A NOTE ON PRODUCTS IN STABLE HOMOTOPY GROUPS OF SPHERES VIA THE CLASSICAL ADAMS SPECTRAL SEQUENCE

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Abstract. In recent years, Liu and his collaborators found many non-trivial products of generators in the homotopy groups of the sphere spectrum. In this paper, we show a result which not only implies most of their results, but also extends a result of theirs.

1. Introduction

The homotopy groups $\pi_*(S^0)$ of the sphere spectrum S^0 form an algebra with multiplication given by composition. The determination of the structure of $\pi_*(S^0)$ is one of the most important problems in stable homotopy theory. We study the problem by considering the p-component ${}_p\pi_*(S^0)$ of the groups at a prime number p. The classical Adams spectral sequence (ASS) and the Adams-Novikov spectral sequence (ANSS) are typical and effective tools for calculating ${}_p\pi_*(S^0)$. We usually use the ANSS to study ${}_p\pi_*(S^0)$ at an odd prime p, and the ASS at the prime two. In recent years, Liu and his collaborators advocated that the ASS is sufficiently effective at $p > 2$ as well as at $p = 2$. Indeed, they derived out many results on the non-triviality of products of generators in ${}_p\pi_*(S^0)$ from the ASS at $p > 2$ by use of the May spectral sequence (MSS). Their method is simple as follows: for a product $\xi \in {}_p\pi_{t-s}(S^0)$ of generators, let $\overline{\xi}$ be an element of the E_2-term ${}^AE_2^{s,t}$ of the ASS, which detects ξ. We also consider an element x in the E_1-term ${}^ME_1^{s,t,*}$ of the MSS, which converges to $\overline{\xi}$. Then, they proceed their argument in the following steps:

1) The element x is not a coboundary of the first May differential $d_1^M\colon {}^ME_1^{s-1,t,*} \to {}^ME_1^{s,t,*}$.
2) For any $r \geq 2$, the domain of the May differential $d_r^M\colon {}^ME_r^{s-1,t,*} \to {}^ME_r^{s,t,*}$ is zero, and
3) For any $r \geq 2$, the domain of the Adams differential $d_r^A\colon {}^AE_r^{s-r,t-r+1} \to {}^AE_r^{s,t}$ is zero by use of the MSS.

The main theorem of this paper Theorem 1.3 is shown in a similar procedure (Proposition 4.4 and Corollary 4.5 for 1) and 2), and the proof of Theorem 1.3 for 3)) for the homotopy groups $\pi_*(V(2))$ of the second Smith-Toda spectrum $V(2)$ (*cf.* (1.1)). The result is new one, and implies most of results shown by Liu and his collaborators as a corollary.

From here on, we assume that the prime number p is greater than five. Let $H_*(X)$ denote the mod p reduced homology groups of a spectrum X represented by the mod p Eilenberg-MacLane spectrum H. The E_2-term ${}^AE_2^{*,*}(X)$ of the ASS converging to the homotopy groups ${}_p\pi_*(X)$ of a spectrum X is the Ext group $\mathrm{Ext}_{\mathcal{A}_*}^{*,*}(\mathbb{Z}/p, H_*(X))$ of the category of $\mathcal{A}_*$-comodules. Here $\mathcal{A}_* = H_*(H)$ denotes the dual of the Steenrod algebra, which is isomorphic as an algebra to the free algebra $P(\xi_i : i \geq 1) \otimes E(\tau_i : i \geq 0)$ over generators ξ_i's and τ_i's. Let $V(k)$ for $k \geq -1$ denotes the k-th Smith-Toda spectrum defined by $H_*(V(k)) = E(\tau_i : 0 \leq i \leq k)$. Then, for $k \leq 3$, $V(k)$ is known to exist if and only if $p \geq 2k+1$ (Smith [32], Toda [33], Ravenel [31]). In particular, if $p \geq 7$, then $V(k)$ for $k \leq 3$ are given by the cofiber sequences

$$\begin{gathered} S^0 \xrightarrow{p} S^0 \xrightarrow{i} V(0) \xrightarrow{j} \Sigma S^0, \quad \Sigma^q V(0) \xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1} V(0), \\ \Sigma^{(p+1)q} V(1) \xrightarrow{\beta} V(1) \xrightarrow{i_2} V(2) \xrightarrow{j_2} \Sigma^{(p+1)q+1} V(1) \quad \text{and} \\ \Sigma^{(p^2+p+1)q} V(2) \xrightarrow{\gamma} V(2) \xrightarrow{i_3} V(3) \xrightarrow{j_3} \Sigma^{(p^2+p+1)q+1} V(2), \end{gathered} \tag{1.1}$$

in which α is the Adams v_1-periodic map, and β and γ are the v_2- and the v_3-periodic maps given by Smith and Toda, respectively. Hereafter, q denotes the integer $2p-2$, and $\pi_*(S^0)$ denotes ${}_p\pi_*(S^0)$. In this paper, we consider the Greek letter elements of $\pi_*(S^0)$ and $\pi_*(V(0))$ defined by

$$\begin{gathered} \alpha_s = j\alpha^s i, \quad \beta_s = jj_1\beta^s i_1 i \quad \text{and} \quad \gamma_s = jj_1j_2\gamma^s i_2 i_1 i \in \pi_*(S^0); \quad \text{and} \\ \beta'_1 = j_1 \beta i_1 i \in \pi_*(V(0)). \end{gathered} \tag{1.2}$$

We moreover consider some other generators:

$$\zeta_n \in \pi_{(p^n+1)q-3}(S^0), \quad j\xi_n \in \pi_{(p^n+p)q-3}(S^0) \quad \text{and} \quad \varpi_n \in \pi_{(p^n+2p+1)q-3}(S^0)$$

given by Cohen [1], Lin [4] and Liu [19]. Lin and Zheng [7] and Liu [15] constructed generators $\lambda_{n,s} \in \pi_{(p^n+sp^2+sp+s)q-7}(S^0)$ for $n \geq 2$ and $3 \leq s < p-2$. We now state our main theorem, which extends the results [20, Theorems 1.2 and 1.3] of Liu's. In this paper, n denotes a fixed integer > 4.

Theorem 1.3. *Let n be an integer greater than four. The following products of elements of $\pi_*(S^0)$ and $\pi_*(V(0))$ are all non-trivial:*

$$\begin{array}{ll} \alpha_1\varpi_n\gamma_s\beta_1,\ j\xi_n\alpha_1\beta_2\gamma_s \in \pi_{(p^n+sp^2+(s+2)p+s)q-9}(S^0) & \textit{for } 3 \leq s < p, \\ \zeta_n\beta_1\beta_2\gamma_s \in \pi_{(p^n+sp^2+(s+2)p+s)q-10}(S^0) & \textit{for } 3 \leq s < p-2, \textit{ and} \\ \beta'_1\lambda_{n,s}\beta_1 \in \pi_{(p^n+sp^2+(s+2)p+s)q-10}(V(0)) & \textit{for } 3 \leq s < p-2. \end{array}$$

The proof is given at the end of the paper.

Corollary 1.4. *Every factor of the elements $\alpha_1\varpi_n\gamma_s\beta_1$, $j\xi_n\alpha_1\beta_2\gamma_s$, $\zeta_n\beta_1\beta_2\gamma_s$ of ${}_p\pi_*(S^0)$ and $\beta'_1\lambda_{n,s}\beta_1$ of $\pi_*(V(0))$ in the theorem is also non-trivial in the homotopy groups.*

We note that the corollary contains almost of all results of Liu and his collaborators on the non-triviality of products of elements of $\pi_*(S^0)$: [2], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [34], [35], [36] and [37].

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2. The Adams spectral sequence for $\pi_*(V(2))$

Hereafter, $P(x_i)$ and $E(x_i)$ denote a polynomial and an exterior algebras on generators x_i over $\mathbb{Z}/p$, respectively. Let $\mathcal{A}_*$ denote the dual of the Steenrod algebra isomorphic to $P(\xi_1, \xi_2, \dots) \otimes E(\tau_0, \tau_1, \dots)$ as a graded algebra, where $\deg \xi_m =$

$2(p^m-1)$ and $\deg \tau_m = 2p^m-1$. It is also a Hopf algebra with the coproduct $\Delta\colon \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ given by

$$\Delta\xi_m = \sum_{i=0}^m \xi_{m-i}^{p^i} \otimes \xi_i \quad \text{and} \quad \Delta\tau_m = \tau_m \otimes 1 + \sum_{i=0}^m \xi_{m-i}^{p^i} \otimes \tau_i$$

($\xi_0 = 1$). Consider the Adams spectral sequence

$$^{A}E_2^{s,t}(V(2)) = \operatorname{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{Z}/p, H_*(V(2))) \Rightarrow \pi_{t-s}(V(2)).$$

The second Smith-Toda spectrum $V(2)$ satisfies $H_*(V(2)) = E(\tau_0,\tau_1,\tau_2) = \mathcal{A}_* \square_{\overline{\mathcal{A}}_*} \mathbb{Z}/p$ for the quotient Hopf algebra $\overline{\mathcal{A}}_* = P(\xi_1,\xi_2,\dots) \otimes E(\tau_3,\tau_4,\dots)$, and we have the isomorphisms

$$\begin{aligned} ^{A}E_2^{s,t}(V(2)) &= \operatorname{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{Z}/p, H^*(V(2))) \\ &= \operatorname{Ext}_{\mathcal{A}_*}^{s,t}(\mathbb{Z}/p, \mathcal{A}_* \square_{\overline{\mathcal{A}}_*} \mathbb{Z}/p) = \operatorname{Ext}_{\overline{\mathcal{A}}_*}^{s,t}(\mathbb{Z}/p, \mathbb{Z}/p) \end{aligned}$$

by the change of rings theorem (*cf.* [31, A1.3.13]). The Ext group is determined as the cohomology of the cobar complex $C_{\overline{\mathcal{A}}_*}^*$ defined by $C_{\overline{\mathcal{A}}_*}^s = \overline{\mathcal{A}}_* \otimes \cdots \otimes \overline{\mathcal{A}}_*$ (the s-fold tensor product of $\overline{\mathcal{A}}_*$) with coboundary $d_s\colon C_{\overline{\mathcal{A}}_*}^s \to C_{\overline{\mathcal{A}}_*}^{s+1}$ given by $d_s(x) = 1 \otimes x + \sum_{i=1}^s (-1)^i \Delta_i(x) + (-1)^{s+1} x \otimes 1$ for $\Delta_i(x_1 \otimes \ldots \otimes x_s) = x_1 \otimes \ldots \otimes \Delta(x_i) \otimes \ldots \otimes x_s$. We consider the following generators:

$$\begin{aligned} h_i &= [\xi_1^{p^i}] \in {}^{A}E_2^{1,p^i q}(V(2)) \text{ and} \\ b_i &= \Big[\textstyle\sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k} \xi_1^{kp^i} \otimes \xi_1^{(p-k)p^i}\Big] \in {}^{A}E_2^{2,p^{i+1}q}(V(2)) \end{aligned} \tag{2.1}$$

for $i \geq 0$, where $[x]$ denotes the cohomology class of a cocycle x of the cobar complex $C_{\overline{\mathcal{A}}_*}^*$. We also have generators

$$\begin{aligned} g_0 &= \langle h_0, h_0, h_1 \rangle \in {}^{A}E_2^{2,(p+2)q}(V(2)) \text{ and} \\ k_0 &= \langle h_0, h_1, h_1 \rangle \in {}^{A}E_2^{2,(2p+1)q}(V(2)) \end{aligned} \tag{2.2}$$

given by the Massey products. By the juggling theorem of the Massey products, we have a well known relation:

$$g_0 h_1 = h_0 k_0 \in {}^{A}E_2^{3,2(p+1)q}(V(2)). \tag{2.3}$$

3. The May spectral sequence

Hereafter, we abbreviate ${}^{A}E_2^{*,*}(V(2))$ to ${}^{A}E_2^{*,*}$. In this section, we study the Adams E_2-term by the May spectral sequence ${}^{M}E_1^{s,t,u} \Rightarrow {}^{A}E_2^{s,t}$ with

$${}^{M}E_1^{*,*,*} = A \otimes H_0 \otimes H \otimes B$$

and differential $d_r^M : {}^{M}E_r^{s,t,u} \to {}^{M}E_r^{s+1,t,u-r}$. Here,

$$\begin{aligned} A = P(a_i : i \geq 3), \quad H_0 = E(h_{i,0} : i > 0), \\ H = E(h_{i,j} : i > 0, j > 0) \quad \text{and} \quad B = P(b_{i,j} : i > 0, j \geq 0) \end{aligned} \tag{3.1}$$

on the generators

$$\begin{gathered} a_i \in {}^{M}E_1^{1,2p^i-1,2i+1}, \\ h_{i,j} \in {}^{M}E_1^{1,2(p^i-1)p^j,2i-1} \quad \text{and} \quad b_{i,j} \in {}^{M}E_1^{2,2(p^i-1)p^{j+1},p(2i-1)}. \end{gathered}$$

We notice that the May E_1-term is a graded commutative algebra and the May differentials are derivations. For each element $x \in {}^{M}E_1^{s,t,u}$, we denote by $\dim x$ and

$\deg x$ the superscripts s and t, respectively. The first May differential d_1^M is given by

$$
\begin{aligned}
d_1^M(a_i) &= \textstyle\sum_{3\le k<i} h_{i-k,k}a_k, \\
d_1^M(h_{i,j}) = \textstyle\sum_{0<k<i} h_{i-k,k+j}h_{k,j} &\quad \text{and} \quad d_1^M(b_{i,j}) = 0.
\end{aligned} \tag{3.2}
$$

By definition of the May E_1-term, the generators $h_{1,i}$, $b_{1,i}$, $\widehat{g}_0 = h_{2,0}h_{1,0}$ and $\widehat{k}_0 = h_{2,0}h_{1,1}$ are obtained by the elements in (2.1) and (2.2). We also have a generator $\widehat{\gamma}_s$, see [8, Th. 1.1].

Lemma 3.3. *In the May E_1-term, we have permanent cycles*

$$h_{1,i}, \quad b_{1,i}, \quad \widehat{g}_0, \quad \widehat{k}_0 \quad \text{and} \quad \widehat{\gamma}_s = a_3^{s-3}h_{3,0}h_{2,1}h_{1,2}$$

for $i \ge 0$ and $3 \le s < p$, which detect h_i, b_i, g_0, k_0 in (2.1) and (2.2), and $\overline{\gamma}_s \in {}^AE_2^{*,*}$, respectively. Here, $\overline{\gamma}_s$ is an element converging to $i_2i_1i\gamma_s \in \pi_{(sp^2+(s-1)p+s-2)q-3}(V(2))$ for the element γ_s in (1.2)

Throughout this paper, the word 'monomial' means a (nonzero) product of algebraic generators of the May E_1-term up to sign, that is, a monomial xy is identified as yx (without sign) for generators x and y. A monomial $x \in {}^ME_1^{*,*,*}$ is expressed as

$$x = \prod_{x_i \in G} x_i \text{ for a subset } G \subset \{a_{k'}, h_{l,k}, b_{l,k} \mid k' \ge 3,\ k \ge 0,\ l \ge 1\}. \tag{3.4}$$

In particular, if $G = \emptyset$, then $x = 1$. A monomial x of ${}^ME_1^{*,*,*}$ has a factorization

$$x = a(x)h_0(x)f(x) \quad \text{for } a(x) \in A,\ h_0(x) \in H_0,\ f(x) \in H \otimes B. \tag{3.5}$$

Let M denote the set of all monomials of ${}^ME_1^{*,*,*}$. We define mappings $c, c', c_k \colon M \to \mathbb{Z}$ for $k \ge 0$ so that

$$
\begin{aligned}
c'(a_i) &= 1, \quad c'(h_{i,j}) = 0, \quad c'(b_{i,j}) = 0, \\
c_k(a_i) &= \begin{cases} 1 & 0 \le k < i \\ 0 & \text{otherwise} \end{cases}, \quad c_k(h_{i,j}) = \begin{cases} 1 & j \le k < i+j \\ 0 & \text{otherwise} \end{cases}, \\
c_k(b_{i,j}) &= \begin{cases} 1 & j < k \le i+j \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

for the generators of ${}^ME_1^{*,*,*}$, and for a monomial $x = \prod_i x_i$,

$$c'(x) = \sum_i c'(x_i), \quad c_k(x) = \sum_i c_k(x_i)$$

and

$$c(x) = \left(\sum_{k\ge 0} c_k(x)p^k\right) q + c'(x). \tag{3.6}$$

Under the notation, we see that

$$\deg x = c(x). \tag{3.7}$$

We note that the part $\sum_{k\ge0} c_k(x)p^k$ of (3.6) is not always the p-adic expansion of c in $\deg x = cq + c'(x)$. We notice that

$$c'(x) = c_0(a(x)) = c_1(a(x)) = c_2(a(x)) = \dim a(x), \quad c_0(h_0(x)) = \dim h_0(x) \tag{3.8}$$

and

$$c_0(x) = c_0(a(x)h_0(x)) = c'(x) + \dim h_0(x) = \dim a(x)h_0(x). \tag{3.9}$$

Furthermore, we have the following relations on $c_k(x)$:

Lemma 3.10. *Let $x \in {}^M E_1^{*,*,*}$ be a monomial. Then,*

1) *For integers s, t and u with $s > t > u$, we have $c_s(x)+c_u(x)-c_t(x) \le \dim x$.*
2) *For $r \ge 0$, $\dim h_0(x) - r \le c_r(x)$.*

Proof. 1) For a monomial $x = \prod_{x_i \in G} x_i$ in (3.4), we put $C_s(x) = \{x_i \in G \mid c_s(x_i) = 1\}$. We notice that $c_s(x) = \#C_s(x)$ and $C_s(x) \cap C_u(x) \subset C_t(x)$. It follows that $c_s(x)+c_u(x)-c_t(x) \le c_s(x)+c_u(x)-\#(C_s(x)\cap C_u(x)) = \#(C_s(x)\cup C_u(x)) \le \dim x$.

2) We note that $\dim h_{i,0} = 1$ and $c_r(h_{i,0}) = 1$ if $i > r$. For a monomial $x = \prod_{x_i \in G} x_i$, we have

$$\dim h_0(x) = \dim \prod_{h_{i,0} \in G, i \le r} h_{i,0} + \dim \prod_{h_{i,0} \in G, i > r} h_{i,0} \le r + c_r(x).$$

□

We introduce a notation:

$$\mathbf{c}_i(x) = (c_{i-1}(x), c_{i-2}(x), \dots, c_0(x)) \tag{3.11}$$

for $i \ge 1$ and a monomial x.

In the Adams spectral sequence, we write

$$\xi = (y)^{\sim}$$

if a permanent cycle y of the E_2-term detects a homotopy element ξ. This is well defined up to higher filtration of the ASS. The Greek letter elements we consider here are

$$\begin{aligned} \alpha_1 = (h_0)^{\sim} \in \pi_{q-1}(S^0), \quad \beta_1 = (b_0)^{\sim} \in \pi_{pq-2}(S^0), \\ \beta_2 = (k_0)^{\sim} \in \pi_{(2p+1)q-2}(S^0); \quad \text{and} \quad \beta_1' = (h_1)^{\sim} \in \pi_{pq-1}(V(0)), \end{aligned} \tag{3.12}$$

and Cohen's [1], Lin's [4] and Liu's elements [19] :

$$\begin{aligned} &\zeta_n = (h_0 b_{n-1})^{\sim} \in \pi_{(p^n+1)q-3}(S^0) \text{ for } n \ge 1, \\ &j\xi_n = (b_0 h_n + h_1 b_{n-1})^{\sim} \in \pi_{(p^n+p)q-3}(S^0) \text{ for } n \ge 3, \quad \text{and} \\ &\varpi_n = (k_0 h_n)^{\sim} \in \pi_{(p^n+2p+1)q-3}(S^0) \text{ for } n \ge 3. \end{aligned} \tag{3.13}$$

Lin and Zheng [7] constructed a generator

$$\lambda_n = \langle \zeta''_{n-1} i_1, \alpha, \beta_1' \rangle = (b_{n-1} g_0)^{\sim} \in \pi_{(p^n+p+2)q-4}(V(1))$$

(Toda bracket), where $\zeta''_{n-1} \in [V(1), V(1)]_{(p^n+1)q-4}$ satisfies $j_1 \zeta''_{n-1} = ijj_1(\zeta_{n-1} \wedge V(1))$. Lin and Zheng [7] and Liu [15] showed that the composite $\lambda_{n,s} = jj_1 j_2 \gamma^s i_2 \lambda_n$ satisfying

$$\lambda_{n,s} = (b_{n-1} g_0 \overline{\gamma}_s)^{\sim} \in \pi_{(p^n+s(p^2+p+1))q-4-s}(S^0) \tag{3.14}$$

is essential for $n \ge 4$ and $3 \le s < p-2$.

For a monomial $x \in {}^M E_1^{*,*,*}$, we denote by $\widetilde{x}$ the set of monomials, each of these has degree $\deg x$. Consider a monomial

$$l_{i,j} \in \{h_{i,j}, b_{i,j-1}\},$$

and we see that $\widetilde{l}_{i,j} = \widetilde{h}_{i,j} = \widetilde{b}_{i,j-1}$. For example,

$$\widetilde{l}_{2,1} = \{h_{2,1}, b_{2,0}, h_{1,2}h_{1,1}, h_{1,1}b_{1,1}, h_{1,2}b_{1,0}, b_{1,1}b_{1,0}, h_{1,1}b_{1,0}^p, b_{1,0}^{p+1}\}$$

and

$$\widetilde{a}_4 = \{a_4, a_3h_{1,3}, a_3b_{1,2}, a_3h_{1,2}b_{1,1}^{p-1}, a_3b_{1,1}^p\}.$$

Lemma 3.15. *For $u > 0$ and $k \geq 0$, we consider a monomial x of ${}^ME_1^{s,c(x),*}$ such that*

$$c_i(x) = \begin{cases} u & k \leq i < n \\ 0 & i \geq n \end{cases}. \tag{3.16}$$

If $l_{a,b}$ with $k < a+b < n$ (resp. a_b with $k < b < n$) is a factor of x, then x has a factor in $\widetilde{l}_{n-b,b}$ (resp. $\widetilde{a}_n$).

Proof. Consider an element $l_{a,b}$ with $k < a+b < n$ such that $x = x_0 l_{a,b}$ for a monomial x_0. Then, $c_{a+b-\varepsilon}(x_0) = c_{a+b-\varepsilon}(x) - \varepsilon = u - \varepsilon$ for $\varepsilon = 0, 1$, which shows that x_0 has a factor $l_{\iota_1, a+b}$ for an integer $\iota_1 > 0$. Therefore, x has a factor $l_{\iota_1,a+b}l_{a,b} \in \widetilde{l}_{a+\iota_1,b}$. Inductively, we see that x has a factorization

$$l_{\iota_\ell,s_\ell}l_{\iota_{\ell-1},s_{\ell-1}}\cdots l_{\iota_1,s_1}l_{a,b} \quad \text{for some } \ell > 0 \text{ and } s_j = a+b+\textstyle\sum_{i=1}^{j-1}\iota_i,$$

which is in $\widetilde{l}_{n-b,b}$ if $\iota_\ell + s_\ell = n$.

The statement for $\widetilde{a}_n$ is verified similarly. □

For sets S_k for $1 \leq k \leq \ell$ of monomials in the May E_1-terms, we consider a set

$$\mathrm{S}_1\mathrm{S}_2\cdots\mathrm{S}_\ell = \{x_1x_2\cdots x_\ell \mid x_k \in \mathrm{S}_k\}$$

of monomials. In particular, we write $\mathrm{S}^e = \mathrm{S}\cdots\mathrm{S}$ (e factors) if $e > 0$, and $\mathrm{S}^0 = \emptyset$ for a set S. We also define

$$\mathrm{S}^{(d)} = \{x \in \mathrm{S} \mid \dim x = d\}$$

and

$$\underline{\dim}\,\mathrm{S} = \begin{cases} 0 & \mathrm{S} = \emptyset, \\ \min\{\dim x \mid x \in \mathrm{S}\} & \text{otherwise.} \end{cases}$$

In particular, we have

$$\underline{\dim}\,\widetilde{l}^e_{n-\iota,\iota} = \begin{cases} 0 & \iota = 0 \text{ and } e > n, \text{ or } e = 0 \\ 2e-1 & \textit{otherwise.} \end{cases} \tag{3.17}$$

Indeed, if $e \geq 1$ and $\widetilde{l}^e_{n-i,i} \neq \emptyset$, then the dimension of a monomial of the subset

$$h_{n-i,i}(\widetilde{l}^{(2)}_{n-i,i})^{e-1} \subset \widetilde{l}_{n-i,i} \tag{3.18}$$

is $2e-1$ and implies $\underline{\dim}\,\widetilde{l}^e_{n-i,i} = 2e-1$ since $h^2_{i,j} = 0$.

Proposition 3.19. *Suppose that a monomial $x \in {}^ME_1^{s,c(x),*}$ satisfies (3.16) for integers $u > 0$ and $k \geq 0$. Then,*

$$x = lz \quad \textit{for } l \in \widetilde{a}_n^{e_0}\widetilde{l}^{e_1}_{n-\iota_1,\iota_1}\cdots\widetilde{l}^{e_m}_{n-\iota_m,\iota_m},$$

in which $k \geq \iota_1 > \iota_2 > \cdots > \iota_m \geq 0$ for $m \geq 0$, $e_0 \geq 0$, $e_i > 0$ for each $i \geq 1$, $\sum_{i=0}^m e_i = u = c_{n-1}(x)$, and z is a monomial which has no factor of the form $l_{\iota_i-\ell,\ell}$ nor a_{ι_i}. Furthermore, $c_i(z) = 0$ for $i \geq k$ and $c_{\iota_i-1}(z) \leq c_{\iota_i}(z)$.

Note that we do not claim the uniqueness of the factorization of the proposition.

Proof. By Lemma 3.15, we have an integer $\iota_0 \le k$ and an element $y_0 \in \widetilde{l}_{n-\iota_0,\iota_0} \cup \widetilde{a}_n$ such that $x = x_0 y_0$. The factor x_0 also satisfies (3.16) for $k \ge 0$ and $u-1$ unless $u = 1$. Inductively, we obtain a factorization

$$x = z y_{u-1} y_{u-2} \dots y_0,$$

for $y_i \in \widetilde{l}_{n-\iota_i,\iota_i} \cup \widetilde{a}_n$ with $\iota_i \le k$, and z has no factor of the form $l_{\iota_i-\ell,\ell}$ nor a_{ι_i}. Put $l = y_{u-1} \cdots y_0$, and we may consider $l \in \widetilde{a}_n^{e_0} \widetilde{l}_{n-\iota_1,\iota_1}^{e_1} \cdots \widetilde{l}_{n-\iota_m,\iota_m}^{e_m}$ and $\iota_1 > \iota_2 > \cdots > \iota_m \ge 0$. We also obtain the equality $\sum_{j=0}^m e_j = u$. The element z satisfies $c_i(z) = 0$ for $i \ge k$, since $c_i(z) = c_i(x) - c_i(y_{u-1} y_{u-2} \dots y_0) = u - u = 0$.

We also have $c_{\iota_i-1}(z) \le c_{\iota_i}(z)$. Indeed, if $c_{\iota_i-1}(z) > c_{\iota_i}(z)$, then z should have a factor $z' \in \widetilde{l}_{\iota_i-\ell,\ell} \cup \widetilde{a}_{\iota_i}$, which implies $y_i z' \in \widetilde{l}_{n-\ell,\ell} \cup \widetilde{a}_n$. Hence we may replace y_i with $y_i z'$ as a factor of l. □

Now consider the internal degree

$$t_0 = (p^n + p^3 + 2p - 1)q + p - 4. \tag{3.20}$$

We put

$$u_s = \deg a_3^s = (sp^2 + sp + s)q + s \quad \text{for } s \ge 0. \tag{3.21}$$

Lemma 3.22. *Consider a monomial x of the May E_1-term ${}^M E_1^{p+5+\varepsilon-s-r,t_0-u_s-r+1,*}$ with $\varepsilon \in \{0,1\}$, $0 \le s \le p-4$, and $r \ge 1$. Then $\mathbf{c}_{n+1}(x)$ in (3.11) is*

$$\begin{array}{rcl} \mathbf{c}_{n+1}^0(s) & = & (1,0,\dots,0,p-1-s,p+1-s,p-1-s) \text{ or} \\ \mathbf{c}_{n+1}^1(s) & = & (0,p-1,\dots,p-1,p,p-1-s,p+1-s,p-1-s). \end{array} \tag{3.23}$$

Proof. We first note that

$$\dim x \le p + 5 - s < 2p - 1 - s \tag{3.24}$$

by $p \ge 7$. We also note that

$$\begin{array}{rcl} \deg x & = & t_0 - u_s - r + 1 \\ & = & (p^n + p^3 - sp^2 + (2-s)p - 1 - s)q + p - 3 - s - r \\ & = & (\sum_{k\ge 0} c_k(x) p^k) q + c'(x) \end{array} \tag{3.25}$$

by (3.6) and (3.7). Consider the factorization (3.5). By (3.8), we obtain $\dim a(x) = c'(x) \equiv p - 3 - s - r \bmod q$. The inequality

$$q + p - 3 - s - r > p + 5 + \varepsilon - s - r = \dim x$$

implies

$$\dim a(x) = c'(x) = p - 3 - s - r. \tag{3.26}$$

Notice that $c_0(x) \equiv -1 - s \bmod p$ by (3.25), $0 \le c_0(x) \le \dim x$ and $c_0(x) = \dim a(x) + \dim h_0(x)$ by (3.9), and we obtain

$$c_0(x) = p - 1 - s \quad \text{and} \quad \dim h_0(x) = 2 + r. \tag{3.27}$$

It follows that

$$\dim f(x) = 6 + \varepsilon - r. \tag{3.28}$$

Since $c_1(x) \equiv 1 - s \bmod p$ by (3.25), and $2 \le r + 1 = \dim h_0(x) - 1 \le c_1(x)$ by (3.27) and Lemma 3.10 2), we deduce

$$c_1(x) = p + 1 - s$$

under the condition (3.24), and so

$$c_2(x) = p-1-s \quad \text{and} \quad c_3(x) \equiv 0 \mod p.$$

We also see that $c_n(x) = 1$ or $= 0$. If $c_n(x) = 1$, then $c_i(x) = 0$ for $3 \le i < n$ by degree reason. Therefore, we have $\mathbf{c}_{n+1}(x) = \mathbf{c}^0_{n+1}(s)$ in this case.

Suppose that $c_n(x) = 0$. Then, we have an integer j with $3 \le j < n$ such that

$$c_i(x) = \begin{cases} 0 & 3 \le i < j \\ p & i = j \\ p-1 & j < i < n \end{cases}.$$

If $j \ne 3$, then Lemma 3.10 1) shows that $p+5+\varepsilon-s-r \ge c_j(x)+c_1(x)-c_3(x) = 2p+1-s$, which contradicts to (3.24). Thus, $j = 3$ and we have $\mathbf{c}_{n+1}(x) = \mathbf{c}^1_{n+1}(s)$. □

Lemma 3.29. *Let x be a monomial such that $\mathbf{c}_{n+1}(x) = \mathbf{c}^1_{n+1}(s)$ in (3.23). Then,*

$$x = lz \text{ for } l \in \widetilde{a}^e_n \widetilde{l}^{e_3}_{n-3,3} \widetilde{l}^{e_1}_{n-1,1} \widetilde{l}^{e_0}_{n,0},$$

where e, e_3, e_1 and e_0 are non-negative integers such that

$$e + e_3 + e_1 + e_0 = p - 1, \tag{3.30}$$

$e_0 \le n$, $e_3 \in \{s, s+1\}$ and $e_1 \in \{0, 1, 2\}$. The factor z satisfies $c_i(z) = 0$ for $i > 3$, $c'(z) \le 3$,

$$\mathbf{c}_4(z) = (1, e_3 - s, 2 + e_3 - s, e_3 + e_1 - s) \tag{3.31}$$

and $\dim z \ge 3$. *Furthermore, $s + r \le \dfrac{4 + w + \varepsilon - c'(z) - \dim z}{2} < 3$, where w denotes the number of i's with $e_i \ne 0$.*

Proof. Consider a factorization

$$x = lz$$

in Proposition 3.19. Since the integer k in Lemma 3.15 is four in our case,

$$l \in \widetilde{a}^e_n \widetilde{l}^{e_4}_{n-4,4} \widetilde{l}^{e_3}_{n-3,3} \widetilde{l}^{e_2}_{n-2,2} \widetilde{l}^{e_1}_{n-1,1} \widetilde{l}^{e_0}_{n,0} \quad \text{for } e \ge 0 \text{ and } e_i \ge 0 \ (0 \le i \le 4), \quad \text{and} \quad c_i(z) = 0 \quad \text{for } i \ge 4.$$

We may assume that $e_0 \le n$. Indeed, if $e_0 > n$, then $\widetilde{l}^{e_0}_{n,0} = \emptyset$. Furthermore, the fact $c_{n-1}(x) = p-1$ implies $e + \sum_{i=0}^4 e_i = p-1$, and so

$$\mathbf{c}_4(z) = \Big(1 + e_4, e_4 + e_3 - s, 2 + \textstyle\sum_{i=2}^4 e_i - s, \sum_{i=1}^4 e_i - s\Big)$$

since $\mathbf{c}_n(l) = \Big(p-1, \dots, p-1, \sum_{i=0}^4 e_i, \sum_{i=0}^3 e_i, \sum_{i=0}^2 e_i, e_1 + e_0, e_0\Big)$. Notice that $c_3(z) > 0 = c_4(z)$ and $c_1(z) > c_2(z)$. Then, the last statement in Proposition 3.19 implies $e_4 = 0$ and $e_2 = 0$. Thus, we obtain (3.30) and (3.31). By (3.31), $c_1(z) = 2 + c_2(z) \ge 2$. If $c_1(z) \ge 3$, then $\dim z \ge 3$. If $c_1(z) = 2$, then $c_2(z) = 0$. Therefore, z has a factor $l_{1,3} \in \widetilde{l}_{1,3}$ and two factors whose coefficient c_1 is one, and so $\dim z \ge 3$.

Proposition 3.19 implies that $2 \ge e_1$ by (3.31) if $e_1 \ne 0$, and that $0 \le c_2(z) = e_3 - s \le c_3(z) = 1$ if $e_3 \ne 0$. We also see $c_2(z) = -s \ge 0$ if $e_3 = 0$. These show $e_1 \in \{0, 1, 2\}$, and $e_3 \in \{s, s+1\}$. Now, $c'(z) = c_1(a(z)) \le c_1(z) \le 3$ by (3.8) and (3.31).

Note that $e_0 \le n$. By (3.17), we compute

$$\begin{array}{rcl} \dim x & \ge & e + 2(e_3 + e_1 + e_0) - w + \dim z \\ & = & e + 2(p-1-e) - w + \dim z \quad \text{(by (3.30))} \\ & = & 2(p-1) - (p-3-s-r-\dim a(z)) - w + \dim z \\ & & \qquad\qquad \text{(by } c'(x) = e + \dim a(z) \text{ and (3.26)).} \end{array}$$

Since $\dim x = p + 5 + \varepsilon - s - r$, $w \le 3$ and $\dim z \ge 3$, we obtain the last inequality. □

4. Proof of the main theorem

In this section, we also abbreviate ${}^AE_2^{*,*}(V(2))$ to ${}^AE_2^{*,*}$. Put $m_s(x) = x\overline{\gamma}_s g_0 h_1 b_0$ for $x \in {}^AE_2^{*,*}$. Then $m_s(h_n) \in {}^AE_2^{s+6,(p^n+sp^2+(s+2)p+s)q+s}$ and $m_s(b_{n-1}) \in {}^AE_2^{s+7,(p^n+sp^2+(s+2)p+s)q+s}$. We notice that

(4.1) the elements $m_s(h_n)$ and $m_s(b_{n-1})$ are permanent cycles,

since

(4.2) $i_2 i_1 i\,(\alpha_1 \varpi_n \gamma_s \beta_1) = (m_s(h_n))^{\sim}$ and $i_2 i_1 i\,(\zeta_n \beta_1 \beta_2 \gamma_s) = (m_s(b_{n-1}))^{\sim}$.

Indeed, we have

$$\begin{array}{rcl} m_s(h_n) & = & h_n\overline{\gamma}_s g_0 h_1 b_0 = b_0 k_0 h_n h_0 \overline{\gamma}_s = (b_0 h_n + h_1 b_{n-1}) k_0 h_0 \overline{\gamma}_s \text{ and} \\ m_s(b_{n-1}) & = & b_{n-1}\overline{\gamma}_s g_0 h_1 b_0 = h_0 b_{n-1} b_0 k_0 \overline{\gamma}_s = h_1 b_{n-1} g_0 \overline{\gamma}_s b_0 \end{array}$$

by (2.3), and also (3.12), (3.13) and (3.14) imply

$$\begin{array}{rcl} i_2 i_1 i(\alpha_1 \varpi_n \gamma_s \beta_1) & = & (h_0 k_0 h_n \overline{\gamma}_s b_0)^{\sim} \\ & = & (-(b_0 h_n + h_1 b_{n-1}) h_0 k_0 \overline{\gamma}_s)^{\sim} \\ & = & -i_2 i_1 i(j \xi_n \alpha_1 \beta_2 \gamma_s) \text{ and} \\ i_2 i_1 i(\zeta_n \beta_1 \beta_2 \gamma_s) & = & (h_0 b_{n-1} b_0 k_0 \overline{\gamma}_s)^{\sim} \\ & = & (h_1 b_{n-1} g_0 \overline{\gamma}_s b_0)^{\sim} \\ & = & i_2 i_1 (\beta_1' \lambda_{n,s} \beta_1) \end{array} \tag{4.3}$$

in $\pi_*(V(2))$. In particular,

$$i_2 i_1 i\,(\alpha_1 \varpi_n \gamma_s \beta_1) = -i_2 i_1 i\,(j \xi_n \alpha_1 \beta_2 \gamma_s)$$

and

$$i_2 i_1 i\,(\zeta_n \beta_1 \beta_2 \gamma_s) = i_2 i_1\,(\beta_1' \lambda_{n,s} \beta_1)$$

up to Adams filtration. In this section, we show that the elements in (4.2) are non-trivial.

Proposition 4.4. *The elements $m_{p-1}(h_n)$ and $m_{p-1}(b_{n-1})$ of the Adams E_2-term are non-trivial.*

Proof. Let $y_\varepsilon \in {}^AE_2^{p+5+\varepsilon,t_0}$ denote $m_{p-1}(h_n)$ if $\varepsilon = 0$, and $m_{p-1}(b_{n-1})$ if $\varepsilon = 1$. We also take an element $\overline{y}_\varepsilon$ in ${}^ME_1^{p+5+\varepsilon,t_0,*}$, which detects y_ε. If $y_\varepsilon = 0$, then there exists $\overline{x}_\varepsilon \in {}^ME_r^{p+4+\varepsilon,t_0,*}$ such that $d_r^M(\overline{x}_\varepsilon) = \overline{y}_\varepsilon$ for some r. We denote by $x_\varepsilon \in {}^ME_1^{p+4+\varepsilon,t_0,*}$ a monomial appearing in a term of a representative of $\overline{x}_\varepsilon$. By Lemma 3.22 at $(s,r) = (0,1)$, the n-tuple $\mathbf{c}_{n+1}(x_\varepsilon)$ is $\mathbf{c}_{n+1}^0(0)$ or $\mathbf{c}_{n+1}^1(0)$ in (3.23). Since $t_0 \equiv p-4 \mod (q)$ by (3.20), we see $c'(x_\varepsilon) = p-4$. Therefore,

$$x_\varepsilon \in \begin{cases} \widetilde{a}_3^{p-4}\widetilde{l}_{1,n}\widetilde{l}_{1,1}^2\widetilde{l}_{3,0}^3 & \mathbf{c}_{n+1}(x_\varepsilon) = \mathbf{c}_{n+1}^0(0), \\ \widetilde{a}_n^{p-4}\widetilde{l}_{1,3}\widetilde{l}_{1,1}^2\widetilde{l}_{n-1,0}^3 & \mathbf{c}_{n+1}(x_\varepsilon) = \mathbf{c}_{n+1}^1(0). \end{cases}$$

Since $\dim x_\varepsilon = p+4+\varepsilon$ and $\underline{\dim}\left(\widetilde{a}_3^{p-4}\widetilde{l}_{1,n}\widetilde{l}_{1,1}^2\widetilde{l}_{3,0}^3\right) = p+5 = \underline{\dim}\left(\widetilde{a}_n^{p-4}\widetilde{l}_{1,3}\widetilde{l}_{1,1}^2\widetilde{l}_{n-1,0}^3\right)$, we have $\varepsilon = 1$. It follows that there is no monomial for x_0, and so ${}^ME_1^{p+3,t_0,*} = 0$. Therefore, $\overline{y}_0$ survives to $y_0 = m_{p-1}(h_n)$.

We consider the case $\varepsilon = 1$. If $\mathbf{c}_{n+1}(x_1) = \mathbf{c}_{n+1}^1(0)$, then

$$x_1 \in a_n^{p-4}h_{1,3}h_{1,1}b_{1,0}h_{n,0}(\widetilde{l}_{n-1,0}^{(2)})^2$$

by (3.18). Put $w_{i,j} = h_{n-1-i,i}h_{i,0}h_{n-1-j,j}h_{j,0}$. Then, we see that $(\widetilde{l}_{n-1,0}^{(2)})^2 = \{w_{i,j} \colon 1 \le i < j \le n-2\}$. It follows that the monomial x_1 is of the form $x_{1,i,j} = a_n^{p-4}h_{1,3}h_{1,1}b_{1,0}h_{n,0}w_{i,j}$. Since $n > 4$, we have

$$d_1^M(x_{1,i,j}) = -4a_n^{p-5}a_4h_{n-4,4}h_{1,3}h_{1,1}b_{1,0}h_{n,0}w_{i,j} + \cdots \neq 0.$$

The images $d_1^M(x_{1,i,j})$ are linearly independent, since so are $w_{i,j}$'s. Therefore, any linear combination of $x_{1,i,j}$'s doesn't survive to the May E_2-term.

For the case $\mathbf{c}_{n+1}(x_1) = \mathbf{c}_{n+1}^0(0)$, we have

$$x_1 \in a_3^{p-4}h_{1,n}h_{1,1}b_{1,0}h_{3,0}(\widetilde{l}_{3,0}^{(2)})^2$$

by (3.18). Since $(\widetilde{l}_{3,0}^{(2)})^2 = \{h_{1,0}h_{2,0}h_{1,2}h_{2,1}\}$,

$$x_1 = a_3^{p-4}h_{1,n}h_{1,1}b_{1,0}h_{3,0}h_{1,0}h_{2,0}h_{1,2}h_{2,1},$$

which converges to $\overline{\gamma}_{p-1}h_1b_0k_0h_n$ in the Adams E_2-term by Lemma 3.3. Therefore $d_r^M(x_1) = 0$ for $r \ge 1$, and so ${}^ME_r^{s+5,t_0,*} = 0$ for $r \ge 2$.

By the above argument, for $r \ge 2$, we obtain $d_r(x) = 0$ for any $x \in {}^ME_r^{p+5,t_0,*}$. Hence $y_1 = m_{p-1}(b_{n-1})$ survives to the Adams E_2-term. $\square$

Corollary 4.5. *The elements $m_s(h_n)$ for $3 \le s < p$ and $m_s(b_{n-1})$ for $3 \le s < p-2$ in the E_2-terms are non-zero.*

Proof. Since $a_3 \in {}^ME_1^{*,*,*}$ survives to ${}^AE_2^{*,*}$, the multiplication by a_3 induces a homomorphism

$$(a_3)_* \colon {}^AE_2^{*,*} \to {}^AE_2^{*,*}. \tag{4.6}$$

Since $a_3^{p-s-1}\widehat{\gamma}_s = \widehat{\gamma}_{p-1}$ in the May E_1-term by Lemma 3.3, we have $(a_3)_*^{p-s-1}(\overline{\gamma}_s) = \overline{\gamma}_{p-1}$, and hence $(a_3)_*^{p-s-1}(m_s(h_n)) = m_{p-1}(h_n)$. Proposition 4.4 implies the non-triviality of the first element.

Since Lemma 3.3 also implies $(a_3)_*^{p-s-1}(b_{n-1}g_0\overline{\gamma}_s) = b_{n-1}g_0\overline{\gamma}_{p-1}$, we obtain the non-triviality of the second elements similarly by Proposition 4.4. $\square$

Remark 4.7. In the May spectral sequence converging to ${}^AE_2^{*,*}(S^0)$, the geneator a_3 in the E_1-term is not permanent, and therefore the map (4.6) is not defined. This is a reason why we consider the second Smith-Toda spectrum $V(2)$ in this paper.

Proof of Theorem 1.3. It suffices to show that

$${}^AE_2^{p+5+\varepsilon-s'-r,t_0-u_{s'}-r+1} = 0 \tag{4.8}$$

for $\varepsilon \in \{0,1\}$, $r \ge 2$ and $s' \ge \varepsilon$. Indeed, if it holds, then the elements $m_{p-1-s'}(h_n)$ and $m_{p-1-s'}(b_{n-1})$ in (4.1) we concern are not in the image of the Adams differential

$$d_r^A \colon {}^AE_r^{p+5+\varepsilon-s'-r,t_0-u_{s'}-r+1} \to {}^AE_r^{p+5+\varepsilon-s',t_0-u_{s'}}, \tag{4.9}$$

and the theorem follows from (4.2) and Corollary 4.5. We show (4.8) by verifying

$$ {}^{M}E_2^{p+5+\varepsilon-s'-r,t_0-u_{s'}-r+1,*} = 0. $$

For a monomial $x \in {}^{M}E_1^{p+5+\varepsilon-s'-r,t_0-u_{s'}-r+1,*}$ with $r \geq 2$, if $c_3(x) = 0$, then $\dim h_0(x) \leq 3$ by Lemma 3.10 2), which contradicts to (3.27). It follows that $\mathbf{c}_{n+1}(x) = \mathbf{c}_{n+1}^1(s')$ by Lemma 3.22, and so $s' + r \leq 2$ by Lemma 3.29. This implies

$$ (s', r) = (0, 2). $$

Therefore, (4.8) holds except for this case.

We will show ${}^{M}E_2^{p+3,t_0-1,*} = 0$. By Lemma 3.29, a monomial x in ${}^{M}E_1^{p+3,t_0-1,*}$ is factorized into

$$ x = lz $$

for $l \in \widetilde{a}_n^e \widetilde{l}_{n-3,3}^{e_3} \widetilde{l}_{n-1,1}^{e_1} \widetilde{l}_{n,0}^{e_0}$ and a monomial z with $\mathbf{c}_4(z) = (1, e_3, 2 + e_3, e_3 + e_1)$, $e_3 \in \{0, 1\}$ and $e_1 \in \{0, 1, 2\}$. We notice that we can tell the least dimension of z from $\mathbf{c}_4(z)$. Since $e = p - 5 - c'(z)$ by (3.8) and (3.20), we have

$$ e_3 + e_1 + e_0 = p - 1 - e = 4 + c'(z) \tag{4.10} $$

by (3.30). These give rise to a table:

(e_3, e_1)	$(0, 0)$	$(0, 1)$	$(0, 2)$	$(1, 0)$	$(1, 1)$	$(1, 2)$
$\mathbf{c}_4(z)$	$(1, 0, 2, 0)$	$(1, 0, 2, 1)$	$(1, 0, 2, 2)$	$(1, 1, 3, 1)$	$(1, 1, 3, 2)$	$(1, 1, 3, 3)$
$\dim z \geq$	3	3	4	3	3	4
w	1	2	2	2	3	3

Here, w is the integer given in Lemma 3.29. We also see that $w - c'(z) - \dim z \in \{0, 1\}$ by the inequality of Lemma 3.29, and hence $w - \dim z \geq 0$. The table shows us that the inequation holds only when $(e_3, e_1) = (1, 1)$, $\dim z = 3$ and $c'(z) = 0$. Then the monomial x is of the form

$$ x_j = a_n^{p-5} h_{n-3,3} h_{n-1,1} h_{n,0} h_{n-j,j} h_{j,0} h_{4,0} h_{2,0} h_{1,1} $$

for $j \geq 5$. Since

$$ d_1^M(x_j) = -5a_n^{p-6} a_4 h_{n-4,4} h_{n-3,3} h_{n-1,1} h_{n,0} h_{n-j,j} h_{j,0} h_{4,0} h_{2,0} h_{1,1} + \cdots \neq 0, $$

the images $d_1^M(x_j)$ are linearly independent. Thus, (4.8) also holds in this case. $\square$

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