A NOTE ON HOPKINS' PICARD GROUPS OF THE STABLE HOMOTOPY CATEGORIES OF L_n-LOCAL SPECTRA

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Abstract. For a stable homotopy category [6], M. Hopkins introduced a Picard group (*cf.* [11], [4]) as a category consisting of isomorphism classes of invertible objects. For the stable homotopy category of L_n-local spectra, M. Hovey and H. Sadofsky [7] showed that the Picard group is actually a group containing the group of integers as a direct summand. We constructed an injection in [8] from the other summand of the Picard group to the direct sum of the E_r-terms $E_r^{r,r-1}$ over $r \geq 2$ of the Adams-Novikov spectral sequence converging to the homotopy groups of the L_n-localized sphere spectrum. In this paper, we show in a classical way that the injection is a bijection under a condition.

1. Introduction

Throughout this paper we fix a prime p. Let $\mathcal{S}_{(p)}$ denote the stable homotopy category consisting of all p-local spectra. For a spectrum E, a spectrum $X \in \mathcal{S}_{(p)}$ is *E-local* if $[C, X]_* = 0$ for C with $E \wedge C = pt$. We denote by $\mathcal{L}_E$ the full subcategory of E-local spectra, and have the Bousfield localization functor $L_E\colon \mathcal{S}_{(p)} \to \mathcal{L}_E$ with respect to E. A spectrum $Q \in \mathcal{L}_E$ is *invertible* in $\mathcal{L}_E$ if there is a spectrum $Q' \in \mathcal{L}_E$ such that $L_E(Q \wedge Q') = L_E S^0$ for the sphere spectrum S^0. M. Hopkins introduced the Picard group $\mathrm{Pic}(\mathcal{L}_E)$ of $\mathcal{L}_E$ consisting of isomorphism classes of invertible spectra (*cf.* [11], [4]). Hereafter, we abuse notation and an invertible spectrum Q denotes a class of $\mathrm{Pic}(\mathcal{L}_E)$ represented by Q.

Let $BP \in \mathcal{S}_{(p)}$ denote the Brown-Peterson spectrum whose homotopy groups are $BP_* = \pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ over generators with degree $|v_i| = 2p^i - 2$. In this paper, we consider the spectrum $E = v_n^{-1}BP$ for a fixed integer $n \geq 1$, and denote by $\mathcal{L}_n$ and L_n traditionally the category $\mathcal{L}_E$ and the functor L_E, respectively. The category and the functor are the same as $\mathcal{L}_{E(n)}$ and $L_{E(n)}$ for the n-th Johnson-Wilson spectrum $E(n)$ (*cf.* [9]), whose homotopy groups are $E(n)_* = \pi_*(E(n)) = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}] \subset v_n^{-1}BP_*$.

We have some results on the Picard group $\mathrm{Pic}(\mathcal{L}_n)$. Hereafter, we set

$$q = 2p - 2.$$

Theorem 1.1 ([7, Prop. 1.4, Lemma 1.5, Th. 5.4, Th. 6.1])**.** *$\mathrm{Pic}(\mathcal{L}_n)$ is a group with multiplication defined by the smash product and $\mathrm{Pic}(\mathcal{L}_n) \cong \mathbb{Z} \oplus \mathrm{Pic}^0(\mathcal{L}_n)$ for a subgroup $\mathrm{Pic}^0(\mathcal{L}_n)$. In particular, $\mathrm{Pic}(\mathcal{L}_n) \cong \mathbb{Z}$ if $q > n^2 + n$ and $\mathrm{Pic}(\mathcal{L}_1) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ if $p = 2$.*

Theorem 1.2 ([3])**.** *$\mathrm{Pic}(\mathcal{L}_2) \cong \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$ if $p = 3$.*

Theorem 1.3 ([7, Th. 2.4])**.** *If $Q \in \mathrm{Pic}^0(\mathcal{L}_n)$, then $E(n)_*(Q) \cong E(n)_*$ as an $E(n)_*(E(n))$-comodule.*

Consider the $v_n^{-1}BP$-based Adams spectral sequence $\{E_r^{s,t}(X), d_r\}$ for a spectrum X converging to homotopy groups $\pi_*(L_nX)$. We notice that the E_2-term $E_2^{s,t}(Q)$ for $Q \in \mathrm{Pic}^0(\mathcal{L}_n)$ is isomorphic to $\mathrm{Ext}^{s,t}_{E(n)_*(E(n))}(E(n)_*, E(n)_*)$ by Theorem 1.3 and [8, Th. 3.3] (see Theorem 2.2). Thus, we see that $E_2^{s,t}(Q) \cong E_2^{s,t}(S^0)$ for an invertible spectrum Q of $\mathrm{Pic}^0(\mathcal{L}_n)$. Let $\mathrm{Pic}^0(\mathcal{L}_n)_k$ be a subgroup consisting of invertible spectra Q of $\mathrm{Pic}^0(\mathcal{L}_n)$ such that $d_s(1_Q) = 0 \in E_s^{s,s-1}(Q)$ for $s < kq+1$, where $1_Q \in E_2^{0,0}(Q) \cong \mathbb{Z}_{(p)}$ denotes the generator. Put

$$G\mathrm{Pic}^0(\mathcal{L}_n) = \bigoplus_{k>0} \left(\mathrm{Pic}^0(\mathcal{L}_n)_k / \mathrm{Pic}^0(\mathcal{L}_n)_{k+1}\right).$$

Then, in [8], we set up a homomorphism

$$\varphi\colon G\mathrm{Pic}^0(\mathcal{L}_n) \to \bigoplus_{k>0} E_{kq+1}^{kq+1,kq}(S^0) \tag{1.4}$$

sending $[Q] \neq 0 (= [L_nS^0])$ to an element $w \neq 0 \in E_{kq+1}^{kq+1,kq}(S^0)$ such that $d_{kq+1}(1_Q) = w1_Q \in E_{kq+1}^{kq+1,kq}(Q)$.

Theorem 1.5 ([8, Th. 2])**.** *The homomorphism φ is a monomorphism.*

For a small n, the E_2-term has a horizontal vanishing line:

Lemma 1.6 (*cf.* [9, (10.10)])**.** *Suppose $n < p-1$. Then, $E_2^{s,t}(S^0) = 0$ for $s > n^2+n$.*

By Theorem 1.5 and Lemma 1.6, we deduce that the second isomorphism in Theorem 1.1 holds in the case where $q = n^2+n$ unless $(p,n) = (2,1)$:

Corollary 1.7. *Suppose that $p > 2$ or $n > 1$. Then, $\mathrm{Pic}(\mathcal{L}_n) \cong \mathbb{Z}$ if $q \geq n^2+n$.*

In this paper, we show the following:

Theorem 1.8. *Suppose that there is an integer r_0 such that $E_2^{rq+2,rq}(S^0) = 0$ for $r > r_0$. Then, for $\omega \in E_{r_0q+1}^{r_0q+1,r_0q}(S^0)$, we have an invertible spectrum X_ω such that $\varphi([X_\omega]) = \omega$.*

Corollary 1.9. *Suppose that $n < p-1$ and let r_0 be the maximal integer less than $(n^2+n)/q$. Then, the composite $G\mathrm{Pic}^0(\mathcal{L}_n) \xrightarrow{\varphi} \bigoplus_{k>0} E_{kq+1}^{kq+1,kq}(S^0) \to E_{r_0q+1}^{r_0q+1,r_0q}(S^0)$ is an epimorphism.*

Corollary 1.10. *For n and r_0 in Corollary 1.9, if $E_{r_0q+1}^{r_0q+1,r_0q}(S^0) \neq 0$, then $\mathcal{L}_n$ admits an exotic invertible spectrum, or an invertible spectrum other than L_nS^k.*

Corollary 1.11. *If $n < p-1$ and $n^2+n \leq 2q$, then $\mathrm{Pic}^0(\mathcal{L}_n)$ is isomorphic to $E_2^{q+1,q}(S^0) \cong \mathrm{Ext}^{q+1,q}_{E(n)_*(E(n))}(E(n)_*, E(n)_*)$.*

In the corollary, if the condition $n^2+n \leq q$ holds, then it is Corollary 1.7. If $q+1 \leq n^2+n$, then the isomorphism $\mathrm{Pic}^0(\mathcal{L}_n) \cong E_2^{q+1,q}(S^0)$ holds in the following cases:

p	5	7	11	13	17	19
n	3	4	5	6	7	8

For the case where $p = 23$ and $n = 9$, we have $n^2+n = 90 > 89 = 2q+1$. We note that we have the isomorphism for the cases $(p,n) = (2,1)$ and $(3,2)$ by Theorems 1.1 and 1.2, though $n = p-1$. We have a conjecture that $E_2^{s,t}(S^0) = 0$ for $s > n^2$

if $n < p-1$. If this is true, then the table extends up to $p = 29$ and $n = 10$ after replacing the condition by $q+1 \le n^2 < 2q+1$.

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2. Recollection from [8]

Let E denote $v_n^{-1}BP$. We recollect some facts from [8, §3-4]: The spectrum E yields a Hopf algebroid $(E_*, E_*(E))$ in a usual way. Here,

$$\begin{aligned} &E_*(E) = E_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E_* \quad \text{and} \\ &E(n)_*(X) = E(n)_* \otimes_{BP_*} BP_*(X) = E(n)_* \otimes_{E_*} E_*(X) \end{aligned}$$

for a spectrum X, where $E_* = v_n^{-1}BP_*$ and $E_*(E)$ is flat over E_*.

An invertible spectrum Q is characterized by its E_*-homology:

Theorem 2.1 ([8, Prop. 3.2] (*cf.* [8, Th. 1.1], [7, Th. 2.4]))**.** *A spectrum $Q \in \mathcal{L}_n$ is invertible if and only if there is an isomorphism $E_*(Q) \cong E_*$ of $E_*(E)$-comodules.*

We have a relation between the E_2-terms of the E-based and the $E(n)$-based Adams spectral sequences:

Theorem 2.2 ([8, Th. 3.3])**.**

$$\mathrm{Ext}^{*,*}_{E_*(E)}(E_*, M) \cong \mathrm{Ext}^{*,*}_{E(n)_*(E(n))}(E(n)_*, E(n)_* \otimes_{E_*} M)$$

for an $E_*(E)$-comodule M, on which v_n acts isomorphically.

The unit map $i\colon S^0 \to E$ yields a cofiber sequence

$$S^0 \xrightarrow{i} E \xrightarrow{j} \overline{E} \xrightarrow{k} S^1, \tag{2.3}$$

which gives rise to the E-based Adams tower

$$\begin{array}{ccccccccccc} S^0 & \xleftarrow{k} & \overline{E} & \xleftarrow{k\wedge 1} & \cdots & \xleftarrow{k\wedge 1} & \overline{E}^m & \xleftarrow{k\wedge 1} & \overline{E}^{m+1} & \xleftarrow{k\wedge 1} & \cdots \\ {\scriptstyle i}\downarrow & \nearrow^{j} & \downarrow{\scriptstyle i\wedge 1} & \nearrow^{j\wedge 1} & & \nearrow^{j\wedge 1} & {\scriptstyle i\wedge 1}\downarrow & \nearrow^{j\wedge 1} & \downarrow{\scriptstyle i\wedge 1} & & \\ E & \xrightarrow[d^0]{} & E\wedge\overline{E} & \xrightarrow[d^1]{} & \cdots & \xrightarrow[d^{m-1}]{} & E\wedge\overline{E}^m & \xrightarrow[d^m]{} & E\wedge\overline{E}^{m+1} & \xrightarrow[d^{m+1}]{} & \cdots, \end{array} \tag{2.4}$$

in which dotted arrows denote degree -1 maps, $\overline{E}^m$ denotes the m-fold smash product of $\overline{E}$ and $d^m = (i \wedge \overline{E}^{m+1})(j \wedge \overline{E}^m) = d^0 \wedge \overline{E}^m$. Hereafter, we denote $f \wedge W\colon X \wedge W \to Y \wedge W$ for a map $f\colon X \to Y$ by $X \wedge W \xrightarrow{f\wedge 1} Y \wedge W$ in diagrams. Let $k^m\colon \overline{E}^m \to S^m$ denote the composite $k(k \wedge \overline{E}) \cdots (k \wedge \overline{E}^{m-1})$ of the above sequence. We have a spectrum $\overline{E}_m$ and maps i_m and j_m fitting in a cofiber sequence

$$\overline{E}^m \xrightarrow{k^m} S^m \xrightarrow{i_m} \Sigma\overline{E}_m \xrightarrow{j_m} \Sigma\overline{E}^m. \tag{2.5}$$

Throughout the paper, we omit suspensions for maps of spectra. So the maps i_m and j_m in (2.5) denote Σi_m and Σj_m for $i_m\colon : S^{m-1} \to \overline{E}_m$ and $j_m\colon \overline{E}_m \to \overline{E}^m$, respectively. In particular, we see that

$$i_1 = i\colon S^0 \to \overline{E}_1 = E. \tag{2.6}$$

Indeed, the cofiber sequence (2.5) for $m = 1$ agrees with the one (2.3) (up to suspension), and so we identify $\overline{E}_1$ with E.

The cofiber sequence (2.5) gives rise to another E-based Adams tower

$$
\begin{array}{ccccccccccc}
\overline{E}_0 = pt & \dashleftarrow & \overline{E}_1 & \overset{k_1^S}{\dashleftarrow} & \cdots & \overset{k_{m-1}^S}{\dashleftarrow} & \overline{E}_m & \overset{k_m^S}{\dashleftarrow} & \overline{E}_{m+1} & \overset{k_{m+2}^S}{\dashleftarrow} & \cdots \\
\downarrow & = & \downarrow{\scriptstyle d=i_1^S} & \nearrow{\scriptstyle j_1^S} & & \nearrow{\scriptstyle j_{m-1}^S} & {\scriptstyle i_m^S}\downarrow & \nearrow{\scriptstyle j_m^S} & \downarrow{\scriptstyle i_{m+1}^S} & & \\
E & \xrightarrow[d^0]{} & E \wedge \overline{E} & \xrightarrow[d^1]{} & \cdots & \xrightarrow[d^{m-1}]{} & E \wedge \overline{E}^m & \xrightarrow[d^m]{} & E \wedge \overline{E}^{m+1} & \xrightarrow[d^{m+1}]{} & \cdots
\end{array} \tag{2.7}
$$

with the same d^m's as (2.4). Here, the maps i_m^S, j_m^S and k_m^S are defined by the commutative diagram

$$
\begin{array}{ccccccc}
S^m & = & S^m & \longrightarrow & * & \longrightarrow & S^{m+1} \\
{\scriptstyle i_{m+1}}\downarrow & & \downarrow{\scriptstyle i_m} & & \downarrow & & \downarrow{\scriptstyle i_{m+1}} \\
\overline{E}_{m+1} & \xrightarrow{k_m^S} & \Sigma\overline{E}_m & \xrightarrow{i_m^S} & \Sigma E \wedge \overline{E}^m & \xrightarrow{j_m^S} & \Sigma\overline{E}_{m+1} \\
{\scriptstyle j_{m+1}}\downarrow & & \downarrow{\scriptstyle j_m} & & \| & & \downarrow{\scriptstyle j_{m+1}} \\
\overline{E}^{m+1} & \xrightarrow{k\wedge 1} & \Sigma\overline{E}^m & \xrightarrow{i\wedge 1} & \Sigma E \wedge \overline{E}^m & \xrightarrow{j\wedge 1} & \Sigma\overline{E}^{m+1}
\end{array}
$$

of cofiber sequences, which is obtained from Verdier's axiom. Note that these maps satisfy

$$
i_m^S = (i \wedge 1)j_m, \quad i_m^S i_m = 0 \quad \text{and} \quad k_m^S i_{m+1} = i_m. \tag{2.8}
$$

A smash product of the tower (2.7) and a spectrum X defines the E-based Adams spectral sequence

$$
E_2^{s,t}(X) = \mathrm{Ext}_{E_*(E)}^{s,t}(E_*, E_*(X)) \Longrightarrow \pi_{t-s}(\operatorname*{holim}_m(\Sigma^{1-m}\overline{E}_m \wedge X)).
$$

Note that $\operatorname*{holim}_m(\Sigma^{1-m}\overline{E}_m \wedge X) = L_n X$. We consider a similar tower

$$
\begin{array}{ccccccccccc}
Q_0 = pt & \dashleftarrow & Q_1 & \overset{k_1^Q}{\dashleftarrow} & \cdots & \overset{k_{m-1}^Q}{\dashleftarrow} & Q_m & \overset{k_m^Q}{\dashleftarrow} & Q_{m+1} & \overset{k_{m+2}^Q}{\dashleftarrow} & \cdots \\
\downarrow & = & \downarrow{\scriptstyle d=i_1^Q} & \nearrow{\scriptstyle j_1^Q} & & \nearrow{\scriptstyle j_{m-1}^Q} & {\scriptstyle i_m^Q}\downarrow & \nearrow{\scriptstyle j_m^Q} & \downarrow{\scriptstyle i_{m+1}^Q} & & \\
E & \xrightarrow[d^0]{} & E \wedge \overline{E} & \xrightarrow[d^1]{} & \cdots & \xrightarrow[d^{m-1}]{} & E \wedge \overline{E}^m & \xrightarrow[d^m]{} & E \wedge \overline{E}^{m+1} & \xrightarrow[d^{m+1}]{} & \cdots
\end{array} \tag{2.9}
$$

over the same bottom sequence as (2.7). Put

$$
Q = \operatorname*{holim}_{k_m^Q} \Sigma^{1-m} Q_m. \tag{2.10}
$$

Then, we have a spectral sequence

$$
E_2^{s,t} = \mathrm{Ext}_{E_*(E)}^{s,t}(E_*, E_*) \Longrightarrow \pi_{t-s}(Q) \tag{2.11}
$$

obtained by applying homotopy groups $\pi_*(-)$ to (2.9), and see the following lemma by [2, Prop. 1.9].

Lemma 2.12. *The spectral sequence (2.11) is isomorphic to the E-based Adams spectral sequence for computing $\pi_*(Q)$ via an isomorphism which is the identity on the E_2-term.*

Proof. By the same argument as [2, Prop. 1.9], we verify that the spectral sequence (2.11) is isomorphic to the modified E-based Adams spectral sequence for computing $\pi_*(Q)$. Now the lemma follows from [2, Prop. 1.9]. $\square$

The lemma implies that the E_r-term of (2.11) agrees with the one of the E-based Adams spectral sequence:

$$E_r^{s,t} \cong E_r^{s,t}(Q).$$

Proposition 2.13. *If a tower (2.9) exists, then Q in (2.10) is an invertible spectrum of $\mathcal{L}_n$. Furthermore, if $d_{rq+1}(1_Q) = \omega 1_Q \neq 0 \in E_{rq+1}^{rq+1,rq} = E_{rq+1}^{rq+1,rq}(Q)$ for the generator $1_Q \in E_2^{0,0} = E_2^{0,0}(Q)$ in the spectral sequence (2.11), then $\varphi(Q) = \omega$ for the monomorphism φ in (1.4).*

Proof. Let $\mu\colon E \wedge E \to E$ be the multiplication of the commutative ring spectrum E, and let $\eta_m\colon Q \to \Sigma^{1-m} Q_m$ denote the canonical map such that $k_{m-1}^Q \eta_m = \eta_{m-1}$. Then, we have a map $\eta\colon E \wedge Q \xrightarrow{E\wedge\eta_1} E \wedge E \xrightarrow{\mu} E$. Let $\mathcal{C} = \{X \in \mathcal{S}_{(p)} \mid X \text{ is finite}, \eta \wedge X\colon E \wedge Q \wedge X \simeq E \wedge X\}$. Then, $\mathcal{C}$ is a thick subcategory of finite spectra. Ravenel [10, 8.3] constructed a finite torsion free spectrum Y, whose E-based Adams E_2-term has a horizontal vanishing line: there exists an integer s_0 such that $E_2^{s,*}(Y) = 0$ for $s > s_0$. By [8, Prop. 4.3], we see that $Y \in \mathcal{C}$. This together with the thick subcategory theorem [5, Th. 7] implies that $S^0 \in \mathcal{C}$, and so $\eta\colon E \wedge Q \simeq E$. This also induces an isomorphism of E_*E-comodules. This fact is verified as follows: Since the cofiber sequence $E \xrightarrow{E\wedge i} E \wedge E \xrightarrow{E\wedge j} E \wedge \overline{E}$ obtained from (2.3) splits by $\mu(E \wedge i) = 1_E$ (the identity map on E), we have a map $\overline{\mu}\colon E \wedge \overline{E} \to E \wedge E$ such that $(E \wedge i)\mu + \overline{\mu}(E \wedge j) = 1$ for the identity $1\colon E \wedge E \to E \wedge E$. We see that $d\eta_1 = dk_1^Q \eta_2 = 0$ by (2.9). Then, we compute $(E \wedge j)(E \wedge \eta_1) = (\mu \wedge \overline{E})(E \wedge i \wedge \overline{E})(E \wedge j)(E \wedge \eta_1) = (\mu \wedge \overline{E})(E \wedge d)(E \wedge \eta_1) = 0$. We also see that $\eta_1 = \eta(i \wedge Q)$ by the commutative diagram

$$\begin{array}{ccccc} Q & \xrightarrow{\eta_1} & E & & \\ {\scriptstyle i\wedge Q}\downarrow & & {\scriptstyle i\wedge E}\downarrow & \diagdown\!\!\diagdown & \\ E \wedge Q & \xrightarrow[E\wedge\eta_1]{} & E \wedge E & \xrightarrow[\mu]{} & E. \end{array}$$

It follows that $(E \wedge i)\eta = (E \wedge i)\mu(E \wedge \eta_1) = (1 - \overline{\mu}(E \wedge j))(E \wedge \eta_1) = E \wedge \eta_1 = (E \wedge \eta)(E \wedge i \wedge Q)$, and we obtain a commutative diagram

$$\begin{array}{ccc} E \wedge Q & \xrightarrow{\eta} & E \\ {\scriptstyle E\wedge i\wedge Q}\downarrow & & \downarrow{\scriptstyle E\wedge i} \\ E \wedge E \wedge Q & \xrightarrow[E\wedge\eta]{} & E \wedge E \end{array}$$

Therefore, η induces a homomorphism $E_*(Q) \to E_*$ of comodules, and the former part of the proposition follows from Theorem 2.1.

The latter follows from Lemma 2.12. □

We call the following finite sub-tower of (2.9) an *m-tower* :

$$(2.14)_m \qquad \begin{array}{ccccccccc} Q_0 = pt & \dashleftarrow & Q_1 & \overset{k_1^Q}{\dashleftarrow} & \cdots & \overset{k_{m-1}^Q}{\dashleftarrow} & Q_m & \overset{k_m^Q}{\dashleftarrow} & Q_{m+1} \\ \downarrow & \diagup\!\!\diagup & \downarrow{\scriptstyle d=i_1^Q} & \nearrow{\scriptstyle j_1^Q} & & {\scriptstyle j_{m-1}^Q}\nearrow & {\scriptstyle i_m^Q}\downarrow & {\scriptstyle j_m^Q}\nearrow & \\ E & \xrightarrow[d^0]{} & E \wedge \overline{E} & \xrightarrow[d^1]{} & \cdots & \xrightarrow[d^{m-1}]{} & E \wedge \overline{E}^m. & & \end{array}$$

Let M be an E-module spectrum with structure map $\nu\colon E\wedge M\to M$. Then, the unit map i of E and the map $d=i_1^Q$ in (2.9) give rise to an exact sequence

$$[E\wedge\overline{E},M]_t\xrightarrow{(i_1^Q)^*}[E,M]_t\underset{\sigma}{\overset{i^*}{\rightleftarrows}}[S^0,M]_t\to 0 \tag{2.15}$$

for each $t\in\mathbb{Z}$ with splitting σ given by $\sigma(x)=\nu(E\wedge x)$ for $x\in\pi_t(M)$ (*cf.* [8, p. 329]). Consider a tower $(2.14)_m$ for an integer m and define a homomorphism $\psi_m\colon\pi_{m+t-1}(M)\to[Q_m,M]_t$ by

$$\psi_m(x)=\sigma(x)(k^Q)^{m-1}, \tag{2.16}$$

where $(k^Q)^s=k_1^Q\cdots k_s^Q$ in (2.9).

Lemma 2.17 ([8, Lemma 4.5])**.** *Suppose that an m-tower $(2.14)_m$ for $m>1$ exists and let M be an E-module spectrum. Then, we have a split short exact sequence*

$$0\to\pi_{m+t-1}(M)\xrightarrow{\psi_m}[Q_m,M]_t\xrightarrow{(j_{m-1}^Q)^*}(\mathrm{Im}\ (d^{m-1})^*)_t\to 0.$$

Here, $(d^{m-1})^*\colon[E\wedge\overline{E}^m,M]_t\to[E\wedge\overline{E}^{m-1},M]_t$ is induced from $d^{m-1}\colon E\wedge\overline{E}^{m-1}\to E\wedge\overline{E}^m$.

Lemma 2.18. *Let $m>1$ and let $\psi_m\colon\pi_{m+t-1}(E\wedge\overline{E}^k)\to[Q_m,E\wedge\overline{E}^k]_t$ be the homomorphism defined by (2.16) with $\nu=\mu\wedge\overline{E}^k$ for the multiplication μ of E and for an integer $k\ge 0$. Then, the diagram*

$$\begin{array}{ccc}\pi_{m+t-1}(E\wedge\overline{E}^k) & \xrightarrow{\psi_m} & [Q_m,E\wedge\overline{E}^k]_t\\ {\scriptstyle d_*}\big\downarrow & & \big\downarrow{\scriptstyle d_*}\\ \pi_{m+t-1}(E\wedge\overline{E}^{k+1}) & \underset{\psi_m}{\to} & [Q_m,E\wedge\overline{E}^{k+1}]_t\end{array}$$

commutes for the induced maps d_* from $d=d^k\colon E\wedge\overline{E}^k\to E\wedge\overline{E}^{k+1}$.

Proof. Since $i^*d_*=d_*i^*\colon[E,E\wedge\overline{E}^k]_*\to\pi_*(E\wedge\overline{E}^{k+1})$ and σ in (2.15) is a splitting for the homomorphism i^*, we see that $i^*(d_*(\sigma(x)))=d_*(x)=i^*(\sigma(d_*(x)))$ for $x\in\pi_{m+t-1}(E\wedge\overline{E}^k)$. Then, we have an element $y\in[E\wedge\overline{E},E\wedge\overline{E}^{k+1}]_*$ such that $(i_1^Q)^*(y)=d_*(\sigma(x))-\sigma(d_*(x))$. Therefore, noticing that $i_1^Q(k^Q)^{m-1}=0$, we compute for $x\in\pi_{m+t-1}(E\wedge\overline{E}^k)$,

$$\begin{aligned}\psi_m d_*(x)\ &\underset{(2.16)}{=\!=}\ \sigma(d_*(x))(k^Q)^{m-1}\ =\ \Big(d_*(\sigma(x))-(i_1^Q)^*(y)\Big)(k^Q)^{m-1}\\ &=\ d_*(\sigma(x))(k^Q)^{m-1}\ \underset{(2.16)}{=\!=}\ d_*\psi_m(x).\end{aligned}$$

□

3. Construction of invertible spectra

Consider a spectral sequence $\{{}^mE_r^{s,t}\}$ obtained by applying the homotopy groups $\pi_*(-)$ to an m-tower $(2.14)_m$. Then, the E_2-term of the spectral sequence is isomorphic to the E_2-term of the E-based Adams spectral sequence for computing $\pi_*(S^0)$ up to $(m-1)$-stage: ${}^mE_2^{s,t}\cong E_2^{s,t}(S^0)$ for $s\le m-1$. Consider a nontrivial element $[w]\in E_{rq+1}^{rq+1,rq}(S^0)$ for $w\in\pi_{rq}(E\wedge\overline{E}^{rq+1})$ and $r>0$. Note that $\pi_{rq}(E\wedge\overline{E}^{rq+1})$ is also an E_1-term of the spectral sequence $\{{}^mE_r^{s,t}\}$. If the element w survives to an element of ${}^mE_{rq+1}^{rq+1,rq}$ for $rq+1\le m$, then we denote it by $\langle w\rangle$.

Lemma 3.1. *Suppose that we have a nontrivial element* $[w] \in E_{rq+1}^{rq+1,rq}(S^0)$ *for* $w \in \pi_{rq}(E \wedge \overline{E}^{rq+1})$ *and* $r > 0$*. Then there exists an* $(rq+2)$*-tower* $(2.14)_{rq+2}$ *such that* $d_{rq+1}(1) = \langle w \rangle \in {}^{rq+2}E_{rq+1}^{rq+1,rq}$ *for* $1 \in {}^{rq+2}E_2^{0,0} \cong E_2^{0,0}(S^0)$ *in the spectral sequence* $\{{}^{rq+2}E_{r'}^{s,t}\}$ *associated to the tower.*

Proof. We take the same rq-tower as (2.7) up to rq-stage. That is, $Q_s = \overline{E}_s$ for $s \le rq+1$ and $\ell_s^Q = \ell_s^S$ for $s \le rq$, where ℓ stands for one of letters i, j and k. Put $i_{rq+1}^Q = i_{rq+1}^S + \psi_{rq+1}(w) \in [\overline{E}_{rq+1}, E \wedge \overline{E}^{rq+1}]_0$ for the maps i_{rq+1}^S in (2.7) and ψ_{rq+1} in (2.16) where $k^Q = k^S$. Then, $i_{rq+1}^Q j_{rq}^Q = i_{rq+1}^Q j_{rq}^S = i_{rq+1}^S j_{rq}^S = d$, since $(j_{rq}^S)^* \psi_{rq+1} = 0$ by Lemma 2.17. Let Q_{rq+2} be a cofiber of i_{rq+1}^Q, and we have an $(rq+1)$-tower. By Lemma 2.18, we see that $d_* \psi_{rq+1}(w) = \psi_{rq+1} d_*(w) = 0$ for $d = d^{rq+1}$, since w is a d_*-cocycle. Since $d_* i_{rq+1}^S = 0$ by (2.7), we deduce $d_* i_{rq+1}^Q = 0$, and hence we obtain a map $i_{rq+2}^Q \colon Q_{rq+2} \to E \wedge \overline{E}^{rq+2}$ such that $d = i_{rq+2}^Q j_{rq+1}^Q$. Thus, we have an $(rq+2)$-tower by setting Q_{rq+3} to be a cofiber of i_{rq+2}^Q.

Now consider the spectral sequence $\{{}^{rq+2}E_r^{s,t}\}$. Note that the generator $1 \in {}^{rq+2}E_2^{0,0}$ is represented by the unit map i. Then, the differentials of the spectral sequence on the generator 1 is given by

$$d_s(1) = [a] \in {}^{rq+2}E_s^{s,s-1} \quad \text{if } i_s^Q i_s = a \in \pi_{s-1}(E \wedge \overline{E}^s)$$

by definition for $s \le rq+1$. Here, i_s is the map of (2.5). By the last relation of (2.8) and (2.6), we see that

$$(k^S)^{s-1} i_s = i \quad \text{for } s \ge 1. \tag{3.2}$$

This together with $i_s^S i_s = 0$ of (2.8) shows that the differential of the spectral sequence acts as $d_s(1) = 0$ for $s \le rq$. Consider (2.15) for $M = E \wedge \overline{E}^{rq+1}$ and $\nu = \mu \wedge \overline{E}^{rq+1}$. Then, we have $\sigma(w)i = i^*\sigma(w) = w$. We compute

$$\psi_{rq+1}(w) i_{rq+1} \underset{(2.16)}{=\!=\!=} \sigma(w)(k^S)^{rq} i_{rq+1} \underset{(3.2)}{=\!=\!=} \sigma(w) i = w,$$

and so, $i_{rq+1}^Q i_{rq+1} = (i_{rq+1}^S + \psi_{rq+1}(w)) i_{rq+1} = \psi_{rq+1}(w) i_{rq+1} = w$. This implies $d_{rq+1}(1) = \langle w \rangle \in {}^{rq+2}E_{rq+1}^{rq+1,rq}$ in the spectral sequence. □

Lemma 3.3. *Suppose that we have a tower* $(2.14)_m$ *for* $m > 1$*. If* $E_2^{m+1,m-1}(S^0) = 0$*, then the tower extends up to* $m+1$ *stage.*

Proof. It suffices to show that the map $i_m^Q \colon Q_m \to E \wedge \overline{E}^m$ in the given tower $(2.14)_m$ is replaced by a map i_m^Q satisfying $i_m^Q j_{m-1}^Q = d^{m-1}$ and $d^m i_m^Q = 0$. Indeed, the relation $d^m i_m^Q = 0$ yields a map $i_{m+1}^Q \colon Q_{m+1} \to E \wedge \overline{E}^{m+1}$ such that $i_{m+1}^Q j_m^Q = d^m$, and we obtain a tower $(2.14)_{m+1}$ by taking Q_{m+2} as a cofiber of i_{m+1}^Q.

By Lemma 2.18, we have the commutative diagram

$$\begin{array}{ccccc}
\pi_{m-1}(E\wedge \overline{E}^m) & \xrightarrow{\psi_m} & [Q_m, E\wedge \overline{E}^m]_0 & \xrightarrow{(j^Q_{m-1})^*} & (\mathrm{Im}\ (d^{m-1})^*)_0 \\
{\scriptstyle (d^m)_*}\downarrow & & \downarrow{\scriptstyle (d^m)_*} & & \downarrow{\scriptstyle (d^m)_*} \\
\pi_{m-1}(E\wedge \overline{E}^{m+1}) & \xrightarrow{\psi_m} & [Q_m, E\wedge \overline{E}^{m+1}]_0 & \xrightarrow{(j^Q_{m-1})^*} & (\mathrm{Im}\ (d^{m-1})^*)_0 \\
{\scriptstyle (d^{m+1})_*}\downarrow & & \downarrow{\scriptstyle (d^{m+1})_*} & & \downarrow{\scriptstyle (d^{m+1})_*} \\
\pi_{m-1}(E\wedge \overline{E}^{m+2}) & \xrightarrow{\psi_m} & [Q_m, E\wedge \overline{E}^{m+2}]_0 & \xrightarrow{(j^Q_{m-1})^*} & (\mathrm{Im}\ (d^{m-1})^*)_0,
\end{array}$$

in which the horizontal sequences are exact by Lemma 2.17. Let $i'_m \in [Q_m, E\wedge\overline{E}^m]_0$ denote the given map i^Q_m. Then, $d^{m-1} = i'_m j^Q_{m-1}$, and so $(j^Q_{m-1})^*(d^m i'_m) = d^m d^{m-1} = 0$ for $d^m i'_m \in [Q_m, E\wedge \overline{E}^{m+1}]_0$. Therefore, we have an element $o \in \pi_{m-1}(E\wedge \overline{E}^{m+1})$ such that $\psi_m(o) = d^m i'_m$. Moreover, $\psi_m(d^{m+1})_*(o) = (d^{m+1})_*\psi_m(o) = d^{m+1}d^m i'_m = 0$. Since ψ_m is a monomorphism, we have $(d^{m+1})_*(o) = 0$. The assumption $E_2^{m+1,m-1}(S^0) = 0$ indicates that the left column of the diagram is exact, and so we have an element $\widetilde{o} \in \pi_{m-1}(E\wedge \overline{E}^m)$ such that $(d^m)_*(\widetilde{o}) = o$. Now set $i^Q_m = i'_m - \psi_m(\widetilde{o})$. Then, $i^Q_m j^Q_{m-1} = i'_m j^Q_{m-1} - (j^Q_{m-1})^*\psi_m(\widetilde{o}) = d^{m-1}$ and

$$\begin{aligned}
d^m i^Q_m &= d^m i'_m - (d^m)_*\psi_m(\widetilde{o}) \underset{2.18}{=\!=} d^m i'_m - \psi_m(d^m)_*(\widetilde{o}) \\
&= d^m i'_m - \psi_m(o) \ = \ 0
\end{aligned}$$

as desired. □

Proof of Theorem 1.8. Consider the (r_0q+2)-tower of Lemma 3.1. For each $m \geq r_0q+2$, the condition of Lemma 3.3 is fulfilled by the assumption $E_2^{rq+2,rq}(S^0) = 0$ for $r > r_0$ and the fact that $E_2^{*,t}(S^0) = 0$ unless $t \equiv 0 \bmod q$. It follows that the tower extends to an infinite tower. Now the theorem follows from Proposition 2.13.

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