there exists an integer s such that Σ^sX represents an element of $\mathrm{Pic}^0(\mathcal{L}_n)$. Moreover, Morava's structure theorem implies that $E_2^{q+1,q}(MJ) = 0$ if $n^2 \le q$ and $n \ge 3$, and so Theorem 1.17 and Proposition 1.22 as well as (1.2) imply the following:

Corollary 1.23. *Let p and n be the integers in (1.10). If $n^2+n \le q$, then $\mathrm{Pic}^0(\mathcal{L}_n)=0$. If $n^2 \le q < n^2+n$, then $\mathrm{Pic}^0(\mathcal{L}_n) = S_J = S_{(n)}$. In other words, the homomorphism $\mathrm{Pic}^0(\mathcal{L}_n) \to \mathrm{Pic}(\mathcal{L}_{K(n)})$ induced from $L_{K(n)}$ is the zero homomorphism if $n^2 \le q$. Furthermore, X is invertible in $\mathcal{L}_n$ if and only if $X \in \mathrm{thick}\langle L_nS^0\rangle$ and $X \wedge MJ \simeq \Sigma^s L_nMJ$ for an integer s and an ideal J of length n in (1.3).*

Now suppose that $q < n^2$. In this case, we have little knowledge about the homomorphisms $(i_J)_*\colon E_2^{q+1,q}(S^0) \to E_2^{q+1,q}(MJ)$. We notice that there is no pair (p,n) satisfying $n^2-2=q$ or $n^2-3=q$. So we consider the cases

1) $(p,n)=(5,3)$, under which $n^2-1=q$, and
2) $(p,n)=(7,4)$, under which $n^2-4=q$.

Note that in these cases, the Smith-Toda spectrum $V(n-1)=MI_n$ exists but it is not a ring spectrum (*cf.* [11]). Here, $I_n=(p,v_1,\dots,v_{n-1})$ is the invariant prime ideal of BP_*. In this paper, we show the following:

Theorem 1.24. *For $(p,n)=(5,3)$ or $(7,4)$, $L_nV(n-1)$ is a ring spectrum.*

Theorem 1.25. *For $(p,n)=(5,3)$ or $(7,4)$, $E_2^{q+1,q}(V(n-1))=0$.*

Theorems 1.17, 1.24 and 1.25 imply

Corollary 1.26. *For $n=3,4$, we have*

$$\mathrm{Pic}^0(\mathcal{L}_3) = \begin{cases} S_{I_3} & p=5 \\ 0 & p \ge 7 \end{cases} \quad \textit{and} \quad \mathrm{Pic}^0(\mathcal{L}_4) = \begin{cases} S_{I_4} & p=7 \\ 0 & p \ge 11. \end{cases}$$

We notice that $\mathrm{Pic}^0(\mathcal{L}_3)$ at the prime five is isomorphic to S_{J_k} for $J_k=(5,v_1,v_2^k)$ with some $k>1$ (at least p^2), but the result may be less interesting and omit here.

In the next section, we show Theorem 1.5 and Propositions 1.16 and 1.22. We verify Theorem 1.24 in section three. Section four is devoted to showing Theorem 1.25.

2. The group S_J

Consider an ideal

$$J = (p^{e_0}, v_1^{s_1p^{e_1}}, \dots, v_{n-1}^{s_{n-1}p^{e_{n-1}}}) \tag{2.1}$$

of BP_*, where $e_i \ge 0$, $s_i \ge 1$ and $p \nmid s_i$. Then, we notice the following:

(2.2) ([15, Th. 1.5]) *J is an invariant ideal of BP_* if and only if $e_0-1 \le e_1$ and $s_i \le p^{e_{i+1}-e_i-e_0+1}$ for $1 \le i < n$.*

Let $\mathrm{thick}\langle L_nS^0\rangle$ denote the thick subcategory of $\mathcal{L}_n$ generated by L_nS^0. Since $\mathcal{L}_n$ is a monogenic stable homotopy category, we see the following:

(2.3) (*cf.* [5, Th. 2.1.3]) *If $X, Y \in \mathrm{thick}\langle L_nS^0\rangle$, then so are $D(X)$ and $X\wedge Y$. Here, $D(X)$ denotes the Spanier-Whitehead dual of X in $\mathcal{L}_n$.*

Lemma 2.4. *Suppose that $C \in \mathrm{thick}\langle L_nS^0\rangle$ satisfies $C \wedge MJ = 0$ for a type n generalized Moore spectrum MJ with J in (2.1). Then, $C=0$.*

Proof. Let $J_k=(p^{e_0}, v_1^{e_1}, \dots, v_{k-1}^{e_{k-1}})$ be an invariant ideal of $E(n)_*$ for each $0 \le k \le n$ ($J_0=(0)$, $J_n=J$). Suppose that $C\wedge MJ_{k+1}=0$. Then, the cofiber sequence $MJ_k \xrightarrow{v_k^{e_k}} MJ_k \to MJ_{k+1}$ gives rise to an isomorphism $C\wedge MJ_k \xrightarrow[\simeq]{C\wedge v_k^{e_k}}$

$C\wedge MJ_k$, which implies $E(n)_*(C\wedge MJ_k)=v_k^{-1}E(n)_*(C\wedge MJ_k)$. Since $C\wedge MJ_k\in$ thick $\langle L_nS^0\rangle$, $E(n)_*(C\wedge MJ_k)$ is a finitely generated $E(n)_*$-module, and hence $C\wedge MJ_k=0$. Inductively, we deduce $C=0$. □

Proposition 2.5 (*cf.* [5, Lemma A.2.6])**.** *Let $X\in$ thick $\langle L_nS^0\rangle$ and $ev\colon D(X)\wedge X\to S^0$ denote the evaluation map. Then, $ev\wedge D(X)\colon D(X)\wedge X\wedge D(X)\to D(X)$ is a retraction.*

Proof. Consider the cofiber sequence

$$D(X)\wedge X\xrightarrow{ev}L_nS^0\xrightarrow{c}C \tag{2.6}$$

of the evaluation map ev. It suffices to show the map $c\wedge D(X)$ trivial.

The evaluation map ev defines a homomorphism

$$ev_W\colon [D(X),W\wedge D(X)]_*\to[D(X)\wedge X,W]_*$$

by $ev_W(f)=(W\wedge ev)(f\wedge X)$. Consider the full subcategory

$$\mathcal{T}_X=\{W\in\mathcal{L}_n\mid ev_W \text{ is an isomorphism}\}$$

of $\mathcal{L}_n$. Then, it is easy to see $\mathcal{T}_X$ thick, and $L_nS^0\in\mathcal{T}_X$. It follows that thick $\langle L_nS^0\rangle\subset\mathcal{T}_X$. By (2.3) and the cofiber sequence (2.6), we see $C\in$ thick $\langle L_nS^0\rangle$, and so $C\in\mathcal{T}_X$. Therefore, we have an isomorphism

$$ev_C\colon [D(X),C\wedge D(X)]_*\to[D(X)\wedge X,C]_*.$$

Since

$$ev_C(c\wedge D(X))=(C\wedge ev)(c\wedge D(X)\wedge X)=c\circ ev=0,$$

we obtain $c\wedge D(X)=0$ as desired. □

Proof of Theorem 1.5. For $[X],[Y]\in S_J$, $X\wedge Y\wedge MJ\simeq X\wedge MJ\simeq L_nMJ$, and so $[X\wedge Y]\in S_J$.

We note that $D(MJ)=\Sigma^aMJ$ for some integer a. Then,

$$\begin{aligned}D(X)\wedge MJ&\simeq D(X\wedge D(MJ))\simeq D(\Sigma^aX\wedge MJ)\\&\simeq L_nD(\Sigma^aMJ)\simeq L_nD(D(MJ))\simeq L_nMJ.\end{aligned}$$

It follows that $[D(X)]\in S_J$.

For $[X]\in S_J$, we show that X is invertible. Consider the cofiber sequence (2.6). Proposition 2.5 gives rise to the decomposition

$$D(X)\wedge X\wedge D(X)\wedge MJ\simeq(D(X)\wedge MJ)\vee\Sigma^{-1}(C\wedge D(X)\wedge MJ),$$

which induces an isomorphism

$$L_nMJ\simeq L_nMJ\vee\Sigma^{-1}(L_nC\wedge MJ)$$

by the above observation. Since $E(n)_*(MJ)$ is a monogenic $E(n)_*$-module, the summand $E(n)_*(C\wedge MJ)$ is zero. Hence $L_nC\wedge MJ=0$. Note that $C\in$ thick $\langle L_nS^0\rangle$ by (2.3). Thus, Lemma 2.4 shows $C=L_nC=0$, and X is invertible in $\mathcal{L}_n$ with $X^{-1}=D(X)$.

Furthermore, the inclution i_J in (1.14) induces the canonical projection $E(n)_*(i_J)\colon E(n)_*(X)\to E(n)_*(X)/J$ of $E(n)_*(E(n))$-comodules, and so we see X exotic. □

Proof of Proposition 1.16. Let $\varphi\colon \mathrm{Pic}^0(\mathcal{L}_n)\to E_2^{q+1,q}(S^0)$ denote the isomorphism in (1.13), and consider the composite $\varphi'\colon S_J\subset\mathrm{Pic}^0(\mathcal{L}_n)\underset{\cong}{\xrightarrow{\varphi}}E_2^{q+1,q}(S^0)$ for the inclusion in Theorem 1.5. Then, for $[X] \in S_J$, $\varphi'([X]) = w$ for w in $d_{q+1}(1_X) = w1_X$ by (1.12). Note that the isomorphism $\eta_J^X : X \wedge MJ \simeq L_nMJ$ showing $[X] \in S_J$ induces an isomorphism of E_2-terms in the commutative diagram

$$\begin{array}{ccc} E_2^{*,*}(X) & \xrightarrow{(i_J)_*} & E_2^{*,*}(X \wedge MJ) \\ {\scriptstyle \eta_*^X}\downarrow\cong & & \cong\downarrow{\scriptstyle (\eta_J^X)_*} \\ E_2^{*,*}(S^0) & \xrightarrow{(i_J)_*} & E_2^{*,*}(MJ), \end{array}$$

where η_*^X denotes the isomorphism in (1.9). We also have the generator $1_{X\wedge MJ} \in E(n)_*(X \wedge MJ) \overset{(\eta_J^X)_*}{\cong} E(n)_*(MJ)$, which gives rise to the permanent cycle $1_{X\wedge MJ} = (\eta_J^X)_*^{-1}(1_{MJ}) \in E_2^{0,0}(X \wedge MJ)$ for the generator $1_{MJ} \in E_2^{0,0}(MJ)$. Since $(i_J)_*(1_X) = 1_{X\wedge MJ} \in E_2^{0,0}(X \wedge MJ)$, the naturality of the differential shows

$$\begin{aligned} (i_J)_*(w) &= (\eta_J^X)_*(i_J)_*(\eta_*^X)^{-1}(w) = (\eta_J^X)_*(i_J)_*(w1_X) \\ &= (\eta_J^X)_*(i_J)_*(d_{q+1}(1_X)) = (\eta_J^X)_*d_{q+1}((i_J)_*(1_X)) \\ &= (\eta_J^X)_*d_{q+1}(1_{X\wedge MJ}) = d_{q+1}(1_{MJ}) = 0. \end{aligned}$$

Therefore, the monomorphism φ' reduces to $\varphi' \colon S_J \to \mathrm{Ker}\ (i_J)_*$.

For any $w \in \mathrm{Ker}\ (i_J)_*$, let X_w denote an inverse spectrum such that $[X_w] \in \mathrm{Pic}^0(\mathcal{L}_n)$ and $d_{q+1}(1_{X_w}) = w1_{X_w} \in E_2^{q+1,q}(X_w)$. Send this relation under the homomorphism $(i_J)_*$, and we obtain

$$d_{q+1}(1_{X_w\wedge MJ}) = (i_J)_*(d_{q+1}(1_{X_w})) = (i_J)_*(w1_{X_w}) = 0 \in E_2^{q+1,q}(X_w \wedge MJ).$$

Thus, $1_{X_w\wedge MJ} \in E_2^{0,0}(X_w \wedge MJ)$ is a permanent cycle and detects a map $i_J^X : S^0 \to X_w \wedge MJ$. Since L_nMJ is a ring spectrum, the map i_J^X extends to the isomorphism $L_nMJ \simeq X_w \wedge MJ$. Thus, $[X_w] \in S_J$, and φ' is the desired isomorphism. □

Proof of Proposition 1.22. Suppose $X \in \bigcap_{J\in\mathcal{I}_n} S_J$. Then, it is shown in [13, Prop. E] that if $\pi_0(L_nMJ)$ is finite for each $J \in \mathcal{I}_n$, then $L_{K(n)}X \simeq L_{K(n)}S^0$. In our case, $H^{rq,rq}K(n)_*$ is finite and equals zero if $rq > n^2$ by [9, (1.8) Cor., (1.9) Th.] (see Lemma 4.2). It follows that $E_2^{rq,rq}(MJ)$ is finite, and so is $\pi_0(L_nMJ)$. Therefore, $X \in S_{(n)}$. The converse is trivial. □

3. Ring structures on the Smith-Toda spectra $L_nV(n-1)$

We begin with the definition of the Smith-Toda spectra $V(k) = MI_{k+1}$ for the pairs (p, k) of a prime number p and a non-negative integer k with $2k < p$ and $k \le 3$. Here, $I_{k+1} = (p, v_1, \dots, v_k)$ denotes the invariant ideal of BP_*. The spectra $V(k)$ are defined by the cofiber sequences

$$\begin{array}{rll} S^0 \xrightarrow{p} S^0 \xrightarrow{i} V(0) \xrightarrow{j} S^1 & \text{for} & p \ge 2, \\ \Sigma^q V(0) \xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1}V(0) & \text{for} & p \ge 3, \\ \Sigma^{q_2} V(1) \xrightarrow{\beta} V(1) \xrightarrow{i_2} V(2) \xrightarrow{j_2} \Sigma^{q_2+1}V(1) & \text{for} & p \ge 5, \text{ and} \\ \Sigma^{q_3} V(2) \xrightarrow{\gamma} V(2) \xrightarrow{i_3} V(3) \xrightarrow{j_3} \Sigma^{q_3+1}V(2) & \text{for} & p \ge 7, \end{array} \tag{3.1}$$

for

$$q_k = 2(p^k - 1) = |v_k| \tag{3.2}$$

($q_1 = q$). Here, $\alpha \in [V(0), V(0)]_q$, $\beta \in [V(1), V(1)]_{q_2}$ and $\gamma \in [V(2), V(2)]_{q_3}$ are the well known v_1-, v_2-, and v_3-periodic maps due to Adams, Smith and Toda, respectively. In particular, we have a cell decomposition

$$V(2) = \big((S^0 \cup_p e^1) \cup_\alpha (e^{q+1} \cup_p e^{q+2})\big) \cup_\beta \Sigma^{q_2+1} \big((e^0 \cup_p e^1) \cup_\alpha (e^{q+1} \cup_p e^{q+2})\big)$$

and $V(3) = V(2) \cup_\gamma \Sigma^{q_3} CV(2)$. Put

$$\mathfrak{Z}(k) = \{i \in \mathbb{Z} \mid e^i \text{ is a cell of } V(k)\}.$$

As stated in [16, p. 59], if $\pi_{i-1}(V(k)) = 0$ for $i = s + a$ with $a \in \mathfrak{Z}(k)$, then $V(k)^{(s-1)} \wedge V(k) \to V(k)$ extends to $V(k)^{(s)} \wedge V(k) \to V(k)$. Here, $W^{(i)}$ denotes the i-skeleton of W. Consider the Adams-Novikov spectral sequence

$${}^nE_2^{s,t} = \mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, BP_*(V(n-1)) \implies \pi_{t-s}(V(n-1)).$$

Let $\mathcal{P}$ denote the dual of the Hopf algebra generated by the reduced power operations, isomorphic to $\mathbb{Z}/p[\xi_1, \xi_2, \dots] \cong \mathbb{Z}/p[t_1, t_2, \dots] \cong BP_*(BP)/(p, v_1, v_2, \dots)$. The isomorphism $BP_*(BP)/I_n \cong \mathcal{P}$ up to dimension $< q_n$ induces another one

$${}^nE_2^{s,t} \cong \mathrm{Ext}^{s,t}_{\mathcal{P}}(\mathbb{Z}/p, \mathbb{Z}/p)$$

for $t - s < q_n$. Toda [16] showed the following:

(3.3)([16, Lemma 2.2]) $\quad \operatorname{rank} \mathrm{Ext}^{s,t}_{\mathcal{P}}(\mathbb{Z}/p, \mathbb{Z}/p) \le \operatorname{rank} \left(P(b_{k,l}) \otimes H^{*,*}(U(L))\right)^{s,t}$.

Here, the module $H^{s,t}(U(L))$ is also determined in [16, p.55] for $t-s \le (p^3+3p^2+2p+1)q - 4$. In particular, for $t - s \le 2q_3 + 2q_2 + 2q + 7 = (2p^2 + 4p + 6)q + 7$ ($= 1591$ if $p = 7$), $H^{s,t}(U(L))$ is additively generated by the elements in the table:

1	h_0	h_1	g_0	k_0	k_0h_0
$(0,0)$	$(1,1)$	$(1,p)$	$(2,p+2)$	$(2,2p+1)$	$(3,2p+2)$
h_2	h_2h_0	g_1	l_1	l_2	l_1h_1
$(1,p^2)$	$(2,p^2+1)$	$(2,p^2+2p)$	$(3,p^2+2p+3)$	$(3,p^2+3p+1)$	$(4,p^2+3p+3)$
k_1	l_3	k_1h_1	l_1h_2	m_1	m_1h_0
$(2,2p^2+p)$	$(3,2p^2+p+2)$	$(3,2p^2+2p)$	$(4,2p^2+2p+3)$	$(4,2p^2+4p+2)$	$(5,2p^2+4p+3)$

Table 3.4

In the table, the pair of integers under each element shows the dimension of it and the degree of it divided by q.

In the following, we study homotopy groups $\pi_*(V(n-1))$ based on the fact given by (3.3):

(3.5) The homotopy group $\pi_{t-s}(V(n-1))$ is a subquotient of $(P(b_{k,l}) \otimes H^{*,*}(U(L)))^{s,t}$.

3.1. **The case for** $(p, n) = (5, 3)$**:** The set of dimensions of cells of $V(2)$ is

$$\mathfrak{Z}(2) = \{0, 1, 9, 10, 49, 50, 58, 59\}.$$

By [16, Th. 4.4], there exists the pairing

$$V(1) \wedge V(2) \to V(2). \tag{3.6}$$

We notice that this follows from the fact $\pi_{i-1}(V(2)) = 0$ for $i = s + a$ with $s \in \{0, 1, 9, 10\}$ and $a \in \mathfrak{Z}(2)$. So we consider the homotopy groups $\pi_{i-1}(V(2))$ for $i = s + a$ with $s \in \{49, 50, 58, 59\}$ and $a \in \mathfrak{Z}(2)$.

By (3.3) together with Table 3.4, the homotopy groups of degrees ≤ 121 are subquotients generated by the following elements:

deg	0	7	38	39	45	54	76	77
	1	$h_{1,0}$	$b_{1,0}$	$h_{1,1}$	$h_{1,0}b_{1,0}$	g_0	$b_{1,0}^2$	$h_{1,1}b_{1,0}$
deg	83	86	92	93	114	115	121	
	$h_{1,0}b_{1,0}^2$	k_0	$g_0b_{1,0}$	k_0h_0	$b_{1,0}^3$	$h_{1,1}b_{1,0}^2$	$h_{1,0}b_{1,0}^3$	

This together with (3.5) shows that $\pi_{i-1}(V(2)) = 0$ for $i = s + a$ with $s \in \{0, 1, 9, 10, 49, 50\}$ and $a \in \mathfrak{Z}(2)$, and $\pi_{115}(V(2))$ is a subquotient of $\mathbb{Z}/5\{h_{1,1}b_{1,0}^2\}$. Thus, by the next lemma, the pairing (3.6) extends to $V(2) \wedge V(2) \to L_3V(2)$ as desired.

Lemma 3.7. *The homotopy groups $\pi_{115}(L_3V(2))$, $\pi_{116}(L_3V(2))$ and $\pi_{117}(L_3V(2))$ are all trivial.*

Proof. In the $E(3)$-based Adams spectral sequence (1.8), the E_2-term $E_2^{*,*}(V(2))$ for the homotopy groups in the lemma are given by

$$H^{5,120}K(3)_*, \quad H^{4,120}K(3)_* \quad \text{and} \quad H^{3,120}K(3)_*.$$

Then, $\operatorname{rank} H^{s,t}K(3)_* \leq \operatorname{rank}(K(3)_* \otimes H^*L(3,3))$ for the module $H^*L(3,3)$ determined in [9, (3.8) Th.]:

$$\begin{aligned} H^3L(3,3) &= A^2\zeta_3 \oplus A^3, \\ H^4L(3,3) &= A^3\zeta_3 \oplus A^4 \quad \text{and} \\ H^5L(3,3) &= A^4\zeta_3 \oplus (A^3\zeta_3)^* \oplus (A^4)^* \end{aligned}$$

for

$$\begin{aligned} A^2 &= \mathbb{Z}/5\{g_i, k_i, b_{1,i}\}, \\ A^3 &= \mathbb{Z}/5\{g_ih_{1,i+1}, \ell_{1,i}, \ell_{2,i}, \ell_{3,i}, \ell_{4,i}, \ell_{5,i}\} \quad \text{and} \\ A^4 &= \mathbb{Z}/5\{m_{i,j}, m'_i\}. \end{aligned}$$

Here,

$$\begin{aligned} \ell_{1,i} &= h_{1,i}h_{2,i}h_{3,i}, & \ell_{2,i} &= h_{1,i}h_{2,i}h_{2,i+2}, \\ \ell_{3,i} &= h_{1,i}h_{2,i}h_{2,i+1} + h_{1,i}h_{1,i+1}h_{3,i}, & \ell_{4,i} &= h_{1,i}h_{2,i+2}h_{3,i+1}, \\ \ell_{5,i} &= \textstyle\sum_i (h_{1,i}h_{2,i+1} - h_{1,i+1}h_{2,i+2})h_{3,i} \end{aligned}$$

and

$$\begin{aligned} m_{i,j} &= h_{1,i}k_ih_{3,j} = g_ih_{1,i+1}h_{3,j} \quad \text{and} \\ m'_i &= h_{1,i+2}h_{1,i}h_{2,i}(h_{3,i} + h_{3,i+1}) \pm h_{1,i}h_{2,0}h_{2,1}h_{2,2}. \end{aligned}$$

These elements have the degrees (modulo $q_3 = 248 = |v_3|$) as follows:

56	88	40	32	192	200	160	-32	8
g_0	k_0	$b_{1,0}$	g_1	k_1	$b_{1,1}$	g_2	k_2	$b_{1,2}$

96	56	16	48	-32	0	96	8
$g_0h_{1,1}$	$\ell_{1,0}$	$\ell_{2,0}$	$\ell_{3,0}$	$\ell_{4,0}$	$\ell_{5,0}$	$m_{0,j}$	m'_0
232	32	80	240	88	0	232	40
$g_1h_{1,2}$	$\ell_{1,1}$	$\ell_{2,1}$	$\ell_{3,1}$	$\ell_{4,1}$	$\ell_{5,1}$	$m_{1,j}$	m'_1
168	160	152	-40	192	0	168	200
$g_2h_{1,0}$	$\ell_{1,2}$	$\ell_{2,2}$	$\ell_{3,2}$	$\ell_{4,2}$	$\ell_{5,2}$	$m_{2,j}$	m'_2

The dual degrees of these elements (elements in $(A^3\zeta_3)^* \mid (A^4)^*$) are

$$\begin{array}{cccccc|cc} 152, & 192, & 232, & 200, & 32, & 0 & 152, & 240; \\ 16, & 216, & 168, & 8, & 160, & 0 & 16, & 208; \\ 80, & 88, & 96, & 40, & 56, & 0 & 80, & 48. \end{array}$$

Thus, there is no element with degree 120. □

3.2. **The case for** $(p, n) = (7, 4)$**:** The dimensions of cells in $V(3)$ are:

$$\mathfrak{Z}(3) = \{0, 1, 13, 14, 97, 98, 110, 111, 685, 686, 698, 699, 782, 783, 795, 796\}.$$

In [16, Th. 4.4], Toda showed the existence of the pairing $V(2\frac{1}{4}) \wedge V(3) \to V(3)$, which follows from the fact

$$\pi_{i-1}(V(3)) = 0 \quad \text{for } i = s + a > 1 \text{ with } s \in \mathfrak{Z}(3)^{(686)} \text{ and } a \in \mathfrak{Z}(3). \tag{3.8}$$

Here, $\mathfrak{Z}(3)^{(t)} = \{s \in \mathfrak{Z}(3) \mid s \leq t\}$.

Let W be a spectrum sitting in the cofiber sequence

$$\Sigma^{738}S^0 \xrightarrow{\beta_1^9} S^0 \xrightarrow{i_W} W \xrightarrow{j_W} \Sigma^{739}S^0, \tag{3.9}$$

in which $\beta_1 \in \pi_{82}(S^0)$ is the well known generator. Then, W is a ring spectrum by [8, Cor. 2.6].

We show the existence of the pairing $\varphi' \colon V(3) \wedge V(3) \to V(3) \wedge W$ in Proposition 3.21, and $\beta_1^8 = 0 \colon V(3) \to L_4V(3)$ in Lemma 3.23 below. The lemma implies the decomposition $L_4V(3) \wedge W = L_4V(3) \vee \Sigma^{739}L_4V(3)$. Therefore, we obtain the composite

$$V(3) \wedge V(3) \xrightarrow{\varphi'} V(3) \wedge W \to L_4V(3)$$

for φ' in (3.22), which yields the desired ring structure on $L_4V(3)$.

We now prove Proposition 3.21 and Lemma 3.23.

Lemma 3.10. *The homotopy groups $\pi_{i-1}(V(3) \wedge W)$ are trivial for $i = s + a$ with $s, a \in \mathfrak{Z}(3) \setminus \mathfrak{Z}(3)^{(686)}$.*

Proof. Since we have an exact sequence

$$\begin{aligned} \pi_{i-739}(V(3)) \xrightarrow{\beta_1^9} \pi_{i-1}(V(3)) &\xrightarrow{(i_W)_*} \pi_{i-1}(V(3) \wedge W) \\ &\xrightarrow[(j_W)_*]{} \pi_{i-740}(V(3)) \xrightarrow[\beta_1^9]{} \pi_{i-2}(V(3)), \end{aligned} \tag{3.11}$$

we study the homotopy groups $\pi_i(V(3))$ under (3.5). Table 3.4 gives rise to the following table of $(H^*U(L) \otimes \mathbb{Z}/7\{b_{1,1}, b_{2,0}\})^{s,t}$ with $t - s \leq (2p^2 + 4p + 6)q + 7 =$

1591:

$\|x\|$	0	11	83(1)	106(24)	178(14)	189(25)	586(12)	587(13)
x	1	h_0	h_1	g_0	k_0	k_0h_0	$b_{1,1}$	h_2
$\|x\|$	597(23)	598(24)	669(13)	670(14)	681(25)	692(36)	753(15)	754(16)
x	$b_{1,1}h_0$	h_2h_0	$b_{1,1}h_1$	$b_{2,0}$	$b_{2,0}h_0$	$b_{1,1}g_0$	$b_{2,0}h_1$	g_1
$\|x\|$	764(26)	775(37)	776(38)	789(51)	848(28)	849(29)	859(39)	872(52)
x	$b_{1,1}k_0$	$b_{1,1}k_0h_0$	$b_{2,0}g_0$	l_1	$b_{2,0}k_0$	l_2	$b_{2,0}k_0h_0$	l_1h_1
$\|x\|$	1172(24)	1173(25)	1183(35)	1184(36)	1255(25)	1256(26)	1257(27)	1258(28)
x	$b_{1,1}^2$	$b_{1,1}h_2$	$b_{1,1}^2h_0$	$b_{1,1}h_2h_0$	$b_{1,1}^2h_1$	$b_{1,1}b_{2,0}$	$b_{2,0}h_2$	k_1
$\|x\|$	1267(37)	1268(38)	1278(48)	1281(51)	1339(27)	1340(28)	1340(28)	1341(29)
x	$b_{1,1}b_{2,0}h_0$	$b_{2,0}h_2h_0$	$b_{1,1}^2g_0$	l_3	$b_{1,1}b_{2,0}h_1$	$b_{2,0}^2$	$b_{1,1}g_1$	k_1h_1
$\|x\|$	1350(38)	1351(39)	1361(49)	1362(50)	1375(63)	1376(64)	1423(29)	1424(30)
x	$b_{1,1}^2k_0$	$b_{2,0}^2h_0$	$b_{1,1}^2k_0h_0$	$b_{1,1}b_{2,0}g_0$	$b_{1,1}l_1$	l_1h_2	$b_{2,0}^2h_1$	$b_{2,0}g_1$
$\|x\|$	1434(40)	1435(41)	1445(51)	1446(52)	1458(64)	1459(65)	1518(42)	1519(43)
x	$b_{2,0}b_{1,1}k_0$	$b_{1,1}l_2$	$b_{2,0}b_{1,1}k_0h_0$	$b_{2,0}^2g_0$	$b_{1,1}l_1h_1$	$b_{2,0}l_1$	$b_{2,0}^2k_0$	$b_{2,0}l_2$
$\|x\|$	1529(53)	1532(56)	1542(66)	1543(67)				
x	$b_{2,0}^2k_0h_0$	m_1	$b_{2,0}l_1h_1$	m_1h_0				

Table 3.12

Here, in the rows $\|x\|$, the numbers $w(u)$ denote the total degrees of the elements x under them:

$$w = \|x\| = t - s, \quad u \equiv w \mod 82 \quad \text{and} \quad 0 \le u < 82,$$

in which $82 = |b_{1,0}| - 2 = \|\beta_1\|$. In order to find a generator $(P(b_{k,l}) \otimes H^{*,*}(U(L)))^{s,t}$ with $i = t - s$, it suffices to find $w(u)$ in the $\|x\|$ rows in Table 3.12 such that

- $i \equiv u$ mod 82 and
- $w \le i$.

If we find such $w(u)$, then $(P(b_{k,l}) \otimes H^{*,*}(U(L)))^{s,t}$ contains an element of the form

$$x b_{1,0}^c$$

for the integer $c = \frac{i-w}{82}$, where x is the element under $w(u)$ in the table.

Furthermore, we notice the relation

$$\|x\| = \|y\| + 1 \quad \text{if } d_r(x) = y$$

of the total degrees in all of the May spectral sequences $P(b_{k,l}) \otimes H^{*,*}(U(L)) \Rightarrow H^*(V(L))$ and $H^*(V(L)) \Rightarrow H^*\mathcal{P} \cong E_2^*(V(3))$ (*cf.* [16]), and the Adams-Novikov spectral sequence $E_2^*(V(3)) \Rightarrow \pi_*(V(3))$. This implies

(3.13) *Consider an element x with total degree $w(u)$ in Table 3.12 and suppose that x yields a permanent cycle in the E_2-term $E_2^*(V(3))$. Then, if x does not survive to the homotopy group $\pi_*(V(3))$, then it is the image of an element of total degree $w+1(u+1)$ under some differential of the above spectral sequences. In particular, x yields an essential homotopy element of $\pi_w(V(3))$ if there is no element with degree $w+1(u+1)$.*

For the integers in $\mathfrak{Z}(3) \setminus \mathfrak{Z}(3)^{(686)}$, we notice the following:

(3.14)

	698	699	782	783	795	796
mod (82)	42	43	44	45	57	58

We first consider $\pi_{s+a-740}(V(3))$ for $s, a \in \mathfrak{Z}(3) \setminus \mathfrak{Z}(3)^{(686)}$. Then, the integers $s+a-740$ are:

656(0), 657(1), 740(2), 741(3), 753(15), 754(16);
658(2), 741(3), 742(4), 754(16), 755(17);
824(4), 825(5), 837(17), 838(18); 826(6), 838(18), 839(19);
850(30), 851(31); 852(32),

and we find elements

$$b_{1,0}^8 \ (656(0)), \quad h_1 b_{1,0}^7 \ (657(1)), \quad b_{2,0}h_1 \ (753(15)) \quad \text{and} \quad g_1 \ (754(16)) \tag{3.15}$$

from Table 3.12.

Next consider similarly $\pi_{s+a-1}(V(3))$ for $s, a \in \mathfrak{Z}(3) \setminus \mathfrak{Z}(3)^{(686)}$. Then, the integers $s+a-1$ are:

1395(1), 1396(2), 1479(3), 1480(4), 1492(16), 1493(17);
1397(3), 1480(4), 1481(5), 1493(17), 1494(18);
1563(5), 1564(6), 1576(18), 1577(19);
1565(7), 1577(19), 1578(20); 1589(31), 1590(32); 1591(33).

Thus, we find

$$h_1 b_{1,0}^{16} \ (1395(1)) \quad \text{and} \quad g_1 b_{1,0}^9 \ (1492(16)). \tag{3.16}$$

(It is stated in [16, Th. 4.4] that $h_1 b_{1,0}^{2p+3}$ is an obstruction, but it is $h_1 b_{1,0}^{2p+2}$ as shown above.)

Let $[\![x]\!]$ denote the homotopy element detected by x. For example, $[\![b_{1,0}]\!] = \iota_3\beta_1 \in \pi_{82}(V(3))$. Hereafter, ι_3 denotes the inclusion

$$\iota_3 = i_{I_4} = i_3 i_2 i_1 i \colon S^0 \xrightarrow{i} V(0) \xrightarrow{i_1} V(1) \xrightarrow{i_2} V(2) \xrightarrow{i_3} V(3) \tag{3.17}$$

to the bottom cell. We notice that

(3.18) *the elements $b_{1,0}^8$, $b_{1,0}^{17}$, $h_1 b_{1,0}^7$ and $h_1 b_{1,0}^{16}$ detect essential homotopy elements*

by (3.13), since h_1 and $b_{1,0}$ detect the generators $\beta_1' \in \pi_{83}(V(0))$ and $\beta_1 \in \pi_{82}(S^0)$, respectively. Therefore, $\beta_1^9([\![b_{1,0}^8]\!]) = [\![b_{1,0}^{17}]\!] \neq 0$ and $\beta_1^9([\![h_1 b_{1,0}^7]\!]) = [\![h_1 b_{1,0}^{16}]\!] \neq 0$ by (3.18), and so in (3.11),

$$[\![b_{1,0}^8]\!],\ [\![h_1 b_{1,0}^7]\!] \notin \mathrm{Ker}\ \beta_1^9 \quad \text{and} \quad [\![h_1 b_{1,0}^{16}]\!] \in \mathrm{Im}\ \beta_1^9. \tag{3.19}$$

We next consider the elements $b_{2,0}h_1$, g_1 and $g_1 b_{1,0}^9$ in (3.15) and (3.16). Note that $g_1 = \langle h_1, h_1, h_2 \rangle \in E_2^{2,*}(V(3))$. By virtue of (3.13) together with Table 3.12, $d_r(g_1) \in E_r^{r+2,755+r}(V(3))$ must be of the form $b_{1,0}^i b_{2,0} h_1$, but this is not the case by degree reason. Therefore, g_1 is a permanent cycle and $g_1 b_{1,0}^9$ detects an essential homotopy element by (3.13) with Table 3.12. Furthermore, this also implies that $b_{1,0}^9 b_{2,0} h_1$ is not a target of the differential by (3.13), if $b_{2,0}h_1$ is a permanent cycle. Thus,

$$[\![b_{2,0}h_1]\!],\ [\![g_1]\!] \notin \mathrm{Ker}\ \beta_1^9 \quad \text{and} \quad [\![h_1 b_{1,0}^{16}]\!] \in \mathrm{Im}\ \beta_1^9. \tag{3.20}$$

Now the lemma follows from (3.11), (3.15), (3.16), (3.19) and (3.20). □

Proposition 3.21. *$V(3) \wedge W$ is a ring spectrum.*

Proof. We first show the existence of a pairing $\varphi'\colon V(3)\wedge V(3)\to V(3)\wedge W$ such that $\varphi'(\iota_3\wedge V(3)) = V(3)\wedge i_W$ for ι_3 in (3.17) by attaching cells as Toda did. By (3.8), we have an extension $V(3)^{(697)}\wedge V(3)\to V(3)$ of the identity $V(3)\to V(3)$. Thus, we have a composite $V(3)^{(697)}\wedge V(3)\to V(3)\xrightarrow{V(3)\wedge i_W} V(3)\wedge W$. Lemma 3.10 certifies the existence of an extension

$$\varphi'\colon V(3)\wedge V(3)\to V(3)\wedge W \tag{3.22}$$

of the composite.

Now the multiplication of $V(3)\wedge W$ is given by

$$(V(3)\wedge W)\wedge(V(3)\wedge W)\xrightarrow[\simeq]{1\wedge T\wedge 1} V(3)\wedge V(3)\wedge W\wedge W\xrightarrow{\varphi'\wedge\mu_W} V(3)\wedge W\wedge W$$
$$\xrightarrow{1\wedge\mu_W} V(3)\wedge W.$$

Here, T denotes the switching map and μ_W denotes the multiplication of the ring spectrum W. □

Assuming Corollary 4.11, which is shown independently, we see the following:

Lemma 3.23. $\beta_1^8\wedge L_4V(3) = 0\in[V(3),L_4V(3)]_{656}$.

Proof. Consider the cofiber sequence

$$\Sigma^{684}V(2)\xrightarrow{\gamma} V(2)\xrightarrow{i_3} V(3)\xrightarrow{j_3}\Sigma^{685}V(2).$$

Since we have a pairing $\varphi_{23}\colon V(2)\wedge V(3)\to V(3)$ such that $i_3=\varphi_{23}(V(2)\wedge\iota_3)$ for the inclusion $\iota_3\colon S^0\to V(3)$ in (3.17),

$$\beta_1^s\wedge i_3 = \beta_1^s\wedge(\varphi_{23}(V(2)\wedge\iota_3)) = \varphi_{23}(V(2)\wedge\iota_3\beta_1^s)\colon\Sigma^{82s}V(2)\to V(3).$$

By Corollary 4.11, $\iota_3\beta_1^4=0$, and so $\beta_1^4\wedge i_3=0$. Consider a commutative diagram

$$\begin{array}{ccccccc}
[V(3),L_4V(3)]_* & \overset{ad}{\cong} & [V(3)\wedge DV(3),L_4S^0]_* & \overset{ad}{\cong} & [DV(3),L_4DV(3)]_* & = & [V(3),L_4V(3)]_* \\
{\scriptstyle i_3^*}\downarrow & & \downarrow{\scriptstyle i_3^*} & & \downarrow{\scriptstyle D(i_3)_*} & & \downarrow{\scriptstyle (j_3)_*} \\
[V(2),L_4V(3)]_* & \overset{ad}{\cong} & [V(2)\wedge DV(3),L_4S^0]_* & \overset{ad}{\cong} & [DV(3),L_4DV(2)]_* & = & [V(3),L_4V(2)]_*,
\end{array}$$

in which ad denotes the adjunction. This together with the relation $\beta_1^4\wedge i_3=0$ gives rise to $\beta_1^4\wedge j_3=0$.

Therefore, we have elements $\xi_4\in[V(2),L_4V(3)]_{1013}$ and $\xi_4^*\in[V(2),L_4V(3)]_{328}$ such that

$$\beta_1^4\wedge L_4V(3)=\xi_4 j_3\quad\text{and}\quad = i_3\xi_4^*:$$

$$\begin{tikzcd}
 & \Sigma^{656}L_4V(3) \arrow[dl, dotted, "\xi_4^*"'] \arrow[d, "\beta_1^4\wedge 1"] & \\
\Sigma^{328}L_4V(2) \arrow[r, "i_3"] & \Sigma^{328}L_4V(3) \arrow[r, "j_3"] \arrow[d, "\beta_1^4\wedge 1"'] & \Sigma^{1013}L_4V(2) \arrow[dl, dotted, "\xi_4"] \\
 & L_4V(3) &
\end{tikzcd}$$

Thus,

$$\beta_1^8\wedge V(3)=\xi_4 j_3 i_3\xi_4^* = 0$$

as desired. □

4. Proof of Theorem 1.25

Put

$$H^{s,t}M = \mathrm{Ext}^{s,t}_{E(n)_*(E(n))}(E(n)_*, M)$$

for an $E(n)_*(E(n))$-comodule M. Then, we have the Miller-Ravenel change of rings theorem

$$\mathrm{Ext}^*_{BP_*(BP)}(BP_*, v_n^{-1}BP_*/I_n) \cong H^*(E(n)_*/I_n) = H^*(K(n)_*)$$

for the invariant ideal $I_n = (p, v_1, \dots, v_{n-1})$ (*cf.* [4, Th. 3.1]). Here, $K(n)_* = E(n)_*/I_n = \mathbb{Z}/p[v_n, v_n^{-1}]$. Ravenel introduced in [9, §1] the exterior complex

$$C(n) = E(h_{i,j} \colon 1 \le i \le n,\ j \in \mathbb{Z}/n)$$

with differential given by

$$d(h_{i,j}) = \sum_{\ell=1}^{i-1} h_{\ell,j} h_{i-\ell,\ell+j}. \tag{4.1}$$

Here, the bidegree of the generator $h_{i,j}$ is $(1, p^j q_i) \in \mathbb{Z} \times (\mathbb{Z}/q_n)$ for q_i in (3.2). From the results in [9, §1], we deduce the following:

Lemma 4.2. *Let J be an invariant ideal $(p, v_1^{e_1}, \dots, v_{n-1}^{e_{n-1}})$ of $E(n)_*$. Then,*

$$\mathrm{rank}\, H^{s,t}E(n)_*/J \le \mathrm{rank}\ (E(n)_*/J \otimes H^*(C(n), d))^{s,t}$$

as $\mathbb{Z}/p$-modules. In particular, $\mathrm{rank}\, H^{s,t}K(n)_* \le \mathrm{rank}\ (K(n)_* \otimes H^*(C(n), d))^{s,t}$.

Let $M^{*,*}$ denote a basis of the $\mathbb{Z}/p$-vector space $C(n)$ consisting of monomials. In the following, we use the word "monomial" for an element of $M^{*,*}$. Note that $M^{n^2,*}$ consists of only one element of degree 0. We denote the element of $M^{n^2,0}$ by g. For $x \in M^{s,*}$, we define $x^* \in M^{n^2-s,*}$ by

$$xx^* = \pm g, \tag{4.3}$$

and obtain an isomorphism

$$C(n)^{q+1,tq} \xrightarrow[\cong]{(-)^*} C(n)^{n^2-q-1,-tq} \tag{4.4}$$

given by $(-)^*(x) = x^*$ for $x \in M^{q+1,tq}$.

Proposition 4.5. *Suppose $n < p-1$ and let $J_k = I_{n-1} + (v_{n-1}^k)$. Then, $H^{n^2,q}E(n)_*/J_k = 0$ for each $1 \le k \le \sum_{i=2}^{n-1} p^i$.*

Proof. Since $|g| = 0 = |v_n|$, we will find a positive integer a such that $|v_{n-1}^a| = q$. This gives an equation

$$a(p^{n-1} - 1) = p - 1 + b(p^n - 1) \in \mathbb{Z}$$

for an integer b. It follows that $a = bp + (b+1)/\left(\sum_{i=0}^{n-2} p^i\right)$ and so $b+1 = u\sum_{i=0}^{n-2} p^i$ for some $u \ge 1$. Therefore,

$$a = u\sum_{i=0}^{n-1} p^i - p$$

for $u \ge 1$. Thus, if $k < \sum_{i=0}^{n-1} p^i - p$, there is no generator in $H^{n^2,q}E(n)_*/J_k$ by Lemma 4.2. □

Remark. More careful computation using [7, (5.18)] makes k in the proposition greater.

We note that

$$E_2^{q+1,q}(V(n-1)) = H^{q+1,q}E(n)_*/I_n = H^{q+1,q}K(n)_*. \tag{4.6}$$

Corollary 4.7. *For $(p,n)=(5,3)$, $H^{9,8}E(3)_*/J_k = 0$ for $k \le 25$. In particular, Theorem 1.25 holds for $(p,n)=(5,3)$.*

Now turn to the case $(p,n)=(7,4)$.

Lemma 4.8. *Let $(p,n)=(7,4)$.*

1) *$C(4)^{3,-12}$ is generated by the elements*

$$\begin{array}{llll} h_{3,1}h_{4,i}h_{4,j} & \text{for } 0 \le i < j \le 3, & & \\ h_{1,1}h_{2,2}h_{4,i} & \text{and} \quad h_{1,3}h_{2,1}h_{4,i} & \text{for } 0 \le i \le 3, & \\ h_{1,1}h_{1,2}h_{1,3}, & h_{1,1}h_{3,1}h_{3,2}, & h_{1,2}h_{3,1}h_{3,3}, & h_{1,3}h_{3,0}h_{3,1}, \\ h_{2,0}h_{2,2}h_{3,1}, & h_{2,1}h_{2,2}h_{3,3} & \text{and} \quad h_{2,1}h_{2,3}h_{3,1}. & \end{array}$$

2) *$C(4)^{8,336} = 0$.*

Proof. Consider the subalgebra

$$\overline{C}(4) = E(h_{i,j} : 1 \le i \le 3,\ j \in \mathbb{Z}/4) \tag{4.9}$$

of $C(4)$. The degree $|x| \in \mathbb{Z}/q_4 = \mathbb{Z}/4800$ of a monomial $x \in \overline{C}(4)$ is expressed as

$$|x| = 12 \times \sum_{i=0}^{3} 7^i a_{4-i} \quad \text{with } 0 \le a_i \le 6,$$

which we write

$$|x| = (a_1a_2a_3a_4).$$

We further assume that the integers a_i satisfy $(a_1a_2a_3a_4) \le (1111)$ under the lexicographic order. We also use a similar notation

$$(a_1a_2a_3a_4)_{\mathbb{N}} = 12 \times \sum_{i=0}^{3} 7^i a_{4-i} \quad \text{for } a_i \ge 0.$$

Note that

$$|x| \equiv ((a_1+k)(a_2+k)(a_3+k)(a_4+k))_{\mathbb{N}} \mod 4800 \quad \text{for an integer } k \ge 0.$$

For a monomial $x \in \overline{C}(4)$, we also introduce notations

$$[x] = a_1 + a_2 + a_3 + a_4 \quad \text{and} \quad (x)_i = a_{4-i}$$

if $|x| = (a_1a_2a_3a_4)_{\mathbb{N}}$.

For the algebraic generators $h_{i,j}$ of $\overline{C}(4)$, we have

$$(4.10)\quad \begin{array}{ll} \text{a)} & h_{1,1}\ (0010),\ h_{2,0}\ (0011),\ h_{2,1}\ (0110),\ h_{3,0}\ (0111),\ h_{3,1}\ (1110), \\ & \qquad h_{3,3}\ (1011); \\ \text{b)} & h_{1,0}\ (0001),\ h_{1,2}\ (0100),\ h_{1,3}\ (1000),\ h_{2,2}\ (1100),\ h_{2,3}\ (1001), \\ & \qquad h_{3,2}\ (1101). \end{array}$$

The generators $h_{i,j}$ in a) and b) satisfy $(h_{i,j})_1 = 1$ and $(h_{i,j})_1 = 0$, respectively.

1) We will find monomials $x \in \overline{C}(4)^{s,-12}$ with $s \leq 3$. Then
$$|x| = (kkk(k-1))_{\mathbb{N}} \quad \text{for an integer } k \geq 1.$$
If $k = 1$, that is, $(1110)_{\mathbb{N}}$, then from (4.10), we find
$$h_{3,1}\ (s=1), \quad h_{2,1}h_{1,3}, \quad h_{1,1}h_{2,2}\ (s=2), \quad \text{and} \quad h_{1,1}h_{1,2}h_{1,3}\ (s=3).$$
Turn to the case $k = 2$ ($|x| = (2221)_{\mathbb{N}}$ and $[x] = 7$). If $h_{1,i}$ is a factor, say $x = h_{1,i}y$, then $[y] = 6$, and so $y = h_{3,j}h_{3,j'}$:
$$h_{1,3}h_{3,0}h_{3,1}, \quad h_{1,2}h_{3,1}h_{3,3}, \quad h_{1,1}h_{3,1}h_{3,2}.$$
If $h_{2,i}$ is a factor, say $x = h_{2,i}y$, then $[y] = 5$, and so $h_{2,j}h_{3,j'}$:
$$h_{2,0}h_{3,1}h_{2,2}, \quad h_{2,1}h_{3,3}h_{2,2}, \quad h_{2,2}h_{3,3}h_{2,1}, \quad h_{2,3}h_{3,1}h_{2,1}.$$
For $k \geq 3$, then $[x] = 4k - 1 \geq 11$, while $[h_{i,j}h_{i',j'}h_{i'',j''}] \leq 9$. Therefore, no element satisfies this.

Now the first statement of the lemma follows from the relation
$$C(4)^{3,-12} = \overline{C}(4)^{3,-12} \oplus \bigoplus_i h_{4,i}\overline{C}(4)^{2,-12} \oplus \bigoplus_{i,j} h_{4,i}h_{4,j}\overline{C}(4)^{1,-12}.$$

2) We will find monomials $x \in \overline{C}(4)^{s,336}$ with $4 \leq s \leq 8$. In this case, it may have a carry-over: $|x| = (0037), \ldots$. Since there are at most six algebraic generators $h_{i,j}$ of $\overline{C}(4)$ with $(h_{i,j})_k = 1$ for each k, none of the carry-over case occurs. Thus, we consider an element $x \in \overline{C}(4)$ with
$$|x| \equiv (kk(k+4)k)_{\mathbb{N}} \mod 4800 \quad \text{for an integer } 0 \leq k \leq 2.$$
This implies that x has a factor of the form $h_{i_1,j_1}h_{i_2,j_2}h_{i_3,j_3}h_{i_4,j_4}$ for h_{i_l,j_l} in (4.10) a). Therefore, $[x] \geq 8$, and so $k \geq 1$. Therefore, $k = 1, 2$.

If $k = 1$, then $[x] = 8$, and $[h_{1,1}h_{2,0}h_{2,1}h_{3,i}] = 8$. Then,
$$\begin{aligned}|h_{1,1}h_{2,0}h_{2,1}h_{3,i}| =& (0242)_{\mathbb{N}}\ (i=0), (1241)_{\mathbb{N}}\ (i=1),\\ & (1232)_{\mathbb{N}}\ (i=2), (1142)_{\mathbb{N}}\ (i=3).\end{aligned}$$
Thus, these yield no solution.

If $k = 2$, then $[x] = 12$ and $(x)_1 = 6$. Since $(x)_1 = 6$, x has all elements in (4.10) a) as a factor. Let h be the product of the six elements in (4.10) a). Then, $[h] = 14$. This contradicts to $[x] \geq [h]$.

Thus, we have the second from
$$\begin{aligned}C(4)^{8,336} =& \overline{C}(4)^{8,336} \oplus \bigoplus_i h_{4,i}\overline{C}(4)^{7,336} \oplus \bigoplus_{i,j} h_{4,i}h_{4,j}\overline{C}(4)^{6,336}\\ & \oplus \bigoplus_{i,j,k} h_{4,i}h_{4,j}h_{4,k}\overline{C}(4)^{5,336} \oplus h_{4,0}h_{4,1}h_{4,2}h_{4,3}\overline{C}(4)^{4,336}.\end{aligned}$$
□

From Lemma 4.8 2), we deduce

Corollary 4.11. *$\pi_{328}(L_4V(3)) = 0$, and in particular, $\iota_3\beta_1^4 = 0 \in \pi_{328}(L_4V(3))$ for the inclusion ι_3 in (3.17).*

Proof. By the spectral sequences,
$$\operatorname{rank} \pi_{328}(L_4V(3)) \leq \operatorname{rank} E_2^{8,336}(V(3)) \leq \operatorname{rank} (K(4)_* \otimes H^*(C(4), d))^{8,336}.$$
Furthermore, the right hand side is also not greater than rank $(K(4)_* \otimes C(4))^{8,336}$, which is 0 by Lemma 4.8. Therefore, we obtain $\pi_{328}(L_4V(3)) = 0$. □

Proof of Theorem 1.25 for $(p,n)=(7,4)$. By Lemma 4.8 together with (4.4), $C(4)^{13,12}$ is generated by the set $M_1\cup M_2\cup M_3(\subset M^{*,*})$ of monomials given by

$$\begin{aligned}
M_1&=\{g(i,j)=(h_{3,1}h_{4,i}h_{4,j})^* \mid 0\le i<j\le 3\},\\
M_2&=\{g_1(i)=(h_{1,1}h_{2,2}h_{4,i})^*,\ g_2(i)=(h_{1,3}h_{2,1}h_{4,i})^* \mid 0\le i\le 3\}\quad\text{and}\\
M_3&=\{g_1=(h_{1,1}h_{1,2}h_{1,3})^*,\ g_2=(h_{1,1}h_{3,1}h_{3,2})^*,\ g_3=(h_{1,2}h_{3,1}h_{3,3})^*,\\
&\qquad g_4=(h_{1,3}h_{3,0}h_{3,1})^*,\ g_5=(h_{2,0}h_{2,2}h_{3,1})^*,\ g_6=(h_{2,1}h_{2,2}h_{3,3})^*\\
&\qquad\qquad g_7=(h_{2,1}h_{2,3}h_{3,1})^*\}.
\end{aligned}$$

On the differential d, we notice that

$$d(x^*)\doteq \sum y^* \text{ for monomials } x \text{ and } y \text{ if and only if } d(y)\doteq x+\dots.$$

Here, $\doteq$ denotes equality up to sign. In particular, $d((h_{i,j}h_{k,l})^*)=0$ if $i+k\ge 5$. Under these facts with (4.1), we obtain

$$\begin{aligned}
d((h_{1,1}h_{2,2}h_{4,i}h_{4,j})^*)&\doteq (h_{3,1}h_{4,i}h_{4,j})^*\doteq g(i,j),\\
d((h_{1,1}h_{2,2}h_{4,i})^*)&\doteq (h_{3,1}h_{4,i})^* \qquad (d(g_1(i))\doteq (h_{3,1}h_{4,i})^*),\\
d((h_{1,1}h_{1,2}h_{1,3}h_{4,i})^*)&\doteq (h_{1,3}h_{2,1}h_{4,i})^*+(h_{1,1}h_{2,2}h_{4,i})^*\doteq g_1(i)+g_2(i),\\
d((h_{1,1}h_{1,2}h_{1,3})^*)&\doteq (h_{2,1}h_{1,3})^*+(h_{1,1}h_{2,2})^* \qquad (d(g_1)\doteq (h_{2,1}h_{1,3})^*+(h_{1,1}h_{2,2})^*),\\
d((h_{1,0}h_{1,3}h_{2,1}h_{3,1})^*)&\doteq (h_{2,3}h_{2,1}h_{3,1})^*+(h_{1,3}h_{3,0}h_{3,1})^*+(h_{1,3}h_{2,1}h_{4,0})^*+(h_{1,3}h_{2,1}h_{4,1})^*\\
&\doteq g_7+g_4+g_2(0)+g_2(1),\\
d((h_{1,0}h_{1,1}h_{2,2}h_{3,1})^*)&\doteq (h_{2,0}h_{2,2}h_{3,1})^*+(h_{3,2}h_{1,1}h_{3,1})^*+(h_{1,1}h_{2,2}h_{4,0})^*+(h_{1,1}h_{2,2}h_{4,1})^*\\
&\doteq g_5+g_2+g_1(0)+g_1(1),\\
d((h_{1,1}h_{1,2}h_{2,2}h_{3,3})^*)&\doteq (h_{2,1}h_{2,2}h_{3,3})^*+(h_{3,1}h_{1,2}h_{3,3})^*+(h_{1,1}h_{2,2}h_{4,2})^*+(h_{1,1}h_{2,2}h_{4,3})^*\\
&\doteq g_6+g_3+g_1(2)+g_1(3),\\
d((h_{1,1}h_{2,1}h_{2,2}h_{2,3})^*)&\doteq (h_{2,1}h_{2,3}h_{3,1})^*+(h_{2,1}h_{2,2}h_{3,3})^*+(h_{1,1}h_{2,2}h_{4,1})^*+(h_{1,1}h_{2,2}h_{4,3})^*\\
&\doteq g_7+g_6+g_1(1)+g_1(3),\\
d((h_{1,3}h_{2,0}h_{2,1}h_{2,2})^*)&\doteq (h_{2,1}h_{2,2}h_{3,3})^*+(h_{2,0}h_{2,2}h_{3,1})^*+(h_{1,3}h_{2,1}h_{4,0})^*+(h_{1,3}h_{2,1}h_{4,2})^*\\
&\doteq g_6+g_5+g_2(0)+g_2(2)\quad\text{and}\\
d((h_{1,1}h_{1,2}h_{2,3}h_{3,1})^*)&\doteq (h_{1,1}h_{3,1}h_{3,2})^*+(h_{2,1}h_{2,3}h_{3,1})^*+(h_{1,2}h_{3,3}h_{3,1})^*\\
&\doteq g_2+g_7+g_3.
\end{aligned}$$

These give rise to an exact sequence

$$0\to A\xrightarrow{d} C(4)^{13,12}\xrightarrow{d} B\to 0$$

for the submodules $A\subset C(4)^{12,12}$ and $B\subset C(4)^{14,12}$ given by

$$\begin{aligned}
A=\mathbb{Z}/7\{&(h_{1,1}h_{2,2}h_{4,i}h_{4,j})^*,(h_{1,1}h_{1,2}h_{1,3}h_{4,k})^*,(h_{1,0}h_{1,3}h_{2,1}h_{3,1})^*,\\
&(h_{1,0}h_{1,1}h_{2,2}h_{3,1})^*,(h_{1,1}h_{1,2}h_{2,2}h_{3,3})^*,(h_{1,1}h_{2,1}h_{2,2}h_{2,3})^*,\\
&(h_{1,1}h_{1,2}h_{2,3}h_{3,1})^*,(h_{1,3}h_{2,0}h_{2,1}h_{2,2})^* \mid 0\le i<j\le 3, 0\le k\le 3\}\quad\text{and}\\
B=\mathbb{Z}/7\{&(h_{3,1}h_{4,i})^*,(h_{2,1}h_{1,3})^*+(h_{1,1}h_{2,2})^* \mid 0\le i\le 3\}.
\end{aligned}$$

Indeed, the images of the generators of A under d are linearly independent, and $\operatorname{rank} A=16$, $\operatorname{rank} B=5$ and $\operatorname{rank} C(4)^{13,12}=21$. This implies $H^{13,12}C(4)=0$ and then the theorem by Lemma 4.2 with (4.6). $\square$

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