PICARD GROUP OF THE $E(2)$-LOCAL STABLE HOMOTOPY CATEGORY AT THE PRIME THREE

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Abstract. Let $\mathcal{L}_2$ denote the stable homotopy category of $v_2^{-1}BP$-local spectra at the prime three. In [2], it is shown that the Picard group of $\mathcal{L}_2$ consisting of isomorphic classes of invertible spectra is isomorphic to either the direct sum of $\mathbb{Z}$ and $\mathbb{Z}/3$ or the direct sum of $\mathbb{Z}$ and two copies of $\mathbb{Z}/3$. In this paper, we conclude the Picard group is isomorphic to the latter group by showing the existence of an exotic invertible spectrum.

1. Introduction

Let $\mathcal{S}_{(p)}$ be the stable homotopy category of spectra localized away from the prime p. The Brown-Peterson spectrum $BP \in \mathcal{S}_{(p)}$ plays a principal role, when we try to understand $\mathcal{S}_{(p)}$. The coefficient ring of BP is the polynomial algebra $\mathbb{Z}_{(p)}[v_1, v_2, \dots]$ over generators v_n $(n \geq 1)$ with degree $|v_n| = 2(p^n - 1)$. For a spectrum E, a spectrum A is *E-local* if $[C, A]_* = 0$ for any C such that $E \wedge C = 0$. Bousfield constructed the localization functor $L_E\colon \mathcal{S}_{(p)} \to \mathcal{L}_E$ to the category of E-local spectra. For $E = v_n^{-1}BP$, we abbreviate L_E and $\mathcal{L}_E$ to L_n and $\mathcal{L}_n$, respectively ([5]). Let $E(n)$ be the Johnson-Wilson spectrum whose coefficient ring is $\mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}]$. Then, it is well known that $L_{E(n)} = L_n$. We also have the $E(n)$-based Adams spectral sequence $\{E_r^*(A)\}$ for a spectrum $A \in \mathcal{S}_{(p)}$ converging to $\pi_*(L_nA)$ with E_2-term $E_2^{s,t}(A) = \operatorname{Ext}^{s,t}_{E(n)_*(E(n))}(E(n)_*, E(n)_*(A))$.

We call a spectrum $Q \in \mathcal{L}_n$ *invertible* in $\mathcal{L}_n$ if there is a spectrum P such that $Q \wedge P = L_nS^0$ for the sphere spectrum $S^0 \in \mathcal{S}_{(p)}$. The spectrum L_nS^0 is a typical example of an invertible spectrum. Invertible spectra are characterized by the $E(n)_*$-homology:

Theorem 1.1. ([2, Th. 1.1], *cf.* [1]) *A spectrum $Q \in \mathcal{L}_n$ is invertible if and only if $E(n)_*(Q) = E(n)_*$ as an $E(n)_*(E(n))$-comodule.*

This implies that the E_2-term $E_2^*(Q)$ of an invertible spectrum Q is isomorphic to $E_2^*(\Sigma^k S^0)$ for some $k \in \mathbb{Z}$ under a degree preserving isomorphism. In this case, the E_2-term of $\Sigma^{-k}Q$ is isomorphic to $E_2^*(S^0)$. We call an invertible spectrum Q *core* if there is a degree preserving isomorphism $E_2^*(Q) \cong E_2^*(S^0)$. Every invertible spectrum is a suspension of a core invertible spectrum. Hereafter, $g_Q \in E_2^{0,0}(Q)$ for a core invertible spectrum Q denotes a generator corresponding to $1 \in \mathbb{Z}_{(p)} = E_2^{0,0}(S^0)$. We notice that g_Q is represented by the unit map $i\colon S^0 \to E(n) = E(n) \wedge Q$ of the ring spectrum $E(n)$. The differentials $d_r(g_Q)$ of the spectral sequences discern core invertible spectra. For instance, $d_r(g_Q) = 0 \in E_r^{r,r-1}(Q)$ for all $r \geq 2$ if and only if $Q = L_nS^0$.

In [1], it is shown that a collection of isomorphism classes of invertible spectra in $\mathcal{L}_n$ forms a set, which is also a group with multiplication defined by $[Q][Q'] =$

$[Q \wedge Q']$ and the unit $[L_nS^0]$. Here, $[X]$ denotes the isomorphism class of X, and below, we will write the classes without square brackets. We call the group the *Picard group* of $\mathcal{L}_n$ and denote it by $\mathrm{Pic}(\mathcal{L}_n)$. The group $\mathrm{Pic}(\mathcal{L}_n)$ splits into the direct sum of $\mathbb{Z}$ generated by L_nS^1 and a subgroup $\mathrm{Pic}^0(\mathcal{L}_n)$ consisting of isomorphism classes of core invertible spectra ([1]). We define a decreasing filtration of $\mathrm{Pic}^0(\mathcal{L}_n)$ by $F_k = \{Q \in \mathrm{Pic}^0(\mathcal{L}_n) : d_r(g_Q) = 0 \in E_r^{r,r-1}(Q) \text{ for } r < kq+1\}$, where $q = 2p-2$. Hopkins and Ravenel (*cf.* [6]) showed that there is an integer r such that the E_r-term has a horizontal vanishing line, and then we have the least integer k_0 such that $F_{k_0} = \{L_nS^0\}$. In [2], we define an abelian group F_k/F_{k+1} as a set of equivalence classes of F_k under a suitable equivalence relation, and put $G\mathrm{Pic}^0(\mathcal{L}_n) = \bigoplus_{k>0} F_k/F_{k+1}$. Note that $E_2^{r,r-1}(Q) = 0$ unless $q \mid (r-1)$, and put

$$(1.2) \qquad G^k = E_{qk+1}^{qk+1,qk}(S^0) \quad (= \mathrm{Ext}^{q+1,q}_{E(n)_*(E(n))}(E(n)_*, E(n)_*) \text{ if } k = 1).$$

We then have a homomorphism $\varphi_k \colon F_k/F_{k+1} \to G^k$ defined by $\varphi_k(Q) = \omega$ for ω in $d_{kq+1}(g_Q) = \omega g_Q \in E_{kq+1}^{kq+1,kq}(Q)$ for $k < k_0$, and $\varphi_k = 0$ for $k \geq k_0$. Let φ denote the direct sum $\bigoplus_{k>0} \varphi_k \colon G\mathrm{Pic}^0(\mathcal{L}_n) \to \bigoplus_{k>0} G^k$.

Theorem 1.3. ([2, Th.1.2]) *The homomorphism φ_k for every $k > 0$ is a monomorphism, and so is φ. In other words, $G\mathrm{Pic}^0(\mathcal{L}_n)$ is isomorphic to a subgroup of $\bigoplus_{k>0} G^k$.*

In some cases, the E_{qk+1}-terms G^k are known, and this theorem implies immediately the facts $\mathrm{Pic}(\mathcal{L}_n) = \mathbb{Z}$ if $n^2 + n < q$ (since $G^k = 0$ for $k > 0$) and $\mathrm{Pic}(\mathcal{L}_1) \subset \mathbb{Z} \oplus \mathbb{Z}/2$ if $p = 2$ (since $G^1 = \mathbb{Z}/2$ and $G^k = 0$ for $k > 1$) (*cf.* [1], [2]). We call an invertible spectrum Q *exotic* unless it is a suspension of L_nS^0. We say that an exotic core invertible spectrum Q is *detected* by $\omega \neq 0 \in G^r$ if $\varphi(Q) = \omega$, and write Q_ω for it. The inclusion $\mathrm{Pic}(\mathcal{L}_1) \subset \mathbb{Z} \oplus \mathbb{Z}/2$ above is shown to be an isomorphism by Hovey and Sadofsky [1] by constructing an exotic invertible spectrum detected by the generator ω_0 of the summand $\mathbb{Z}/2$. In fact, the spectrum Q_{ω_0} is a suspension of $L_1V(\frac{1}{2})$, where $V(\frac{1}{2})$ is the Toda spectrum, which is also known as the 'question mark complex'.

Turn to the case $n = 2$ and $p = 3$. Note that $q = 4$ in (3.2). From [10], we read off that $\bigoplus_{k>0} G^k = G^1 = \mathbb{Z}/3 \oplus \mathbb{Z}/3$ generated by

$$(1.4) \qquad \begin{aligned} \omega_1 &= \eta(v_2^{-1}h_1b_0/3v_1) \equiv v_2^{-1}h_1b_0^2 \mod (3, v_1) \quad \text{and} \\ \omega_2 &= \eta(v_2^{-1}\xi\zeta/3v_1) = h_0\chi \equiv v_2^{-1}\xi b_0\zeta_2 \mod (3, v_1). \end{aligned}$$

For the generators, see section three. It follows that $\mathrm{Pic}(\mathcal{L}_2) \subset \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$. We constructed an exotic invertible spectrum detected by ω_1 to show that $\mathrm{Pic}(\mathcal{L}_2) \supset \mathbb{Z} \oplus \mathbb{Z}/3$ in [2]. The spectrum Q_{ω_1} is closely related to the Toda spectrum $V(1\frac{1}{2})$, though we do not know whether or not it is an $E(2)$-localization of a finite spectrum.

In this paper, we show the existence of Q_{ω_2}, which does not seem to be related to any Toda spectrum $V(a)$.

Theorem 1.5. *There exists an invertible spectrum detected by ω_2.*

Corollary 1.6. $\mathrm{Pic}(\mathcal{L}_2) = \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$.

We call a finite spectrum U an *$E(n)$-bouquet* if $E(n) \wedge U = E(n) \vee \bigvee_{k=1}^u \Sigma^{e_k} E(n)$ for $e_k \not\equiv 0 \mod q$. Let g and g_k ($1 \leq k \leq u$) denote the generators of the $E(n)_*$-summands of $E(n)_*(U)$. Here, the generator g is represented by $S^0 \xrightarrow{i} E(n) \xrightarrow{\iota^U}$

$E(n)\wedge U$ for the inclusion ι^U. Let $\omega \in E_r^{r,r-1}(S^0)$ be a generator. We call an $E(n)$-bouquet U an *ω-bouquet* if it satisfies the condition:

$$d_r(g) = \omega g \neq 0 \in E_r^{r,r-1}(U). \tag{1.7}$$

Proposition 1.8. *Suppose that there exists an ω-bouquet. Then, the invertible spectrum Q_ω exists.*

This is proved in the next section by constructing an ∞-tower as we did in [2].

Proposition 1.9. *For ω_2 in (1.4), there exists an ω_2-bouquet.*

This is the main part of this paper and proved in section three. Theorem 1.5 now follows from Propositions 1.8 and 1.9.

2. An invertible spectrum associated with an ω-bouquet

Let E denote the n-th Johnson-Wilson spectrum $E(n)$ with multiplication $\mu\colon E\wedge E\to E$ and unit map $i\colon S^0\to E$. We then have the cofiber sequences $S^0 \xrightarrow{i} E \xrightarrow{j} \overline{E} \xrightarrow{k} S^1$, and spectra D_s sitting in the cofiber sequence $\overline{E}^s \xrightarrow{k^s} S^s \xrightarrow{\overline{k}_s} \Sigma D_s \xrightarrow{\overline{i}_s} \Sigma\overline{E}^s$, where $\overline{E}^s$ and k^s denote the s-fold smash product of $\overline{E}$ with itself and the composite $k(k\wedge\overline{E})\cdots(k\wedge\overline{E}^{s-1})$, respectively. The spectra D_s also fit into cofiber sequences

$$D_s \xrightarrow{i_s} E\wedge\overline{E}^s \xrightarrow{j_s} D_{s+1} \xrightarrow{k_s} \Sigma D_s. \tag{2.1}$$

Now these yield the exact couple $(D_1^s(A), E_1^s(A)) = (\pi_*(D_s\wedge A), \pi_*(E\wedge\overline{E}^s\wedge A))$ for a spectrum A, that defines the E-based Adams spectral sequence for computing $\pi_*(L_nA)$. Consider the sequence

$$E \xrightarrow{d_1} E\wedge\overline{E} \xrightarrow{d_1} \cdots \xrightarrow{d_1} E\wedge\overline{E}^s \xrightarrow{d_1} E\wedge\overline{E}^{s+1} \xrightarrow{d_1} \cdots \tag{2.2}$$

with $d_1 = i_{s+1}j_s\colon E\wedge\overline{E}^s \to E\wedge\overline{E}^{s+1}$. Let m denote a positive integer or ∞. We call a sequence of spectra $\{A_s\}_{0\le s\le m}$ an *m-tower* if the spectra fit into the cofiber sequences

$$A_s \xrightarrow{i_s^A} E\wedge\overline{E}^s \xrightarrow{j_s^A} A_{s+1} \xrightarrow{k_s^A} \Sigma A_s \tag{2.3}$$

for $0\le s<m$. The sequence $\{D_s\}_{s\ge 0}$ is a typical example of an ∞-tower in which $i_s^D = i_s$, $j_s^D = j_s$ and $k_s^D = k_s$.

Lemma 2.4. *Let $\{A_s\}_{s\ge0}$ be an ∞-tower, and put $A_\infty = \lim_s \Sigma^{-s}A_s$. Then, A_∞ is an invertible spectrum.*

Proof. By [2, Th. 1.1] (see Theorem 1.1), it suffices to show that $E_*(A_\infty) = E_*$. This follows immediately from a similar argument as [11, Prop. 5.5], since the spectral sequence associated with the ∞-tower has a horizontal vanishing line. □

The following lemma is well known (*cf.* [2, Lemma 4.5]).

Lemma 2.5. *Let $\{A_s\}_{0\le s\le m}$ be an m-tower and $\widetilde{E}$ an E-module spectrum. Then, we have a split short exact sequence*

$$0\to \pi_{t+m-1}(\widetilde{E}) \xrightarrow{\varphi^*} [A_m,\widetilde{E}]_t \xrightarrow{(j_{m-1}^A)^*} (\mathrm{Im}\ d_1)_t \to 0.$$

Here, φ^* is induced from the composite $A_m \xrightarrow{k_{m-1}^A} A_{m-1} \xrightarrow{k_{m-2}^A} \cdots \xrightarrow{k_1^A} A_1 = E \xleftarrow{i} S^0$, and $(\mathrm{Im}\ d_1)_t \subset [E\wedge\overline{E}^{m-1},\widetilde{E}]_t$.

Let U be an E-bouquet. Then, we have the split cofiber sequence

$$E \underset{\sigma^U}{\overset{\iota^U}{\rightleftarrows}} E \wedge U \underset{\tau^U}{\overset{\kappa^U}{\rightleftarrows}} E \wedge \overline{U} \tag{2.6}$$

for a bouquet $\overline{U}$, which induces split cofiber sequences in the commutative diagram

$$\begin{array}{ccccc} E \wedge \overline{E}^s & \underset{\sigma^U_s}{\overset{\iota^U_s}{\rightleftarrows}} & E \wedge \overline{E}^s \wedge U & \underset{\tau^U_s}{\overset{\kappa^U_s}{\rightleftarrows}} & E \wedge \overline{E}^s \wedge \overline{U} \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_1 \wedge U} & & \downarrow{\scriptstyle d_1 \wedge \overline{U}} \\ E \wedge \overline{E}^{s+1} & \underset{\sigma^U_{s+1}}{\overset{\iota^U_{s+1}}{\rightleftarrows}} & E \wedge \overline{E}^{s+1} \wedge U & \underset{\tau^U_{s+1}}{\overset{\kappa^U_{s+1}}{\rightleftarrows}} & E \wedge \overline{E}^{s+1} \wedge \overline{U}. \end{array} \tag{2.7}$$

Proof of Proposition 1.8. Let a be the order of $\omega \in E_r^{r,r-1}(S^0)$ and U denote the ω-bouquet. Put $a_s = 1 + \delta_{s,r}(a-1)$ for the Kronecker delta $\delta_{s,r}$. Then, we have elements $g^s_U \in \pi_{s-1}(D_s \wedge U)$ for $s > 0$ such that $k^U_s g^{s+1}_U = a_s g^s_U$, $g^1_U = \iota^U i$ and $i^U_r g^r_U = \iota^U_r \widetilde{\omega} \in \pi_{r-1}(E \wedge \overline{E}^r \wedge U)$. Here, l^U_s for $l = i, j, k$ denotes $l_s \wedge U$, and $\widetilde{\omega}$ denotes an element representing ω. We notice that $\iota^U_r \widetilde{\omega}$ represents $\omega g \in E_r^{r,r-1}(U)$.

By the induction on m, we construct an m-tower $\{Q_s\}_{0 \le s \le m}$ along with elements $g^s_Q \in \pi_{s-1}(Q_s)$ and maps $f_s \colon Q_s \to D^s \wedge U$ for $s \le m$ such that

1) $k^Q_{s-1} g^s_Q = a_{s-1} g^{s-1}_Q$, $g^1_Q = i$ and $i^Q_r g^r_Q = \widetilde{\omega} \in \pi_{r-1}(E \wedge \overline{E}^r)$,
2) $f_s g^s_Q = g^s_U$ for $s < m$, and
3) these fit into the commutative diagrams

$$\begin{array}{ccccccc} Q_s & \xrightarrow{i^Q_s} & E \wedge \overline{E}^s & \xrightarrow{j^Q_s} & Q_{s+1} & \xrightarrow{k^Q_s} & \Sigma Q_s \\ {\scriptstyle f_s}\downarrow & & \downarrow{\scriptstyle \iota^U_s} & & \downarrow{\scriptstyle f_{s+1}} & & \downarrow{\scriptstyle f_s} \\ D_s \wedge U & \xrightarrow{i^U_s} & E \wedge \overline{E}^s \wedge U & \xrightarrow{j^U_s} & D_{s+1} \wedge U & \xrightarrow{k^U_s} & \Sigma D_s \wedge U \end{array} \tag{2.8}$$

of cofiber sequences for $s < m$.

We start from setting $Q_0 = *$, $f_0 = 0$, $Q_1 = E$, $g^1_Q = i$ and $f_1 = \iota^U_0 = \iota^U$.

Suppose that there exist an m-tower $\{Q_s\}_{0 \le s \le m}$ with (2.8). Note that i^Q_m is not defined at this stage. So we put $i^Q_m = \sigma^U_m i^U_m f_m$. Then, it suffices to show that

$$\iota^U_m i^Q_m = i^U_m f_m \quad \text{and} \quad f_m g^m_Q = g^m_U. \tag{2.9}$$

Indeed, since the former is the left square of the diagram (2.8)3) for $s = m$, we obtain Q_{m+1} and f_{m+1} fitting into (2.8)3). The latter implies that $i^Q_m g^m_Q = 0$, which yields g^{m+1}_Q. We will redefine f_m so that (2.9) holds. We see that $k^U_{m-1}(f_m g^m_Q - g^m_U) = 0$, which yields an element $y \in \pi_{m-1}(E \wedge \overline{E}^{m-1} \wedge U)$ such that $j^U_{m-1} y = f_m g^m_Q - g^m_U$. Since $i^* \colon [E, E \wedge \overline{E}^{m-1} \wedge U]_{m-1} \to \pi_{m-1}(E \wedge \overline{E}^{m-1} \wedge U)$ is an epimorphism, we have an element $\widetilde{y}$ such that $\widetilde{y} i = y$. Put $f = f_m - j^U_{m-1} \widetilde{y} k^Q_1 \cdots k^Q_{m-1}$. Then, f sits in the place of f_m in (2.8)3) for $s = m-1$, and $f g^m_Q = f_m g^m_Q - j^U_{m-1} \widetilde{y} i = g^m_U$. So we replace f_m by f, and show the first equality of (2.9).

Consider the element $o_m = \iota^U_m i^Q_m - i^U_m f_m = \iota^U_m \sigma^U_m i^U_m f - i^U_m f$. Then, we see that $o_m j^Q_{m-1} = 0$, since we see that $i^Q_m j^Q_{m-1} = d_1$ by diagram chasing. By Lemma 2.5, we have an element $x \in \pi_{m-1}(E \wedge \overline{E}^m \wedge U)$ such that $\varphi^*(x) = o_m$, and then an element $\widetilde{x} \in [E, E \wedge \overline{E}^m \wedge U]_{m-1}$ such that $\widetilde{x} i = x$ and $\widetilde{x} k^Q_1 \cdots k^Q_{m-1} = o_m$. Put $\overline{a}_s = a_{\min\{s-1,r\}}$, and we compute $o_m g^m_Q = \overline{a}_m \widetilde{x} i = \overline{a}_m x$. It follows that

$$\varphi^*(o_m g^m_Q) = \overline{a}_m o_m \in [Q_m, E \wedge \overline{E}^m \wedge U]_0 \tag{2.10}$$

Since $a_m i^U_m f g^m_Q = a_m i^U_m g^m_U = i^U_m k^U_m g^{m+1}_U = 0$, $a_m o_m g^m_Q = a_m \iota^U_m \sigma^U_m i^U_m f g^m_Q - a_m i^U_m f g^m_Q = 0$. Now, (2.10) implies $o_m = 0$, since $a_m \overline{a}_m \neq 0$ and $[Q_m, E\wedge \overline{E}^m \wedge U]_0$ is torsion free.

In particular, we have Q_r, f_r and g^r_Q such that $f_r(g^r_Q) = g^r_U$. Then, $\iota^U_r i^Q_r g^r_Q = i^U_r f_r g^r_Q = i^U_r g^r_U = \iota^U_r \widetilde{\omega}$, which completes the verification of (2.8), since ι^U_r is a monomorphism. □

3. Construction of an ω_2-bouquet

From now on, we work in the $E(2)$-local category at the prime number three. Let $E^*_r(A)$ for a spectrum A denote the E_r-term of $E(2)$-based Adams spectral sequence converging to $\pi_*(L_2A)$. Then the E_2-term $E^{s,t}_2(A)$ is a cohomology of the cobar complex $C^{s,t}(A) = \Omega^{s,t}E(2)_*(A) = \big(E(2)_*(A)\otimes_{E(2)_*} E(2)_*(E(2))^{\otimes s}\big)_t$. We abbreviate $C^{s,t}(S^0)$ to $C^{s,t}$. We have the cochains $x \in C^{2,8}$, $f_0 \in C^{3,0}$ and $f_0 z \in C^{4,0}$ so that

$$d(x) \equiv v_1^2 f_0 \mod (3) \quad \text{and} \quad d(f_0) = 3f_0 z \tag{3.1}$$

by [10, (3.5) and Lemma 3.6]. Furthermore, these cochains x and f_0 yield the generators $\xi \in E^{2,8}_2(V(1))$ and $-v_2^{-1}\psi_0 \in E^{3,0}_2(V(0))$, respectively (for generators, see also [7, 6.3.24. Th.], [8, Th. 5.8] and [9, Th. 2.11]). Here, $V(0)$ and $V(1)$ denote the Smith-Toda spectra. In particular, $V(0)$ is the modulo three Moore spectrum. In order to get information on the E_2-term $E^*_2(S^0)$ from [10], we consider the generalized Greek letter map ([3, p.483])

$$\eta\colon E^{s,t}_2(M^2) \longrightarrow E^{s+2,t}_2(S^0) \tag{3.2}$$

defined by a composite of connecting homomorphisms $\delta\colon E^{s+1,t}_2(N^1) \to E^{s+2,t}_2(S^0)$ and $\delta'\colon E^{s,t}_2(M^2) \to E^{s+1,t}_2(N^1)$ associated to the cofiber sequences

$$L_2S^0 \longrightarrow L_0S^0 \longrightarrow N^1 \quad \text{and} \quad N^1 \longrightarrow L_1N^1 \longrightarrow M^2$$

defining the chromatic spectra N^1 and M^2. The element ω_2 is now defined by $\omega_2 = \eta(\xi\zeta_2/3v_1)$.

We further consider the element $\chi = \eta(\xi/3v_1^2) \in E^{4,0}_2(S^0)$ for $\xi/3v_1^2 \in H^{2,0}M^2_0$ represented by $x/3v_1^2$ given in [10, Prop. 4.7]. The relations in (3.1) show that χ is represented by the cocycle $f_0 z$. We also have the element $h_0 \in E^{1,4}_2(S^0)$ represented by t_1, which detects the element $\alpha_1 \in \pi_3(S^0)$.

Lemma 3.3. *$h_0\chi = \omega_2$ in $E^{5,4}_2(S^0)$.*

Proof. Since $d(v_1) = 3t_1$ in the cobar complex $C^{1,4}$, we obtain $\omega_2 = \eta(\xi\zeta_2/3v_1) = \delta(v_1\chi/3) = h_0\chi$ by (3.1). □

Let Y be the cofiber of the generator $\alpha_1 \in \pi_3(S^0)$ and X the 8-skeleton of BP. Then we have cofiber sequences (*cf.* [8])

$$\begin{array}{c} S^3 \xrightarrow{\alpha_1} S^0 \xrightarrow{\iota} Y \xrightarrow{\kappa} S^4, \quad S^0 \xrightarrow{\iota_1} X \xrightarrow{\kappa_1} \Sigma^4 Y \xrightarrow{\lambda_1} S^1, \quad \text{and} \\ Y \xrightarrow{\iota_2} X \xrightarrow{\kappa_2} S^8 \xrightarrow{\lambda_2} \Sigma Y. \end{array} \tag{3.4}$$

Note that

$$E(2)_*(Y) = E(2)_*[x]/(x^2) \quad \text{and} \quad E(2)_*(X) = E(2)_*[x]/(x^3)$$

for a generator x with $|x| = 4$, with $E(2)_*(E(2))$-comodule structure given by $\psi(x) = x + t_1$. The second and the third cofiber sequences in (3.4) induce short exact sequences of $E(2)_*$-homologies, in which $(\kappa_1)_*(a+bx+cx^2) = b+2cx$ and $(\kappa_2)_*(a+bx+cx^2) = c$. It follows that the connecting homomorphisms $(\lambda_1)_*\colon E_2^{s,t}(Y) \to E_2^{s+1,t+4}(S^0)$ and $(\lambda_2)_*\colon E_2^{s,t}(S^0) \to E_2^{s+1,t+8}(Y)$ act as follows:

$$(\lambda_1)_*([a+xb]) = [t_1|a+\tfrac{1}{2}t_1^2|b] \quad \text{and} \quad (\lambda_2)_*([c]) = [t_1^2|c+2xt_1|c] \tag{3.5}$$

Since the generators $\alpha_1 \in \pi_3(S^0)$ and $\beta_1 \in \pi_{10}(S^0)$ are detected by $h_0 = [t_1] \in E_2^{1,4}(S^0)$ and $b_0 = [t_1|t_1^2 + t_1^2|t_1] \in E_2^{2,12}(S^0)$. It follows that

$$\lambda_1\iota = \alpha_1 \quad \text{and} \quad \lambda_1\lambda_2 = \beta_1. \tag{3.6}$$

We also see that

$$\iota_1 = \iota_2\iota. \tag{3.7}$$

Lemma 3.8. *$E_2^{s,*}(X) = 0$ for $s > 5$. Furthermore, $E_2^{5,4}(X) = 0 = E_2^{5,8}(X)$ and $E_2^{1,4}(X) = 0$.*

Proof. Let M_1X^1 and MX^2 be the spectra defined by the cofiber sequences

$$\begin{gathered} N_1X^0 \to L_1N_1X^0 \to M_1X^1, \quad L_2X \to L_0X \to NX^1 \quad \text{and} \\ NX^1 \to L_1NX^1 \to MX^2 \end{gathered} \tag{3.9}$$

for $N_1X^0 = L_2V(0) \wedge X$. Then, we have a cofiber sequence $M_1X^1 \xrightarrow{\phi} MX^2 \xrightarrow{3} MX^2$, which induces the long exact sequence

$$H^{s,t}M_1^1 \xrightarrow{\phi_*} E_2^{s,t}(MX^2) \xrightarrow{3} E_2^{s,t}(MX^2) \xrightarrow{\delta} H^{s+1,t}M_1^1. \tag{3.10}$$

Here, $H^{s,t}M_1^1$ denotes $E_2^{s,t}(M_1X^1)$ and is determined in [4]. Since $E_2^{s,t}(L_1NX^1) = 0 = E_2^{s,t}(L_0X)$ for $s > 1$, a similar map to (3.2) yields an isomorphism

$$E_2^{s+2,t}(X) = E_2^{s,t}(MX^2) \quad \text{for } s > 1. \tag{3.11}$$

By [4, Th. 10.2], $H^{s,*}M_1^1 = 0$ for $s > 3$, and so is $E_2^{s,*}(MX^2)$ by [3, Remark 3.11] on (3.10), and the first statement of the lemma follows from (3.11).

For $s = 3$, the result $H^{4,*}M_1^1 = 0$ by [4, Th. 10.2] with (3.10) implies that $E_2^{3,*}(MX^2) = E_2^{5,*}(X)$ is a direct sum of $\mathbb{Q}/Z_{(3)}$ obtained by the generators of the image of ϕ_*. In particular, $H^{3,4}M_1^1 = 0$ by [4, Th. 9.3] shows $E_2^{5,4}(X) = 0$.

Suppose that $E_2^{5,8}(X)$ has a summand $\mathbb{Q}/Z_{(3)}$. By [10], neither of $E_2^{5,8}(S^0)$ nor $E_2^{4,-4}(S^0)$ contains $\mathbb{Q}/Z_{(3)}$ as a summand. It follows that $(\kappa_1)_*$ assigns the summand to the one in $E_2^{5,4}(Y)$. Since $E_2^{5,4}(X) = 0$, the summand is pulled back to $E_2^{4,-4}(S^0)$ under $(\lambda_2)_*$, which is a contradiction, and $E_2^{5,8}(X) = 0$.

Since $E_2^{1,*}(L_0X) = 0$, we have an epimorphism $\delta\colon E_2^{0,4}(NX^1) \to E_2^{1,4}(X)$. It is easy to see that $E_2^{0,4}(NX^1) = \mathbb{Q}/\mathbb{Z}_{(3)}$ generated by $\{(v_1 - 3x)/3^i\}$, whose image of the connecting homomorphism is zero. Therefore, $E_2^{1,4}(X) = 0$. □

Corollary 3.12. $\pi_3(L_2X) = 0$.

Lemma 3.13. *$\iota_*(\chi) \in E_2^{4,0}(Y)$ is a permanent cycle.*

Proof. Consider the second cofiber sequence in (3.4). We compute $(\lambda_1)_*\iota_*(\chi) = (\alpha_1)_*(\chi)$ by (3.6), which equals ω_2 by Lemma 3.3. This is a permanent cycle since $\xi\zeta_2/3v_1 \in E_2^{3,0}(M^2)$ is a permanent cycle by [10, Lemma 6.2]. Lemma 3.3 also shows the relation $(\iota_1)_*(\omega_2) = 0$ in the E_2-term. It implies the same relation in homotopy by Lemma 3.8, and the homotopy element ω_2 is pulled back to an element under the map ι_1, which is detected by $\iota_*(\chi)$. □

Since $\pi_{-11}(L_2S^0) = 0$ by [10], the last cofiber sequence of (3.4) induces a monomorphism $\pi_{-4}(L_2Y) \xrightarrow{(\iota_2)_*} \pi_{-4}(L_2X)$. Lemma 3.13 together with (3.7) implies the following

Lemma 3.14. *$(\iota_1)_*(\chi) \in E_2^{4,0}(X)$ survives to a essential element $\widehat{\chi} \in \pi_{-4}(L_2X)$.*

Lemma 3.15. *There is an element $\overline{\chi} \in [X, L_2X]_{-4}$ such that $(\iota_1)^*(\overline{\chi}) = \widehat{\chi}$ and $E(2)_*(\overline{\chi}) = 0$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc}
\pi_0(L_2X) & \xrightarrow{\kappa^*} & [Y, L_2X]_{-4} & \xrightarrow{\iota^*} & \pi_{-4}(L_2X) & \xrightarrow{\alpha_1^*} & \pi_{-1}(L_2X) \\
{\scriptstyle i_*}\downarrow & & {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} & & \\
E(2)_0(X) & \xrightarrow{\kappa^*} & [Y, E(2) \wedge X]_{-4} & \xrightarrow{\iota^*} & E(2)_{-4}(X) & &
\end{array}$$

of exact sequences, in which the lower sequence is a split short exact one. Since $\widehat{\chi}$ is detected by an element of $E_2^{4,0}(X)$, $\alpha_1^*(\widehat{\chi})$ is detected by an element of $E_2^{4s+1,4s}(X)$ for $s \geq 1$, which is zero by Lemma 3.8. It follows that $\alpha_1^*(\widehat{\chi}) = 0$, and $\widehat{\chi}$ is pulled back to an element $\chi' \in [Y, L_2X]_{-4}$. If $i_*(\chi') \neq 0 \in [Y, E(2) \wedge X]_{-4}$, we have an element $\theta \in E(2)_0(X)$ such that $\theta\kappa = i_*(\chi')$, since $\iota^* i_*(\chi') = i_*(\widehat{\chi}) = 0$. Consider the Adams spectral sequences obtained by applying $\pi_*(-)$ and $[Y, -]_*$ to the cofiber sequences (2.1). Since $d_1(i_*(\chi')) = 0$, $d_1(\theta) = 0$, and so $\theta \in E_2^{0,0}(X)$. By Lemma 3.8, $d_5(\theta) \in E_2^{5,4}(X) = 0$, and so $j_*(\theta) = 0$ and we have an element $\theta' \in \pi_0(L_2X)$ such that $i_*(\theta') = \theta$. Now replace χ' by $\chi' - \theta'\kappa$, and we see that $i_*(\chi') = 0$.

We play the same game in the diagram

$$\begin{array}{ccccccc}
\pi_4(L_2X) & \xrightarrow{\kappa_2^*} & [X, L_2X]_{-4} & \xrightarrow{\iota_2^*} & [Y, L_2X]_{-4} & \xrightarrow{\lambda_2^*} & \pi_3(L_2X) \\
{\scriptstyle i_*}\downarrow & & {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle i_*} & & \\
E(2)_4(X) & \xrightarrow{\kappa_2^*} & [X, E(2) \wedge X]_{-4} & \xrightarrow{\iota_2^*} & [Y, E(2) \wedge X]_{-4} & &
\end{array}$$

By Corollary 3.12, χ' is pulled back to $\overline{\chi} \in [X, L_2X]_{-4}$. Since $E_2^{5,8}(X) = 0$, a similar argument shows that $i_*(\overline{\chi}) = 0$.

Now $\iota_1^*(\overline{\chi}) = \iota^*\iota_2^*(\overline{\chi}) = \iota^*(\chi') = \widehat{\chi}$, and $E(2) \wedge \overline{\chi} = (\mu \wedge X)(E(2) \wedge i_*(\overline{\chi})) = 0$, as desired. □

The spectra Y and X are introduced by Ravenel to define the small descent spectral sequence, and we here consider its first differential

$$\partial_1 = \iota_2\kappa_1. \tag{3.16}$$

Let W denote the cofiber of $\beta_1 \in \pi_{10}(S^0)$. Then, we have cofiber sequences

$$S^{10} \xrightarrow{\beta_1} S^0 \xrightarrow{\iota_W} W \xrightarrow{\kappa_W} S^{11} \quad \text{and} \quad W \xrightarrow{i_W} X \xrightarrow{\partial_1} \Sigma^4 X \xrightarrow{k_W} \Sigma W \tag{3.17}$$

so that

$$k_W\iota_2 = \iota_W\lambda_1 \in [Y, W]_4. \tag{3.18}$$

Lemma 3.19. *In $E_2^5(W)$, $(k_W)_*((\iota_1)_*(\chi)) = (\iota_W)_*(\omega_2)$.*

Proof. We compute $(k_W)_*((\iota_1)_*(\chi)) = (k_W\iota_2)_*\iota_*(\chi) = (\iota_W)_*(\lambda_1\iota)_*\chi = (\iota_W)_*(\alpha_1\chi) = (\iota_W)_*(\omega_2)$ by (3.7), (3.18), (3.6) and Lemma 3.3. □

Consider the fiber U of the map $\partial_1 + \overline{\chi} \in [X, X]_{-4}$ for the element $\overline{\chi}$ of Lemma 3.15, which fits into the cofiber sequence

$$U \xrightarrow{i_U} X \xrightarrow{\partial_1 + \overline{\chi}} \Sigma^4 X \xrightarrow{k_U} \Sigma U. \tag{3.20}$$

Then, $E(2) \wedge (\partial_1 + \overline{\chi}) = E(2) \wedge \partial_1$ by Lemma 3.15, and we see the following

Lemma 3.21. *$E(2) \wedge U = E(2) \wedge W$. In particular $E(2) \wedge U = E(2) \vee \Sigma^{11} E(2)$.*

Proof of Proposition 1.9. Lemma 3.21 shows that U is an $E(2)$-bouquet.

For the generator ι_1 of $\pi_0(X)$, $(\partial_1)_*(\iota_1) = 0$ by (3.4) and (3.16), and $\overline{\chi}_*(\iota_1) = (\iota_1)^*(\overline{\chi}) = \widehat{\chi}$ by Lemma 3.15. It follows that $(\partial_1 + \overline{\chi})_*(\iota_1) = \widehat{\chi} \in \pi_{-4}(X)$, and then $(k_U)_*(\widehat{\chi}) = 0 \in \pi_{-1}(U)$. On the other hand, $\widehat{\chi}$ is detected by $(\iota_1)_*(\chi)$ by Lemma 3.14, and so is $(k_U)_*(\widehat{\chi})$ by $(k_U)_*((\iota_1)_*(\chi)) \in E_2^5(U)$, which equals $(\iota_W)_*(\omega_2) = \omega_2 g$ by Lemma 3.19 and Lemma 3.21. Therefore, $\omega_2 g \in E_2^5(U)$ is killed, and the killer belongs to $E_5^{0,0}(U) = \mathbb{Z}_{(3)}$. This shows the condition (1.7) for ω_2 as desired. □

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