PRODUCTS OF GREEK LETTER ELEMENTS DUG UP FROM THE THIRD MORAVA STABILIZER ALGEBRA

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Abstract. In [3], Oka and the second author considered the cohomology of the second Morava stabilizer algebra to study nontriviality of the products of beta elements of the stable homotopy groups of spheres. In this paper, we use the cohomology of the third Morava stabilizer algebra to find nontrivial products of Greek letters of the stable homotopy groups of spheres: $\alpha_1\gamma_t$, $\beta_2\gamma_t$, $\langle\alpha_1, \alpha_1, \beta^p_{p/p}\rangle\gamma_t\beta_1$ and $\langle\beta_1, p, \gamma_t\rangle$ for t with $p \nmid t(t^2-1)$ for a prime number $p > 5$.

1. Introduction

Greek letter elements are well known generators of the stable homotopy groups of spheres localized at a prime p. Studying products among these elements is an interesting subject, and studied by several authors. For example, at an odd prime p, all products of alpha elements are trivial. In [3], we used $H^*S(2)$ to study nontriviality of the product of beta elements. In this paper, we use $H^*S(3)$ to find relations of Greek letters. The multiplicative structure of $H^*S(3)$ is given by Yamaguchi [7], but unfortunately, it has some typos. So here, our computation is based on Ravenel's.

Let $\beta_{p/p}$ be the generator of the E_2-term $E_2^{2,p^2q}(S)$ of the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(S)$ of the sphere spectrum S. Hereafter, $q = 2p-2$ as usual. A relation given by Toda implies that $\beta_{p/p}$ dies in the Adams-Novikov spectral sequence at a prime $p > 2$. At the prime two, $\beta^2_{2/2} = 0$ by [2, Prop. 8.22], while at the prime numbers three and five, Ravenel showed that $\beta^p_{p/p}$ survives to a homotopy element of $\pi_*(S)$ and $\alpha_1\beta^p_{p/p} = 0$ for the generator α_1 of $\pi_{q-1}(S)$. Here, we show the following

Theorem 1.1. *At a prime $p > 3$, $\beta^p_{p/p}$ survives to $\pi_{(p^3-1)q-2}(S)$ and $\alpha_1\beta^p_{p/p} = 0$.*

Corollary 1.2. *At a prime $p > 3$, the Toda bracket $\langle\alpha_1, \alpha_1, \beta^p_{p/p}\rangle$ $\big(= \alpha_1\beta_{p^2/p^2}\big)$ is defined.*

Remark 1.3. It is already known that $\alpha_1\beta_{p^2/p^2}$ survives in the Adams-Novikov spectral sequence by the work of R. Cohen [1]. Corollary 1.2 states that the Cohen's element is a Toda bracket $\langle\alpha_1, \alpha_1, \beta^p_{p/p}\rangle$.

We notice that at the prime 3, Ravenel showed these in [4].

Let β_1, β_2 and γ_t $(t > 0)$ be the generators of Coker J of dimensions $pq-2$, $(2p+1)q-2$ and $(tp^2+(t-1)p+t-2)q-3$, respectively.

Theorem 1.4. *Let $p > 5$, and t be a positive integer with $p \nmid t(t^2-1)$. Then, the elements $\alpha_1\gamma_t$, $\beta_2\gamma_t$, $\langle\alpha_1, \alpha_1, \beta^p_{p/p}\rangle\beta_1\gamma_t$ and $\langle\beta_1, p, \gamma_t\rangle$ generate subgroups of the* *stable homotopy groups of spheres isomorphic to $\mathbb{Z}/p$. Besides, even in the case $p|(t+1)$, $\beta_1\gamma_t$ and $\langle\beta_1, p, \gamma_t\rangle$ are generators of order p.*

Note that $\langle\beta_1, p, \gamma_t\rangle = \langle\gamma_t, p, \beta_1\rangle$. We also notice that if $t = 1$, then $\langle\gamma_1, p, \beta_1\rangle = 0$, while $\beta_2\gamma_1$ is non-trivial (see section five).

From here on, we assume that the prime number p is greater than three.

2. $H^*S(3)$ revisited

We begin with recalling some notation from Ravenel's green book [4]. Let BP denote the Brown-Peterson spectrum. Then, the pair

$$(BP_*, BP_*(BP)) = (\mathbb{Z}_{(p)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots])$$

is a Hopf algebroid. Here, the degrees of v_i and t_i are $2p^i - 2$. The structure maps act as follows:

$$\begin{aligned}
\eta_R(v_1) &= v_1 + pt_1 \\
\eta_R(v_2) &\equiv v_2 + v_1 t_1^p + pt_2 \mod (p^2, v_1^p) \\
\eta_R(v_3) &\equiv v_3 + v_2 t_1^{p^2} + v_1 t_2^p + pt_3 \\
&\quad -pv_1v_2^{p-1}(t_2 + t_1^{p+1}) \mod (p^2, v_1^2, v_2^p) \\
\Delta(t_1) &= t_1 \otimes 1 + 1 \otimes t_1 \\
\Delta(t_2) &= t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 - v_1 b_{10} \\
\Delta(t_3) &\equiv t_3 \otimes 1 + t_2 \otimes t_1^{p^2} + t_1 \otimes t_2^p + 1 \otimes t_3 \mod (v_1, v_2) \\
\Delta(t_4) &\equiv t_4 \otimes 1 + t_3 \otimes t_1^{p^3} + t_2 \otimes t_2^{p^2} + t_1 \otimes t_3^p + 1 \otimes t_4 \\
&\quad -v_3 b_{12} \mod (v_1, v_2) \\
\Delta(t_5) &= t_5 \otimes 1 + t_4 \otimes t_1^{p^4} + t_3 \otimes t_2^{p^3} + t_2 \otimes t_3^{p^2} + t_1 \otimes t_4^p + 1 \otimes t_5 \\
&\quad -v_3 b_{22} - v_4 b_{13} \mod (p, v_1, v_2)
\end{aligned} \tag{2.1}$$

for

(2.2)

$$\begin{aligned}
b_{1k} &= \frac{1}{p}\left(\Delta(t_1)^{p^{k+1}} - t_1^{p^{k+1}} \otimes 1 - 1 \otimes t_1^{p^{k+1}}\right) = \frac{1}{p}\sum_{i=1}^{p^{k+1}-1} \binom{p^{k+1}}{i} t_1^i \otimes t_1^{p^{k+1}-i} \quad \text{and} \\
b_{2k} &= \frac{1}{p}\left(\Delta(t_2)^{p^{k+1}} - t_2^{p^{k+1}} \otimes 1 - t_1^{p^{k+1}} \otimes t_1^{p^{k+2}} - 1 \otimes t_2^{p^{k+1}} - v_1^{p^{k+1}} b_{1k+1}\right).
\end{aligned}$$

Let $K(3)_* = F_p[v_3, v_3^{-1}]$ have the BP_*-module structure given by $v_i v_3^s = v_3^s v_i = v_3^{s+1}$ if $i = 3$, and $= 0$ otherwise, and

$$\begin{aligned}
\Sigma(3) &= K(3)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} K(3)_* \\
&= K(3)_*[t_1, t_2, \dots]/(v_3 t_i^{p^3} - v_3^{p^i} t_i : i > 0) \quad \text{(by [4, 6.1.16])}
\end{aligned}$$

is the Hopf algebra with structure inherited from $BP_*(BP)$. Define the Hopf algebra $S(3)$ by $S(3) = \Sigma(3) \otimes_{K(3)_*} F_p$, where $K(3)_*$ acts on F_p by $v_3 \cdot 1 = 1$. Then,

$$S(3) = F_p[t_1, t_2, \dots]/(t_i^{p^3} - t_i : i > 0).$$

Now we abbreviate $\mathrm{Ext}_{S(3)}(F_p, F_p)$ to $H^*S(3)$.

Consider integers d_i $(= d_{3,i}$ in [4, 6.3.1])

$$d_i = \begin{cases} 0 & i \le 0, \\ \max(i, pd_{i-3}) & i > 0. \end{cases}$$

Then, there is a unique increasing filtration of the Hopf algebroid $S(3)$ with deg $t_i^{p^j} = d_i$ for $0 \le j < 3$.

Theorem 2.3. (Ravenel[4, 6.3.2]) *The associated Hopf algebra $E^0S(3)$ is isomorphic to the truncated polynomial algebra of height p on the elements $t_i^{p^j}$ for $i > 0$ and $j \in \mathbb{Z}/3$, with coproduct defined by*

$$\Delta(t_i^{p^j}) = \begin{cases} \sum_{k=0}^{i} t_k^{p^j} \otimes t_{i-k}^{p^{k+j}} & i \le 3, \\ t_i^{p^j} \otimes 1 + 1 \otimes t_i^{p^j} + b_{i-3,j+2} & i > 3. \end{cases}$$

Let $L(3)$ be the Lie algebra without restriction with basis $x_{i,j}$ for $i > 0$ and $j \in \mathbb{Z}/3$ and bracket given by

$$[x_{i,j}, x_{k,l}] = \begin{cases} \delta_{i+j}^{l} x_{i+k,j} - \delta_{k+l}^{j} x_{i+k,l} & \text{for } i+k \le 3, \\ 0 & \text{otherwise,} \end{cases}$$

where $\delta_j^i = 1$ if $i \equiv j$ mod 3 and 0 otherwise, and $L(3,k)$ the quotient of $L(3)$ obtained by setting $x_{i,j} = 0$ for $i > k$. Then, Ravenel noticed in [4, 6.3.8]:

Theorem 2.4. *$H^*(L(3,k))$ for $k \le 3$ is the cohomology of the exterior complex $E(h_{i,j})$on one-dimensional generators $h_{i,j}$ with $i \le k$ and $j \in \mathbb{Z}/3$, with coboundary*

$$d(h_{i,j}) = \sum_{s=1}^{i-1} h_{s,j} h_{i-s,s+j}.$$

From now on, we abbreviate $h_{i,j}$ to h_{ij}, and h_{1j} to h_j.

Under the above filtration, Ravenel constructed the May spectral sequences

Theorem 2.5. (Ravenel [4, 6.3.4, 6.3.5]) *There are spectral sequences*

(a) $E_2 = H^*(L(3,3)) \Longrightarrow H^*(E_0S(3))$ *and*

(b) $E_2 = H^*(E_0S(3)) \Longrightarrow H^*(S(3))$.

Since these spectral sequences collapse, $H^*S(3)$ is additively isomorphic to $H^*L(3,3)$. Therefore, we have a projection

$$\pi\colon H^*S(3) \to E^0H^*S(3) = H^*(E_0S(3)) = H^*L(3,3). \tag{2.6}$$

Note that the Massey product $\langle h_i, h_{i+1}, h_{i+2}, h_i \rangle$ is homologous to $v_3^{(2-p)p^i} b_{i+2}$ represented by $v_3^{(2-p)p^i} b_{1,i+2}$ of (2.2), and π assigns the Massey product to $b_{i+2} \in H^*L(3,3)$. Ravenel determined in [4, 6.3.34] the additive structure of $H^*L(3,3)$. In particular, we have the following:

Theorem 2.7. *$H^*L(3,3)$ contains submodules generated by the linear independent elements:*

$$h_1k_1\zeta_3, \quad b_0k_1\zeta_3, \quad h_0l, \quad k_0l, \quad h_0b_0b_2l \quad \text{and} \quad h_1l.$$

Here, $l = h_2h_{21}h_{30}$, $k_i = h_{2i}h_{i+1}$ $(i = 0,1)$, $b_0 = h_1h_{32} + h_{21}h_{20} + h_{31}h_1$, $b_2 = h_0h_{31} + h_{20}h_{22} + h_{30}h_0$ and $\zeta_3 = h_{30} + h_{31} + h_{32}$.

Proof. In the table of the proof of [4, 6.3.34], we find the elements

$$h_0, \quad h_1, \quad k_0, \quad b_0, \quad b_2, \quad l, \quad l' = h_0h_{22}h_{31}, \quad \text{and} \quad \zeta_3,$$

as well as the first element $h_1k_1\zeta_3$ of the theorem. We also have the element $-h_1k_1h_{30} = h_1h_2h_{21}h_{30}$ in the table, which is the last element h_1l of the theorem.

Besides $h_1k_1h_{31}$ and $h_1k_1h_{32}$ are in the table too. We see that $b_0k_1 = -h_1k_1h_{31} + h_1k_1h_{32}$ and so the second element is given by $b_0k_1\zeta_3 = -h_1k_1h_{31}\zeta_3 + h_1k_1h_{32}\zeta_3$.

The element $h_0b_0b_2l\zeta_3$ is computed as

$$\begin{aligned} & h_0h_2h_{21}h_{30}(h_1h_{32} + h_{21}h_{20} + h_{31}h_1)(h_0h_{31} + h_{20}h_{22} + h_{30}h_0)(h_{30} + h_{31} + h_{32}) \\ = \ & -2h_0h_1h_2h_{20}h_{21}h_{22}h_{30}h_{31}h_{32}. \end{aligned}$$

Therefore, $h_0b_0b_2l$ is the dual of the generator $-\frac{1}{2}\zeta_3$, and the elements $h_0b_0b_2l$ and h_0l are generators. Similarly, a computation

$$\begin{aligned} k_0ll'\zeta_3 &= h_{20}h_1h_2h_{21}h_{30}h_0h_{22}h_{31}(h_{30} + h_{31} + h_{32}) \\ &= h_0h_1h_2h_{20}h_{21}h_{22}h_{30}h_{31}h_{32} \end{aligned}$$

shows that k_0l is the dual of the generator $l'\zeta_3$. □

Lemma 2.8. *In $H^*L(3,3)$, $h_0k_1 = 0$ and $k_0k_1 = 0$.*

Proof. From the proof of [4, 6.3.34], we read off the relations $h_0k_1 = e_{30}h_2$ and $k_0k_1 = e_{30}g_1$ in $H^*L(3,2)$. Since e_{30} cobounds h_{30} in $H^*L(3,3)$, the lemma follows. □

3. Greek letter elements

Let $E_r^{s,t}(X)$ denote the E_r-term of the Adams-Novikov spectral sequence converging to the homotopy group $\pi_{t-s}(X)$ of a spectrum X. Then the E_2-term is $\mathrm{Ext}_{BP_*(BP)}(BP_*, BP_*(X))$. We here consider the Ext-group $\mathrm{Ext}_{BP_*(BP)}(BP_*, M)$ for a $BP_*(BP)$-comodule M as the cohomology of the cobar complex $\Omega^*_{BP_*(BP)}M$ (*cf.* [2]). Consider a sequence $A = (a_0, a_1, \dots, a_n)$ of non-negative integers so that the sequence $p^{a_0}, v_1^{a_1}, \dots, v_n^{a_n}$ is invariant and regular. For such a sequence A, Miller, Ravenel and Wilson introduced in [2] n-th Greek letter elements $\eta^{(n)}_{s(A)}$ in the Adams-Novikov E_2-term $E_2^{n,t(A)}(S)$ by

$$\eta^{(n)}_{s(A)} = \delta_{A,1}\cdots\delta_{A,n}(v_n^{a_n}) \in E_2^{n,t(A)}(S) \tag{3.1}$$

for $v_n^{a_n} \in \mathrm{Ext}^{0,2a_n(p^n-1)}_{BP_*(BP)}(BP_*, BP_*/I(A,n))$. Here, $s(A) = a_n/a_{n-1}, a_{n-2}, \cdots, a_0$ and $t(A) = 2a_n(p^n-1) - 2\sum_{i=0}^{n-1} a_i(p^i-1)$, $I(A,k)$ denotes the ideal of BP_* generated by $p^{a_0}, v_1^{a_1}, \dots, v_{k-1}^{a_{k-1}}$, and $\delta_{A,k+1}$ is the connecting homomorphism associated to the short exact sequence

$$0 \to BP_*/I(A,k) \xrightarrow{v_k^{a_k}} BP_*/I(A,k) \to BP_*/I(A,k+1) \to 0.$$

In particular, we write $\alpha = \eta^{(1)}$, $\beta = \eta^{(2)}$ and $\gamma = \eta^{(3)}$. So far, only when $n \le 3$, many conditions for that Greek letter elements survives to homotopy elements are known. We abbreviate $\eta^{(n)}_{s(A)}$ to $\eta^{(n)}_{a_n}$ if $A = (1, \dots, 1, a_n)$ as usual. For example, we consider β-elements defined by

$$\begin{aligned} &\beta_s = \delta_{(1,1),1}(\beta'_s) \in E_2^{2,t(1,1,s)}(S) \\ &\qquad \text{for } \beta'_s = \delta_{(1,1),2}(v_2^s) \in E_2^{1,t(1,1,s)}(V(0)), \text{ and} \\ &\beta_{p^i/p^i} = \beta_{p^i/p^i,1} = \delta_{(1,p^i),1}\delta_{(1,p^i),2}(v_2^{p^i}) \in E_2^{2,t(1,p^i,p^i)}(S). \end{aligned} \tag{3.2}$$

Hereafter we assume that the prime p is greater than three. We have the Smith-Toda spectrum $V(k)$ for $k = 0, 1, 2$ defined by the cofiber sequences

$$\begin{aligned} & S \xrightarrow{p} S \xrightarrow{i} V(0) \xrightarrow{j} \Sigma S, \\ (3.3) \qquad & \Sigma^q V(0) \xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1} V(0) \quad \text{and} \\ & \Sigma^{(p+1)q} V(1) \xrightarrow{\beta} V(1) \xrightarrow{i_2} V(2) \xrightarrow{j_2} \Sigma^{(p+1)q+1} V(1). \end{aligned}$$

Here, $\alpha \in [V(0), V(0)]_q$ is the Adams map and $\beta \in [V(1), V(1)]_{(p+1)q}$ is the v_2-periodic element due to L. Smith. Note that the BP_*-homology of these spectra are $BP_*(V(k)) = BP_*/I_{k+1}$ for the ideal I_k of BP_* generated by v_i for $0 \le i < k$ with $v_0 = p$. We consider the Bousfield-Ravenel localization functor L_3 with respect to $v_3^{-1}BP$. The E_2-term $E_2^*(L_3V(2))$ of $L_3V(2)$ is isomorphic to $K(3)_* \otimes H^*S(3)$, whose structure is given in [4] (see also [7]), and we consider the composite

$$r\colon E_2^*(S) \xrightarrow{\iota_*} E_2^*(V(2)) \xrightarrow{\eta} E_2^*(L_3V(2)) \xrightarrow{\rho} H^*(S(3)) \xrightarrow{\pi} H^*L(3,3).$$

Here the first map is induced from the inclusion $\iota\colon S \to V(2)$ to the bottom cell, the second is from the localization map, the third is obtained by setting $v_3 = 1$ and the last is the projection (2.6).

Lemma 3.4. *The map r assigns the Greek letter elements as follows:*

$$\begin{aligned} r(\alpha_1) &= h_0, \quad r(\beta_1) = -b_0, \quad r(\beta_2) = 2k_0, \\ r(\gamma_t) &= -t(t^2-1)l - t(t-1)k_1\zeta_3 \quad \text{and} \quad r(\beta_{p/p}) = -b_1. \end{aligned}$$

We also have $\beta_1' = h_1 - v_1^{p-1}h_0 \in E_2^{1,pq}(V(0))$ for the generators h_i of $E_2^{1,p^iq}(V(0))$ represented by $t_1^{p^i}$.

Proof. First we consider the images of the Greek letter elements under the map $\iota_*\colon E_2^*(S) \to E_2^*(V(2))$. In the cobar complex $\Omega^*_{BP_*(BP)}BP_*$, by (2.1), $d(v_1) = pt_1$, $d(v_2^{p^i}) \equiv v_1^{p^i}t_1^{p^{i+1}} - v_1^{p^{i+1}}t_1^{p^i}$ mod (p) for $i \ge 0$, $d(v_2^2) \equiv 2v_1v_2t_1^p + v_1^2t_1^{2p}$ mod (p, v_1^p), and $d(v_3^t) \equiv tv_2v_3^{t-1}t_1^{p^2} + \binom{t}{2}v_2^2v_3^{t-2}t_1^{2p^2} + \binom{t}{3}v_2^3v_3^{t-3}t_1^{3p^2}$ mod (p, v_1, v_2^4), which imply

$$\begin{aligned} \delta_{(1),1}(v_1) &= [t_1], \quad \delta_{(1,1),2}(v_2) = [t_1^p - v_1^{p-1}t_1], \\ \delta_{(1,1),2}(v_2^2) &= [2v_2t_1^p + v_1t_1^{2p} + v_1^{p-1}y], \quad \delta_{(1,p),2}(v_2^p) = [t_1^{p^2} - v_1^{p^2-p}t_1^p] \quad \text{and} \\ \delta_{(1,1,1),3}(v_3^t) &= [tv_3^{t-1}t_1^{p^2} + \tbinom{t}{2}v_2v_3^{t-2}t_1^{2p^2} + \tbinom{t}{3}v_2^2v_3^{t-3}t_1^{3p^2} + v_2^3z] = \overline{\gamma}_t, \end{aligned}$$

for cochains $y \in \Omega^1_{BP_*(BP)}BP_*/(p)$ and $z \in \Omega^1_{BP_*(BP)}BP_*/(p, v_1)$. Here, $[x]$ denotes a cohomology class represented by a cocycle x. The first one shows $\alpha_1 = h_0$, and the second gives the last statement of the lemma. We further see that $d(t_1^{p^k}) = -pb_{1k-1}$ for $k \ge 1$ and $d(v_k) \equiv pt_k$ mod $I((2,1,1),k)$ for $k = 2, 3$ by (2.1) in $\in \Omega^1_{BP_*(BP)}BP_*$. Moreover, $[b_{1k}]$'s are assigned to b_k in $H^*L(3,3)$ under the projection π, and we obtain

$$\begin{aligned} & r\delta_{(1,p^{k-1}),1}(h_k - v_1^{p^k-p^{k-1}}h_{k-1}) = -b_{k-1} \quad \text{for } k = 1, 2, \\ & r\delta_{(1,1),1}([2v_2t_1^p + v_1t_1^{2p}]) = 2k_0, \\ & \delta_{(1,1,1),2}(\overline{\gamma}_t) = [t(t-1)v_3^{t-2}t_2^p \otimes t_1^{p^2} + \tbinom{t}{2}v_3^{t-2}t_1^p \otimes t_1^{2p^2} + w] = \gamma_t' \quad \text{and} \\ & r\delta_{(1,1,1),1}(\gamma_t') = t(t-1)(t-2)h_{30}k_1 + t(t-1)r\delta_{(1,1,1),1}([t_2^p \otimes t_1^{p^2} + \tfrac{1}{2}t_1^p \otimes t_1^{2p^2}]). \end{aligned}$$

Here, w is a linear combination of terms in the ideal $(v_1, v_2)^2$. Thus the relations other than $r(\gamma_t)$ follows.

We note that b_{20} in (2.2) corresponds to $h_{21}h_{30}+h_{31}h_{21}$ by $\Delta(t_5)^p$ in (2.1). Since $d(t_2^p) = -t_1^p\otimes t_1^{p^2}+v_1^pb_{11}-pb_{20}$ by (2.1), we obtain $r\delta_{(1,1,1),1}([t_2^p\otimes t_1^{p^2}+\frac{1}{2}t_1^p\otimes t_1^{2p^2}]) = -(h_{21}h_{30}+h_{31}h_{21})h_2+h_{21}b_1 = -3l-k_1\zeta_3$, which shows the relation on $r(\gamma_t)$. □

Recall the cofiber sequences (3.3) and the v_3-periodic element $\gamma \in [V(2),V(2)]_{q_3}$ ($q_3 = (p^2+p+1)q$) due to H. Toda. Then, the Greek letter elements in homotopy are defined by

(3.5) $$\alpha_t = j\alpha^t i, \quad \beta_t = j\beta'_t \quad \text{for } \beta'_t = j_1\beta^t i_1 i \quad \text{and} \quad \gamma_t = jj_1j_2\gamma^t i_2i_1i$$

for $t>0$, and the Greek elements in the E_2-term survives to the same named one in homotopy by the Geometric Boundary Theorem (*cf.* [4]).

Proof of Theorem 1.4. We begin with noticing that the element b_i in $H^*L(3,3)$ is the image of the Massey product $\langle h_i,h_{i+1},h_{i+2},h_i\rangle$ under π, which is homologous to b_i represented by b_{1i} in (2.2). We further note that the Toda brackets $\langle\alpha_1,\alpha_1,\beta_{p/p}^p\rangle$ and $\langle\beta_1,p,\gamma_t\rangle$ are detected by α_1b_2 and $h_1\gamma_t$ of $E_2^*(S)$, respectively. Indeed, in the first bracket, $d_{2p-1}(b_2) = \alpha_1\beta_{p/p}^p$ by Corollary 4.4 below, and in the second bracket, $\langle\beta_1,p,\gamma_t\rangle = j\langle\beta'_1,p,\gamma_t\rangle$. Under the condition on t, Lemmas 3.4, 2.7 and 2.8 imply that each element of $\alpha_1\gamma_t$, $\beta_2\gamma_t$, $\alpha_1b_2\gamma_t\beta_1$ and $h_1\gamma_t$, as well as $\beta_1\gamma_t$, generates a submodule isomorphic to $\mathbb{Z}/p$ of the E_2-term $E_2^*(S)$. These are, of course, permanent cycles, and nothing kills them in the Adams-Novikov spectral sequence since each element has a filtration degree less than $2p-1$. □

4. $\beta_{p/p}^p$ IN THE HOMOTOPY OF SPHERES

Let X and $\overline{X}$ be the $(p-1)q$- and $(p-2)q$-skeletons of the Brown-Peterson spectrum BP. Then, we have the cofiber sequences

(4.1) $$S \xrightarrow{\iota} X \xrightarrow{\kappa} \Sigma^q\overline{X} \xrightarrow{\lambda} S^1 \quad \text{and} \quad \overline{X} \xrightarrow{\iota'} X \xrightarrow{\kappa'} S^{(p-1)q} \xrightarrow{\lambda'} \Sigma\overline{X}.$$

Then,

$$BP_*(X) = BP_*[x]/(x^p) \quad \text{and} \quad BP_*(\overline{X}) = BP_*[x]/(x^{p-1})$$

as subcomodules of $BP_*(BP)$, where x corresponds to t_1. From [4, Chap.7], we read off the following:

(4.2) $b_1^p = 0 \in E_2^{2p,p^3q}(X)$, *which implies*

$$E_2^{2s+e,tq}(X) = 0 \quad \text{if } s\ge p \text{ and } t<(s-1)p^2+(s+1+e)p.$$

Lemma 4.3. $b_0\colon E_2^{2s+e,tq}(S) \to E_2^{2s+2+e,(t+p)q}(S)$ *is monomorphic if* $s\ge p$ *and* $t\le (s-1)p^2+(s+e)p$.

Proof. Note that $b_0 = \lambda\lambda'$, and the lemma follows from (4.2) and the exact sequences

$$E_2^{2s+e,(t+p-1)q}(X) \xrightarrow{\kappa'} E_2^{2s+e,tq}(S) \xrightarrow{\lambda'} E_2^{2s+1+e,(t+p-1)q}(\overline{X})$$
$$E_2^{2s+e+1,(t+p)q}(X) \to E_2^{2s+e+1,(t+p-1)q}(\overline{X}) \xrightarrow{\lambda} E_2^{2s+2+e,(t+p)q}(S)$$

induced from the cofiber sequences in (4.1). □

Ravenel showed that $d_{2p-1}(\beta_{p^2/p^2}) \equiv \alpha_1\beta^p_{p/p}$ mod Ker β_1^p in the Adams-Novikov spectral sequence for $\pi_*(S)$ (*cf.* [4, 6.4.1]). Here, the mapping β_1^p on $E_2^{2p+1,(p^3+1)q}(S)$ is a monomorphism by Lemma 4.3:

Corollary 4.4. *In the Adams-Novikov spectral sequence for* $\pi_*(S)$*,* $d_{2p-1}(\beta_{p^2/p^2}) = \alpha_1\beta^p_{p/p} \in E_{2p-1}^{2p+1,(p^3+1)q}(S) = E_2^{2p+1,(p^3+1)q}(S)$.

Proof of Theorem 1.1. Consider the first cofiber sequence in (4.1). Since the Adams-Novikov E_2-term $E_2^{sq+3,(p^3+s)q}(X)$ vanishes for $s > 0$ by (4.2), the element $\iota_*(\beta_{p^2/p^2}) \in E_2^{2,p^3q}(X)$ survives to a homotopy element ${}^X\beta_{p^2/p^2} \in \pi_*(X)$. In general, we see that

(4.5) *Let* $\bar\iota\colon S \to \overline{X}$ *denote the inclusion to the bottom cell. Then,* $\lambda_*\bar\iota(x) = \alpha_1 x$ *for* $x \in E_2^*(S)$.

Put $\overline{\beta}_{p/p} = \bar\iota_*(\beta_{p/p}) \in E_2^{2,p^2q}(\overline{X})$, and we see that $\lambda_*(\overline{\beta}^p_{p/p}) = \alpha_1\beta^p_{p/p}$, and so we see that $\overline{\beta}^p_{p/p}$ detects an essential homotopy element $\kappa_*({}^X\beta_{p^2/p^2}) \in \pi_*(\overline{X})$ by Corollary 4.4 and [5], which we also denote by $\overline{\beta}^p_{p/p}$.

Now turn to the second cofiber sequence in (4.1). The relation $b_1^p = 0$ of (4.2) yields a cochain $y = \sum_{i=0}^{p-1} x^i y_i \in \Omega^{2p-1}BP_*(X)$ such that $d(y) = b_1^p$, where $y_i \in \Omega^{2p-1}BP_*$. It follows that $d(\overline{y}) = b_1^p - d(x^{p-1})y_{p-1} \in \Omega^{2p}BP_*(\overline{X})$ for $\overline{y} = \sum_{i=0}^{p-2} x^i y_i \in \Omega^{2p-1}BP_*(\overline{X})$. In particular $d(y_{p-1}) = 0 \in \Omega^{2p-1}BP_*$ and $d(y_{p-2}) = (1-p)t_1 \otimes y_{p-1}$. By definition, these imply $\lambda'_*(y_{p-1}) = b_1^p$. Consider the exact sequence obtained by applying the homotopy groups to the second cofiber sequence. Then, $\iota'_*(\overline{\beta}^p_{p/p}) = 0$ by (4.2), and so $\overline{\beta}^p_{p/p}$ must be pulled back to an element $\xi \in \pi_*(S)$ detected by y_{p-1}. Since $b_0 = \lambda\lambda'$, $b_0 y_{p-1} = h_0 b_1^p$, and $\langle h_0, \dots, h_0\rangle y_{p-1} = h_0\langle h_0, \dots, h_0, y_{p-1}\rangle$, we see that

$$b_1^p \equiv \langle h_0, \dots, h_0, y_{p-1}\rangle \not\equiv 0 \in E_2^{2p,p^3q}(S) \quad \text{mod } \ker h_0.$$

Put $b_1^p = \langle h_0, \dots, h_0, y_{p-1}\rangle + c$ for $c \in \ker h_0 \subset E_2^{2p,p^3q}(S)$. Then, $b_1^p - c$ survives to $\beta^p_{p/p} \in \pi_*(S)$.

The element $\alpha_1\beta^p_{p/p}$ is detected by $h_0(b_1^p - c) = h_0 b_1^p$ in the Adams-Novikov E_2-term, which is killed by b_2 by Corollary 4.4. □

5. Remarks

5.1. **A relation on Toda bracket.** The relation $\langle\beta_s, p, \gamma_t\rangle = \langle\gamma_t, p, \beta_s\rangle$ follows immediately from results of Toda: By definition, $\langle\beta_s, p, \gamma_t\rangle = j\beta_{(s)}\gamma_{(t)}i$ and $\langle\gamma_t, p, \beta_s\rangle = j\gamma_{(t)}\beta_{(s)}i$ for $\beta_{(s)} = j_1\beta^s i_1$ and $\gamma_{(t)} = j_1 j_2\gamma^t i_2 i_1$. Since $V(2)$ and $V(3)$ are $V(0)$-module spectra, $\theta(\beta) = 0$ and $\theta(\gamma) = 0$ by [6, Lemma 2.3]. Similarly, $\theta(i_k) = 0$ and $\theta(j_k) = 0$ for $k = 1, 2$. Therefore, [6, Lemma 2.2] implies $\theta(\beta_{(s)}) = 0$ and $\theta(\gamma_{(t)}) = 0$. Therefore, $\beta_{(s)}\gamma_{(t)} = \gamma_{(t)}\beta_{(s)}$ by [6, Cor. 2.7] as desired.

5.2. **On the action of γ_1.** Note that $\gamma_1 = \alpha_1\beta_{p-1}$. Then, $\alpha_1\gamma_1 = \alpha_1^2\beta_{p-1} = 0$, $\langle\alpha_1, \alpha_1, \beta^p_{p/p}\rangle\beta_1\gamma_1 = -\alpha_1\langle\alpha_1, \alpha_1, \beta^p_{p/p}\rangle\beta_1\beta_{p-1} = -\langle\alpha_1, \alpha_1, \alpha_1\rangle\beta^p_{p/p}\beta_1\beta_{p-1} = 0$ since $\langle\alpha_1, \alpha_1, \alpha_1\rangle = 0$, and $\langle\gamma_1, p, \beta_1\rangle = \beta_{p-1}\langle\alpha_1, p, \beta_1\rangle = \beta_{p-1}j\underline{\alpha j_1}\beta i_1 i = 0$.

For $t \geq 2$,

$$\begin{aligned}\beta_t &= \delta_{(1,1),1}\delta_{(1,1),2}(v_2^t) = \delta_{(1,1),1}([tv_2^{t-1}t_1^p + \tbinom{t}{2}v_1v_2^{t-2}t_1^{2p} + v_1^2x]) \\ &\equiv [t(t-1)v_2^{t-2}t_2 \otimes t_1^p - tv_2^{t-1}b_0 + \tbinom{t}{2}v_2^{t-2}t_1 \otimes t_1^{2p}] \mod (p, v_1) \\ &\equiv t(t-1)v_2^{t-2}k_0 - tv_2^{t-1}b_0 \mod (p, v_1)\end{aligned}$$

and $\alpha_1\beta_2\beta_{p-1} \in E_2^5(S^0)$ is projected to $h_0(2k_0 - 2v_2b_0)(2v_2^{p-3}k_0 + v_2^{p-2}b_0) = -2v_2^{p-2}h_0k_0b_0 - 2h_0v_2^{p-1}b_0^2$ in $E_2^5(V(2))$ under the induced map i_* from the inclusion $i\colon S^0 \to V(2)$ to the bottom cell. Here, $k_0 = [t_2 \otimes t_1^p + \frac{1}{2}t_1 \otimes t_1^{2p}]$. Then, this element is detected by $-2v_2^{p-2}k_0 \in E_1^3 = E_2^{2,(p^2+p-1)q}(X \wedge V(2))$ in the small descent spectral sequence. The killer of this element, if any, stays in the E_1-terms $E_1^2 = E_2^{2,(p^2+p)q}(X \wedge V(2))$, $E_1^1 = E_2^{3,(p^2+2p-1)q}(X \wedge V(2))$ and $E_1^0 = E_2^{4,(p^2+2p)q}(X \wedge V(2))$. These are zero, and we see that the product is not zero.

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