A BETA FAMILY IN THE HOMOTOPY OF SPHERES

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ABSTRACT. Let p be a prime number greater than three. In the p-component of stable homotopy groups of spheres, Oka constructed a beta family from a v_2-periodic map on a four cell complex. In this paper, we construct another beta family in the groups at a prime p greater than five from a v_2-periodic map on a eight cell complex.

1. INTRODUCTION

We fix a prime number p greater than three, and work in the stable homotopy category $\mathcal{S}_{(p)}$ of spectra localized at the prime p. Let S and BP in $\mathcal{S}_{(p)}$ denote the sphere and the Brown-Peterson spectra. It is an important problem to understand the homotopy groups $\pi_*(S)$, whose structure is little known. On the other hand, we know the structures of $\pi_*(BP) = BP_*$ and $BP_*(BP)$:

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots] \quad \text{and} \quad BP_*(BP) = BP_*[t_1, t_2, \dots]$$

and $BP_*(BP)$ is a Hopf algebroid over BP_*. Here, the generators have degrees $|v_k| = |t_k| = 2(p^k - 1)$. Furthermore, we have the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(X)$ of a spectrum X with E_2-term

$$E_2^{s,t}(X) = \mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, BP_*(X)),$$

and the spectral sequence for $X = S$ acts as a go-between between BP and S. Here we consider the homotopy groups $\pi_*(S)$ through the spectral sequence. In the E_2-term $E_2^{2,*}(S)$, Miller, Ravenel and Wilson [1] defined the beta elements $\widehat{\beta}_{s/t,r}$ for suitable triples (s, t, r) of positive integers. Consider the spectra and the maps defined by the cofiber sequences:

$$\begin{gathered} S \xrightarrow{p^r} S \xrightarrow{i_r} M(r) \xrightarrow{j_r} \Sigma S \quad \text{and} \\ \Sigma^{up^{r-1}q} M(r) \xrightarrow{A^u_{r-1}} M(r) \xrightarrow{i_{r,up^{r-1}}} M(r, up^{r-1}) \xrightarrow{j_{r,up^{r-1}}} \Sigma^{up^{r-1}q} M(r), \end{gathered} \tag{1.1}$$

where A_r denotes an element such that $BP_*(A_r) = v_1^{p^r}$ for $r \geq 0$ (*cf.* [6, Th. 6.2], see also (2.6)), and A_0 is known as the Adams map and denoted by α. Hereafter, $q = 2p - 2$. We note that $BP_*(M(r)) = BP_*/(p^r)$ and $BP_*(M(r, up^{r-1})) = BP_*/(p^r, v_1^{up^{r-1}})$ are $BP_*(BP)$-comodules. The cofiber sequences in (1.1) induce the connecting homomorphisms $\partial_r \colon E_2^{s,t}(M(r)) \to E_2^{s+1,t}(S)$ and $\partial_{r,up^{r-1}} \colon E_2^{s,t}(M(r, up^{r-1})) \to E_2^{s+1,t-up^{r-1}q}(M(r))$. Then, the beta element for a triple (s, t, r) is defined by

$$\widehat{\beta}_{s/t,r} = \partial_r \partial_{r,t}(v_2^s) \in E_2^{2,(s(p+1)-t)q}(S)$$

for $v_2^s \in E_2^{0,s(p+1)q}(M(r,t))$. We abbreviate $\widehat{\beta}_{s/t,1}$ and $\widehat{\beta}_{s/1}$ to $\widehat{\beta}_{s/t}$ and $\widehat{\beta}_s$, respectively, as usual. It is an interesting problem which of them survives in the spectral sequence. So far, the following elements are known to be permanent cycles:

a) $\widehat{\beta}_s$ for $s \ge 1$ in [12],
b) $\widehat{\beta}_{sp/t}$ for $s \ge 1$ and $t \le p$, and $t \le p-1$ if $s = 1$ in [2], [3],
c) $\widehat{\beta}_{sp^2/t}$ for $s \ge 1$ and $t \le 2p$, and $t \le 2p-2$ if $s = 1$ in [2], [4],
d) $\widehat{\beta}_{sp^2/t}$ for $s \ge 1$ and $t \le p^2 - 2$ in [11],
e) $\widehat{\beta}_{sp^n/t}$ for $s \ge 1$, $n \ge 3$, $1 \le t \le 2^{n-2}p$, and $t \le 2^{n-3}p$ if $s = 1$, in [6],
f) $\widehat{\beta}_{sp^2/p,2}$ for $s \ge 2$ in [4], and
g) $\widehat{\beta}_{sp^n/up,2}$ for $s \ge 1$, $n \ge 3$, $1 \le u \le 2^{n-2}$, and $u \le 2^{n-3}$ if $s = 1$, in [6],

We note that we have $\widehat{\beta}_{sp^n/t}$ for $t \le p^n$ in the E_2-term, and Ravenel showed that $\widehat{\beta}_{p^n/p^n}$ cannot be a permanent cycle for $n \ge 1$ [9, 6.4.2. Th.]. Thus, $\widehat{\beta}_{sp^n/t}$ for $2^{n-2}p < t \le p^n$ if $s > 1$, and for $2^{n-3}p < t < p^n$ if $s = 1$ were left undetermined.

In [11], we modified the definition: Let (s,t,r) be a triple of positive integers such that $t = up^{r-1} - c$ for integers u and c and $v_1^c v_2^s \in E_2^{2,(s(p+1)+c)q}(M(r, up^{r-1}))$. Then, the beta element for (s,t,r) is defined by

$$\widehat{\beta}_{s/t,r} = \partial_r \partial_{r,up^{r-1}}(v_1^c v_2^s) \in E_2^{2,(s(p+1)+c-up^{r-1})q}(S).$$

We notice that $\widehat{\beta}_{s/t,r}$ is determined uniquely for any choice of integers u and c. In this paper we modify it further.

Definition 1.2. Let s, u and r be positive integers and c non-negative one such that $v_1^c v_2^s$ belongs to $E_2^{0,(s(p+1)+c)q}(M(r, up^{r-1}))$. We denote by $b(s,c;u,r)$, a set of elements x of $E_2^{0,(s(p+1)+c)q}(M(r, up^{r-1}))$ such that $x \equiv v_1^c v_2^s \bmod (p, v_1^{c+1})$. We define the *beta element* by

$$\widehat{\beta}_{s/up^{r-1}-c,r} = \partial_r \partial_{r,up^{r-1}}(b(s,c;u,r)) \subset E_2^{2,(s(p+1)+c-up^{r-1})q}(S).$$

We notice that this beta element is not an element but a set, and $\widehat{\beta}_{s/up^{r-1}-c,r} = \widehat{\beta}_{s/t,r}$ if $s = up^{r-1} - c$. We further abuse a term.

Definition 1.3. We say that a beta element $\widehat{\beta}_{s/up^{r-1}-c,r}$ *survives* to the homotopy groups $\pi_*(S)$ if an element of $\widehat{\beta}_{s/up^{r-1}-c,r}$ is a permanent cycle.

In this paper, we consider the beta elements $\widehat{\beta}_{s/t,r}$ for $r = 1, 2$, and so the following spectra and maps of (1.1):

$$\begin{array}{ll} M = M(1), \quad \overline{M} = M(2), \quad K_u = M(1,u) \quad \text{and} \quad \overline{K}_u = M(2,up); \quad \text{and} \\ k = k_1, \quad \overline{k} = k_2, \quad \alpha = A_0, \quad A = A_1, \quad k_u = k_{1,u} \quad \text{and} \quad \overline{k}_u = k_{2,up} \end{array} \tag{1.4}$$

for $k = i, j$. Thus, from now on, i_u and j_u denote $i_{1,u}$ and $j_{1,u}$.

The above definitions make Oka's method developed in [6] and [7] simple: Let $f_{s,u} \in \pi_*(K_u)$ be an element such that $\eta_*(f_{s,u}) = v_2^s \in BP_*(K_u)$ for the unit map $\eta\colon S \to BP$ of the ring spectrum BP, and put

$$\mathfrak{B}_{Oka}(s,u) = \{\widehat{\beta}_{sp^n/t,\varepsilon+1} : n \ge \varepsilon,\ \varepsilon = 0,1,\ p^\varepsilon \mid t \ge 1,\ t \le 2^{n-\varepsilon}u\}.$$

Theorem 1.5. (Oka [6], [7]) *If* $f_{s,u} \in \pi_*(K_u)$ *exists, then every element of* $\mathfrak{B}_{Oka}(s,u)$ *survives to* $\pi_*(S)$.

Since Oka also showed that $f_{sp,u} \in \pi_*(K_u)$ for $s \geq 1$ and $u \leq p$, and $u < p$ if $s = 1$ in [2, Th. C] and [3, Th. CII], the theorem implies that $\mathfrak{B}_{Oka}(sp, u)$ yields generators of $\pi_*(S)$ including the elements in b), c), e) and g) in the above table.

Let W be the cofiber of the generator $\beta_1 \in \pi_{pq-2}(S)$, and we have a cofiber sequence

$$S^{pq-2} \xrightarrow{\beta_1} S^0 \xrightarrow{i_W} W \xrightarrow{j_W} S^{pq-1}. \tag{1.6}$$

In [10], we introduce another method to give a beta family from $f_{s,u} \in \pi_*(W \wedge K_u)$ such that $\eta_*(f_{s,u}) = v_2^s \in BP_*(W \wedge K_u)$. In this paper, we merge these methods. For an element $f_{p^i,u} \in \pi_*(W \wedge K_u)$, consider a family

$$\mathfrak{B}(p^i, u) = \{\widehat{\beta}_{sp^{i+n}/t,\varepsilon+1} : s \geq 1,\ n \geq \varepsilon,\ \varepsilon = 0, 1,\ p^\varepsilon \mid t \geq 1,\ t \leq 2^{n-\varepsilon}u - p^\varepsilon(2-\varepsilon)\}.$$

Theorem 1.7. *If $f_{p^i,u} \in \pi_*(W \wedge K_u)$ exists, then every element of $\mathfrak{B}(s, u)$ survives to $\pi_*(S)$.*

In [11, Th. 1.7], we showed the existence of $f_{p^2,p^2} \in \pi_*(W \wedge K_{p^2})$ for $p > 5$, though there does not exist $f_{p^2,p^2} \in \pi_*(K_{p^2})$ shown by Ravenel.

Corollary 1.8. *Let $p > 5$. Then, $\mathfrak{B}(p^2, p^2)$ yields a beta family of $\pi_*(S)$.*

This improves Oka's results if the prime number p is greater than five. Indeed, $\bigcup_{s\geq 1} \bigcup_{u=1}^{p} \mathfrak{B}_{Oka}(sp^2, u) \subset \mathfrak{B}(p^2, p^2)$.

2. Recollection on finite ring spectrum

In this section, we recall some results of Oka's. We call a spectrum E *a ring spectrum* if it admits a multiplication $\mu\colon E \wedge E \to E$ and a unit $\iota\colon S \to E$ such that $\mu(\iota \wedge 1) = 1 = \mu(1 \wedge \iota)$ and $\mu(\mu \wedge 1) = \mu(1 \wedge \mu)$. A ring spectrum E is *commutative* if $\mu T = \mu$ for the switching map $T\colon E \wedge E \to E \wedge E$. The homotopy groups $E_* = \pi_*(E)$ of E have a multiplication given by $ab = \mu(a \wedge b)$ for $a, b \in E_*$, which makes E_* a ring. Oka [7] (*cf.* [8]) defined $\mathrm{Mod}(E)$ and $\mathrm{Der}(E)$ by

$$\begin{array}{rcl} \mathrm{Mod}(E) & = & \{f \in [E, E]_* \mid \mu(f \wedge 1) = f\mu\} \quad \text{and} \\ \mathrm{Der}(E) & = & \{f \in [E, E]_* \mid \mu(f \wedge 1) + \mu(1 \wedge f) = f\mu\}. \end{array}$$

We call an element of $\mathrm{Der}(E)$ a *derivation* of E.

Theorem 2.1. (Oka [8, Lemma 1.3]) *For the unit ι, the induced homomorphism $\iota^*\colon \mathrm{Mod}(E) \to E_*$ is a ring isomorphism. Its inverse $\kappa\colon E_* \to \mathrm{Mod}(E)$ is given by $\kappa(f) = \mu(f \wedge 1)$.*

Consider a spectrum K_u in (1.4). Then, Oka showed that

Theorem 2.2. (Oka [7, Th. 2.5]) *K_u has a commutative and associative multiplication m_u.*

Theorem 2.3. (Oka [7, Lemma 2.3]) *$\mathrm{Mod}(K_u)$ is a commutative subring of $[K_u, K_u]_*$ and a commutator $[f, g]$ belongs to $\mathrm{Mod}(K_u)$ for $f \in \mathrm{Mod}(K_u)$ and $g \in \mathrm{Der}(K_u)$. In particular,*

$$f^p g = g f^p \quad \textit{for } f \in \mathrm{Mod}(K_u) \textit{ and } g \in \mathrm{Der}(K_u).$$

Let $\delta'_u = i_u j_u \in [K_u, K_u]_{-uq-1}$. Then, it fits into a cofiber sequence

$$\Sigma^{uq} K_u \xrightarrow{\widetilde{i}_u} K_{2u} \xrightarrow{\widetilde{j}_u} K_u \xrightarrow{\delta'_u} \Sigma^{uq+1} K_u \tag{2.4}$$

by (1.1) with 3×3 Lemma.

Theorem 2.5. (Oka [7, Th. 2.5]) $\delta'_u \in \mathrm{Der}(K_u)$.

It is well known that $\delta = ij \in \mathrm{Der}(M)$ and $\alpha \in \mathrm{Mod}(M)$, and so $\alpha^p\delta = \delta\alpha^p \in [M,M]_{pq-1}$. It gives rise to not only the element A but also δ_u in a commutative diagram

$$\begin{tikzcd}
\Sigma^{upq-1}M \arrow[r, "\delta"] \arrow[d, "\alpha^{up}"'] & \Sigma^{upq}M \arrow[r, "\pi"] \arrow[d, "\alpha^{up}"] & \Sigma^{upq}\overline{M} \arrow[r] \arrow[d, "A^u"] & \Sigma^{upq}M \arrow[d, "\alpha^{up}"] \\
\Sigma^{-1}M \arrow[r, "\delta"] \arrow[d, "i_{up}"'] & M \arrow[r, "\pi"] \arrow[d, "i_{up}"] & \Sigma^{pq}\overline{M} \arrow[r] \arrow[d, "\bar{i}_u"] & M \arrow[d, "i_{up}"] \\
\Sigma^{-1}K_{up} \arrow[r, "\delta_u"] \arrow[d, "j_{up}"'] & K_{up} \arrow[r, "\overline{\pi}"] \arrow[d, "j_{up}"] & \overline{K}_u \arrow[r] \arrow[d, "\bar{j}_u"] & K_{up} \arrow[d, "j_{up}"] \\
\Sigma^{upq}M \arrow[r, "-\delta"] & \Sigma^{upq+1}M \arrow[r, "\pi"] & \Sigma^{upq+1}\overline{M} \arrow[r] & \Sigma^{upq+1}M,
\end{tikzcd} \tag{2.6}$$

in which rows and columns are cofiber sequences. By [6, Lemma 4.5, Th. 4.2], we have the following

Theorem 2.7. (Oka [7, p.425]) *The map δ_u in the above diagram is a derivation of K_{up}.*

Proof. The matrices for the map $\delta_u \wedge 1$ and the switching map T are given by

$$\tau(\delta_u \wedge 1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \delta & -1 & 0 & 0 \\ \delta' & 0 & -1 & 0 \\ \delta\delta' & \delta' & -\delta & 1 \end{pmatrix}$$

by [6, Lemma 4.5] and [6, Th. 4.2], and so the first row of the matrix for $(\delta_u \wedge 1) + (\delta_u \wedge 1)T$ is $(\delta\ 0\ 0\ 0)$. Since the multiplication m_u is the projection to the first summand, we see that $m_u(\delta_u \wedge 1 + 1 \wedge \delta_u) = m_u((\delta_u \wedge 1) + T(\delta_u \wedge 1)T) = m_u((\delta_u \wedge 1) + (\delta_u \wedge 1)T) = \delta_u m_u$ as desired. □

The following lemma is a folklore:

Lemma 2.8. *There exist self-maps $\widetilde{\alpha}\colon \Sigma^q K_u \to K_u$ and $\widetilde{A}\colon \Sigma^{pq}\overline{K}_u \to \overline{K}_u$ such that $BP_*(\widetilde{\alpha}) = v_1$ and $BP_*(\widetilde{A}) = v_1^p$.*

Lemma 2.9. $A\bar{i}\beta_1 = 0 \in \pi_{2pq-2}(\overline{M})$.

Proof. Consider the cobar complex $\{(C^s, d)\}_{s\geq 0}$, whose cohomology is the E_2-term $E_2^*(\overline{M})$ of the Adams-Novikov spectral sequence converging to $\pi_*(\overline{M})$. Then, $C^s = \Gamma/(p^2) \otimes_A \Gamma^{s-1}$, where $(A, \Gamma) = (BP_*, BP_*BP)$, and the differential d of the complex is given by derivation with $d(v) = \eta_R(v) - \eta_L(v) \in C^1 = \Gamma/(p^2)$ for $v \in C^0 = A/(p^2)$ and $d(x) = 1 \otimes x - \Delta(x) + x \otimes 1 \in C^2$ for $x \in C^1$. We also use the formulas on the structure maps of the Hopf algebroid given by the formulas of Quillen and Hazewinkel:

$$\begin{gathered}\eta_R(v_1) = v_1 + pt_1, \quad \eta_R(v_2) \equiv v_2 + v_1t_1^p + pt_2 - (p+1)v_1^pt_1 \mod (p^2) \\ \Delta(t_1) = 1 \otimes t_1 + t_1 \otimes 1 \quad \text{and} \quad \Delta(t_2) = 1 \otimes t_2 + t_1 \otimes t_1^p + t_1 \otimes 1 + v_1 b_{10}.\end{gathered}$$

Here, b_{10} denotes the cocycle defined by $d(t_1^p) = pb_{10}$. Let b_0 denote the cohomology class of b_{10}. Then, by definition, $\widehat{\beta}_1 = \partial\partial_1(v_2) = b_0$. ($b(1,1)$ consists of only one element v_2 by degree reason.) Therefore, $A\bar{i}\beta_1$ is detected by $v_1^pb_0 \in E_2^{2,2pq}(\overline{M})$.

We compute that $d(c) = v_1^p b_{10} \in C^2$ for $c = -v_1^{p-1}t_2 + v_1^{p-3}(v_1t_1 - pt_1^2)\eta_R(v_2) + v_1^{2p-3}\left(\frac{p+1}{2}v_1t_1^2 - \frac{p}{3}t_1^3\right)$. It follows that $v_1^p b_0 = 0 \in E_2^{2,2pq}(\overline{M})$. We further see that $E_2^{2+rq,2pq+rq}(\overline{M}) = 0$ for $r \geq 1$ by the vanishing line (*cf.* [1, Lemma 1.16, Remark 1.17]). Hence $A\bar{i}\beta_1 = 0 \in \pi_{2pq-2}(\overline{M})$. □

3. The ring spectrum $W \wedge K_u$

In this section, we fix an integer u and K denotes $K_u = M(1,u)$, which is a commutative ring spectrum with multiplication $m = m_u$ by Theorem 2.2. The spectrum W in (1.6) admits a multiplication $m_W\colon W \wedge W \to W$ such that $m_W(i_W \wedge 1_W) = 1_W = m_W(1_W \wedge i_W)$ by [5, Example 2.9].

Consider the spectrum $WM = W \wedge M$ and the multiplication $m_{WM} = (m_W \wedge m_M)(1_W \wedge T \wedge 1_M)\colon WM \wedge WM \to WM$. Then, we see that

$$m_{WM}(i' \wedge i') = i'm_W, \tag{3.1}$$

for $i' = 1_W \wedge i$. We further see the following lemma by [6, Cor. 4.3]:

Lemma 3.2. *WM is a commutative ring spectrum with multiplication m_{WM}.*

Proof. Since $p = 0 \in [WM \wedge WM]_0$ and $\beta_1 \wedge 1 = 0 \in [M \wedge WM, M \wedge WM]_{pq-2}$, WM is a split ring spectrum. □

Put $WMK = W \wedge M \wedge K$, and consider a multiplication $m_{WMK} = (m_{WM} \wedge m)(1_{WM} \wedge T \wedge 1_K)\colon WMK \wedge WMK \xrightarrow{1_{WM}\wedge T\wedge 1_K} WM \wedge WM \wedge K \wedge K \xrightarrow{m_{WM}\wedge m} WMK$ on WMK. Since the smash product of commutative ring spectra is a commutative ring spectrum, we have the following

Corollary 3.3. *WMK is a commutative ring spectrum with multiplication m_{WMK}.*

Consider the spectrum $WK = W \wedge K$ and a multiplication m_{WK} on WK defined by $m_{WK} = (m_W \wedge m)(1_W \wedge T \wedge 1_K)$ for the switching map $T\colon K \wedge W \to W \wedge K$. We have the split cofiber sequence

$$WK \underset{\sigma}{\overset{\widehat{i}}{\rightleftarrows}} WMK \xrightarrow{\widehat{j}} \Sigma WK, \tag{3.4}$$

in which $\widehat{i} = 1_W \wedge i \wedge 1_K\colon WK \to WMK$ and σ denotes a splitting.

Lemma 3.5. $\widehat{i}m_{WK} = m_{WMK}(\widehat{i} \wedge \widehat{i})$.

Proof. This follows from computation:

$$\begin{aligned}
\widehat{i}m_{WK} &= (i' \wedge 1_K)(m_W \wedge m)(1_W \wedge T \wedge 1_K) \\
&= (m_{WM} \wedge m)(i' \wedge i' \wedge 1_{K\wedge K})(1_W \wedge T \wedge 1_K) \quad \text{by (3.1)} \\
&= (m_{WM} \wedge m)(1_{WM} \wedge T \wedge 1_K)(\widehat{i} \wedge \widehat{i}) \ = \ m_{WMK}(\widehat{i} \wedge \widehat{i}).
\end{aligned}$$

□

Lemma 3.6. *The spectrum WK is a commutative ring spectrum with multiplication m_{WK}.*

Proof. Apply σ in (3.4) to Lemma 3.5, and we have

$$m_{WK} = \sigma m_{WMK}(\widehat{i} \wedge \widehat{i}). \tag{3.7}$$

Then, noticing that $m_{WMK}T' = m_{WMK}$ by Corollary 3.3,

$$m_{WK}T \ = \ \sigma m_{WMK}(\widehat{i} \wedge \widehat{i})T \ = \ \sigma m_{WMK}T'(\widehat{i} \wedge \widehat{i}) \ = \ \sigma m_{WMK}(\widehat{i} \wedge \widehat{i}) \ = \ m_{WK}.$$

Here, $T\colon WK\wedge WK\to WK\wedge WK$ and $T'\colon WMK\wedge WMK\to WMK\wedge WMK$ are the switching maps.

The associativity of it is verified as follows:

$$\begin{aligned}
m_{WK}(m_{WK}\wedge 1_{WK}) &= \sigma m_{WMK}(\widehat{i}\wedge\widehat{i})\,(m_{WK}\wedge 1_{WK}) \quad \text{by (3.7)}\\
&= \sigma m_{WMK}(m_{WMK}\wedge 1_{WMK})(\widehat{i}\wedge\widehat{i}\wedge\widehat{i}) \quad \text{(by Lemma 3.5)}\\
&= \sigma m_{WMK}(1_{WMK}\wedge m_{WMK})(\widehat{i}\wedge\widehat{i}\wedge\widehat{i}) \quad \text{(by Corollary 3.3)}\\
&= \sigma m_{WMK}(\widehat{i}\wedge\widehat{i})\,(1_{WK}\wedge m_{WK}) \quad \text{(by Lemma 3.5)}\\
&= m_{WK}(1_{WK}\wedge m_{WK}).
\end{aligned}$$

□

Consider the homomorphism

$$\varphi_W\colon [K,K]_*\to[WK,WK]_*$$

given by $\varphi_W(f)=1_W\wedge f$. Then, an easy computation shows

Lemma 3.8. *The homomorphism φ_W induces ones $\varphi_W\colon \mathrm{Mod}(K)\to\mathrm{Mod}(WK)$ and $\varphi_W\colon \mathrm{Der}(K)\to\mathrm{Der}(WK)$.*

4. Construction of homotopy elements

We begin with a general result corresponding to Oka's theorems [6, Th. 7.1, Th. 7.2] and [7, Construction III]. The proof is almost identical.

Proposition 4.1. *Let $f\in\pi_*(WK_u)$ be an element such that $\eta_*(f)\equiv v_2^s \bmod (v_1)$ for the unit map η of BP. Then,*

1) *for $n\ge 0$, there is an element $f_n\in\pi_*(WK_{2^nu})$ such that $\eta_*(f_n)\equiv v_2^{sp^n} \bmod (v_1)$, and*
2) *for $n\ge 1$, let u' be a positive integer such that $u'p\le 2^{n-1}u$. Then, there are an element $f'_{n-1}\in\pi_*(WK_{u'p})$ such that $\eta_*(f'_{n-1})\equiv v_2^{sp^{n-1}} \bmod (v_1)$, and an element $\overline{f}_n\in\pi_*(W\overline{K}_{u'})$ such that $\eta_*(\overline{f}_n)\equiv v_2^{sp^n} \bmod (p,v_1)$. Here, $W\overline{K}_u=W\wedge\overline{K}_u$.*

Proof. Put $f_0=f$, and suppose that $f_n\in\pi_*(WK_{2^nu})$. Then, $\kappa(f_n)\in\mathrm{Mod}(WK_{2^nu})$ for κ in Theorem 2.1. By Theorems 2.2 and 3.8, $\delta'=\varphi_W(\delta'_{2^nu})\in\mathrm{Der}(WK_{2^nu})$, and so we have $\kappa(f_n)^p\delta'=\delta'\kappa(f_n)^p$ by Theorem 2.3. Thus we obtain a map $\widetilde{f}_{n+1}$, which makes the diagram

$$\begin{array}{ccccc}
WK_{2^{n+1}u} & \longrightarrow & WK_{2^nu} & \xrightarrow{\delta'} & WK_{2^nu}\\
{\scriptstyle\widetilde{f}_{n+1}}\downarrow & & \downarrow{\scriptstyle\kappa(f_n)^p} & & \downarrow{\scriptstyle\kappa(f_n)^p}\\
WK_{2^{n+1}u} & \longrightarrow & WK_{2^nu} & \xrightarrow{\delta'} & WK_{2^nu}
\end{array}$$

commutes. Now put $f_{n+1}=(i_W\wedge i_{2^nu}i)^*(\widetilde{f}_{n+1})$ to complete the induction.

Turn to the second. By 1), we have $f_{n-1}\in\pi_*(WK_{2^{n-1}u})$, which gives rise to $f'_{n-1}\in\pi_*(WK_{u'p})$ as in [7, Construction I]. Theorems 2.7, 3.8 and 2.3 imply that

$$\kappa(f'_{n-1})^p\delta''=\delta''\kappa(f'_{n-1})^p.$$

for $\delta'' = \varphi_W(\delta_{u'})$, where $\delta_{u'}$ is the map in (2.6). We then have a map $\widetilde{\overline{f}}_n$ fitting into the commutative diagram

$$\begin{array}{ccccc}
W\overline{K}_{u'} & \longrightarrow & WK_{u'p} & \xrightarrow{\delta''} & WK_{u'p} \\
{\scriptstyle \widetilde{\overline{f}}_n}\downarrow & & \downarrow{\scriptstyle \kappa(f'_{n-1})^p} & & \downarrow{\scriptstyle \kappa(f'_{n-1})^p} \\
W\overline{K}_{u'} & \longrightarrow & WK_{u'p} & \xrightarrow{\delta''} & WK_{u'p}
\end{array}$$

and $\overline{f}_n = (i_W \wedge \overline{i}_{u'} i)^*(\widetilde{\overline{f}}_n)$. $\square$

In [10], we show the following lemma:

Lemma 4.2. ([10, Lemma 2.11]) *For $u > 2$, there exists an element $\omega_u \in \pi_{(p+2)q-1}(WK_u)$ such that $(j_W)_*(\omega_u) = i_u\alpha^2 i \in \pi_{2q}(K_u)$. Moreover, $v_1^2 g \in E_2^0(WK_u) = E_2^0(K_u) \oplus gE_2^0(K_u)$ detects it.*

A similar lemma follows from Lemma 2.9.

Lemma 4.3. *For $u > 1$, there exists an element $\overline{\omega}_u \in \pi_{2pq-1}(W\overline{K}_u)$ such that $(j_W)_*(\overline{\omega}_u) = \overline{i}_u A \overline{i} \in \pi_{pq}(\overline{K}_u)$. Moreover, $v_1^p g \in E_2^0(W\overline{K}_u) = E_2^0(\overline{K}_u) \oplus gE_2^0(\overline{K}_u)$ detects it.*

Proof of Theorem 1.7. By Proposition 4.1, we have elements $(f_{p^i,u})_n \in \pi_*(WK_{2^n u})$ and $\overline{(f_{p^i,u})}_n \in \pi_*(W\overline{K}_{u'})$ for $u'p \le 2^{n-1}u$ such that $\eta_*((f_{p^i,u})_n) \equiv v_2^{p^{i+n}} \bmod (v_1)$ and $\eta_*(\overline{(f_{p^i,u})}_n) \equiv v_2^{p^{i+n}} \bmod (p, v_1)$. Consider the composites

$$\begin{aligned}
B_{sp^{i+n}/2^n u-r} &= \widetilde{\alpha}^{r-2}(j_W \wedge 1_K)\kappa((f_{p^i,u})_n)^s\omega_{2^n u} \quad (r \ge 2), \quad \text{and} \\
B_{sp^{i+n}/u'p-rp,2} &= \widetilde{A}^{r-1}(j_W \wedge 1_{\overline{K}})\kappa(\overline{(f_{p^i,u})_n})^s\overline{\omega}_{u'} \quad (r \ge 1),
\end{aligned}$$

where $\widetilde{\alpha}$, $\widetilde{A}$, ω_u and $\overline{\omega}_u$ are the elements of Lemmas 2.8, 4.2 and 4.3. Then, $\eta_*(B_{sp^{i+n}/2^n u-r}) \in b(sp^{i+n}, r; 2^n u, 1)$ and $\eta_*(B_{sp^{i+n}/u'p-rp,2}) \in b(sp^{i+n}, rp; u'p, 2)$. Since j, $j_{2^n u}$ and $\overline{j}_{u'}$ correspond to ∂_1, $\partial_{1,2^n u}$ and $\partial_{2,u'p}$, respectively, the elements $jj_{2^n u}B_{sp^{i+n}/2^n u-r}$ and $j\overline{j}_{u'}B_{sp^{i+n}/u'p-rp,2}$ are detected by elements of $\widehat{\beta}_{sp^{i+n}/2^n u-r}$ and $\widehat{\beta}_{sp^{i+n}/u'p-rp,2}$, as desired. $\square$

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