THE BETA ELEMENTS $\beta_{tp^2/r}$ IN THE HOMOTOPY OF SPHERES

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Abstract. In [1], Miller, Ravenel and Wilson defined generalized beta elements in the E_2-term of the Adams-Novikov spectral sequence converging to the stable homotopy groups of spheres, and in [5], Oka showed that the beta elements of the form $\beta_{tp^2/r}$ for positive integers t and r survives to the stable homotopy groups at a prime $p > 3$, when $r \le 2p-2$ and $r \le 2p$ if $t > 1$. In this paper, we expand the condition so that $\beta_{tp^2/r}$ for $r \le p^2 - 3$ survives to the stable homotopy groups.

1. Introduction

Let BP be the Brown-Peterson spectrum at a prime p, and consider the Adams-Novikov spectral sequence converging to homotopy groups $\pi_*(X)$ of a spectrum X with E_2-term $E_2^{s,t}(X) = \mathrm{Ext}^{s,t}_{BP_*(BP)}(BP_*, BP_*(X))$. Here,

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots] \quad \text{and} \quad BP_*(BP) = BP_*[t_1, t_2, \dots]$$

for $v_i \in BP_{2p^i-2}$ and $t_i \in BP_{2p^i-2}(BP)$. In [1], Miller, Ravenel and Wilson defined generalized Greek letter elements in the E_2-term of the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(S^0)$ of the sphere spectrum S^0 at each prime p. For the beta elements, we consider the mod p Moore spectrum M and finite spectra V_a for $a > 0$ defined by the cofiber sequences

$$S^0 \xrightarrow{p} S^0 \xrightarrow{i} M \xrightarrow{j} S^1 \quad \text{and} \quad \Sigma^{aq} M \xrightarrow{\alpha^a} M \xrightarrow{i_a} V_a \xrightarrow{j_a} \Sigma^{aq+1} M, \tag{1.1}$$

where $p \in \pi_0(S^0) = \mathbb{Z}_{(p)}$, $\alpha \in [M, M]_q$ is the Adams map, and

$$q = 2p - 2.$$

Since j and j_a induces trivial maps on the BP_*-homologies, these cofiber sequences yield short exact sequences

$$\begin{gathered} 0 \longrightarrow BP_* \xrightarrow{p} BP_* \xrightarrow{i_*} BP_*/(p) \longrightarrow 0 \quad \text{and} \\ 0 \longrightarrow BP_*/(p) \xrightarrow{v_1^a} BP_*/(p) \xrightarrow{i_{a*}} BP_*/(p, v_1^a) \longrightarrow 0, \end{gathered} \tag{1.2}$$

where

$$BP_*(M) = BP_*/(p) \quad \text{and} \quad BP_*(V_a) = BP_*/(p, v_1^a). \tag{1.3}$$

The beta elements are now defined by

$$\beta_{s/a-b} = \delta\delta_a(v_1^b v_2^s) \in E_2^{2,(sp+s-a+b)q}(S^0) \tag{1.4}$$

for $s > 0$ and $a > b \ge 0$, if $v_1^b v_2^s \in E_2^{0,(sp+s+b)q}(V_a)$, where δ and δ_a are the connecting homomorphisms associated to the short exact sequences (1.2). We abbreviate $\beta_{s/1}$ to β_s as usual. Now assume that the prime p is greater than three.

Then, L. Smith [8] showed that every β_s for $s > 0$ survives to a homotopy element $\beta_s \in \pi_{(sp+s-1)q-2}(S^0)$, and S. Oka showed the following beta elements survivors:

$$\begin{array}{ll} \beta_{tp/r} & \text{for } t > 0 \text{ and } r \le p \text{ with } (t,r) \ne (1,p) \text{ in [3], [4],} \\ \beta_{tp^2/r} & \text{for } t > 0 \text{ and } r \le 2p-2 \text{ in [3], and} \\ \beta_{tp^2/r} & \text{for } t > 1 \text{ and } r \le 2p \text{ in [5].} \end{array}$$

Let W denote the cofiber of the beta element $\beta_1 \in \pi_{pq-2}(S^0)$, and we have a cofiber sequence

$$S^{pq-2} \xrightarrow{\beta_1} S^0 \xrightarrow{i_W} W \xrightarrow{j_W} S^{pq-1}. \tag{1.5}$$

Then, $E_2^{s,tq}(W \wedge V_a) = E_2^{s,tq}(V_a)$. In [7], we show that

Theorem 1.6. ([7]) *Suppose that $v_2^s \in E_2^{0,s(p+1)q}(W \wedge V_a)$. If the element v_2^s survives to $\pi_*(W \wedge V_a)$, then $\beta_{st/r}$ for $t > 0$ and $0 < r < a-1$ survives to $\pi_*(S^0)$.*

A proof of the theorem is outlined as follows: There is an element $\widetilde{\alpha} \in \pi_{(p+2)q-1}$ $(W \wedge V_{r+2})$ such that $(j_W)_*(\widetilde{\alpha}) = i_{r+2}\alpha^2 i \in \pi_{2q}(V_{r+2})$, since $\alpha^2 i\beta_1 = 0$. Consider a map $p_{r+2}\colon V_a \to V_{r+2}$ that induces the projection $BP_*/(p, v_1^a) \to BP_*/(p, v_1^{r+2})$ on the BP_*-homology, and write $v_2^s = (p_{r+2})_*(v_2^s) \in \pi_*(W \wedge V_{r+2})$. Since $W \wedge V_{r+2}$ is a ring spectrum by [2], we have an element $(v_2^s)^t\widetilde{\alpha} \in \pi_*(W \wedge V_{r+2})$ so that $(j_W)_*((v_2^s)^t\widetilde{\alpha}) \in \pi_*(V_{r+2})$ is detected by $v_1^2 v_2^{st} \in E_2^0(V_{r+2})$. Now the theorem follows from the definition (1.4) and the Geometric Boundary Theorem (*cf.* [6]).

In this paper, we show the following theorem:

Theorem 1.7. *Let p be a prime greater than three. Then, the element $v_2^{p^2} \in E_2^0(W \wedge V_{p^2-1})$ is a permanent cycle.*

Corollary 1.8. *Let p be a prime greater than three. Then, the beta elements $\beta_{tp^2/r} \in E_2^{2,(tp^2(p+1)-r)q}(S^0)$ for $t > 0$ and $0 < r < p^2-2$ are permanent cycles.*

2. Adams-Novikov E_2-terms

Ravenel constructed a ring spectrum $T(m)$ for each integer $m \ge 0$ characterized by $BP_*(T(m)) = BP_*[t_1, \dots, t_m]$ (*cf.* [6]). He then showed the change of rings theorem $E_2^{s,t}(T(m) \wedge X) = \operatorname{Ext}^{s,t}_{\Gamma(m+1)}(BP_*, BP_*(X))$ for the Hopf algebroid $\Gamma(m+1) = BP_*(BP)/(t_1, \dots, t_m)$, and determine $E_2^{s,t}(T(m))$ in [6, Th. 7.2.6, Cor. 7.2.7] below dimension $2(p^{m+3} - p^2)$. In particular, below dimension $(p^3+p^2)q$,

$$\begin{array}{rcl} E_2^{0,*}(T(1)) & = & \mathbb{Z}_{(p)}[v_1] \\ E_2^{1,*}(T(1)) & = & k(1)_*\{v_2^s h_{20} : p \nmid s \ge 0\} \oplus h_{20}\mathbb{Z}/p^2[v_1, v_2^p] \\ \bigoplus_{s\ge 2} E_2^{s,t}(T(1)) & = & k(2)_*\{v_3^s b_{20} : s \ge 0\} \otimes E(h_{20}) \otimes P(b_{20}). \end{array} \tag{2.1}$$

Here, E and P denote an exterior and a polynomial algebras over $\mathbb{Z}/p$ and

$$k(m)_* = \mathbb{Z}/p[v_m] \tag{2.2}$$

denotes the BP_*-algebra with trivial v_i-action for $i \ne m$. Consider the connecting homomorphism $\delta\colon E_2^{s,t}(T(1) \wedge M) \to E_2^{s+1,t}(T(1))$ associated to the first cofiber sequence in (1.1). We then see that there are elements v_2^s and $v_3^s h_{21}$ in $E_2^*(T(1) \wedge M)$

such that $\delta(v_2^s) = sv_2^{s-1}h_{20}$ and $\delta(v_3^s h_{21}) = v_3^s b_{20}$. Since $p = 0\colon E_2^{s,t}(T(1)) \to E_2^{s,t}(T(1))$ for $s > 0$, we obtain

$$
\begin{array}{rcl}
E_2^{0,*}(T(1)\wedge M) & = & k(1)_*[v_2] \\
E_2^{1,*}(T(1)\wedge M) & = & h_{20}k(1)_*[v_2] \oplus h_{21}k(2)_*[v_3] \\
\bigoplus\limits_{s\geq 2} E_2^{s,*}(T(1)\wedge M) & = & b_{20}k(2)_*[v_3]\otimes E(h_{20},h_{21})\otimes P(b_{20})
\end{array}
\tag{2.3}
$$

below $(p^3+p^2)q$. Here, the generators have the bidegrees as follows:

$$
|v_2| = (0,(p+1)q), \quad |v_3| = (0,(p^2+p+1)q), \quad |h_{20}| = (1,(p+1)q), \\
|h_{21}| = (1,(p^2+p)q) \quad \text{and} \quad |b_{20}| = (2,(p^2+p)q).
$$

Consider a spectrum X_k constructed by Ravenel [6], in which X_k is denoted by $T(0)_{(k)}$, with BP_*-homology $BP_*(X_k) = BP_*[t_1]/(t_1^{p^k})$ as a $BP_*(BP)$-comodule. We abbreviate X_1 to X. Then, we have a diagram

$$
\begin{tikzcd}
X_{k-1} \arrow[d, "\iota_k"'] & \Sigma^{p^{k-1}q}\overline{X}_k \arrow[l, dashed, "\lambda_k"'] \arrow[d, "\iota'_k"'] & X_{k-1} \arrow[l, dashed, "\lambda'_k"'] \\
X_k \arrow[ur, "\kappa_k"'] & X_k \arrow[ur, "\kappa'_k"'] &
\end{tikzcd}
\tag{2.4}
$$

in which each triangle is a cofiber sequence with inclusion ι_k or ι'_k. Since λ_k and λ'_k induce the zero maps on BP_*-homologies, applying the Adams-Novikov E_2-terms $E_2^*(-\wedge M)$ to the diagram gives rise to an exact couple that defines the small descent spectral sequence:

$$
\begin{array}{c}
{}^{SD}E_1^* = E_2^*(X_k\wedge M)\otimes E(h_{k-1})\otimes P(b_{k-1}) \\
\Longrightarrow E_2^*(X_{k-1}\wedge M),
\end{array}
\tag{2.5}
$$

where $h_{k-1}\in {}^{SD}E_1^{1,0,p^{k-1}q}$ and $b_{k-1}\in {}^{SD}E_1^{2,0,p^kq}$ are represented by the cocycles $t_1^{p^{k-1}}$ and $y_{k-1} = \sum_{k=1}^{p-1}\frac{1}{p}\binom{p}{k}t_1^{kp^{k-1}}\otimes t_1^{(p-k)p^{k-1}}$ of the cobar complex

$$\Omega^* = \Omega^*_{BP_*(BP)}BP_*/(p)$$

for computing $E_2^*(M)$, respectively. Let $\delta_k\colon E_2^{s,t}(\overline{X}_k\wedge M)\to E_2^{s+1,t+p^{k-1}q}(X_{k-1}\wedge M)$ and $\delta'_k\colon E_2^{s,t}(X_{k-1}\wedge M)\to E_2^{s+1,t+p^{k-1}(p-1)q}(\overline{X}_k\wedge M)$ denote the connecting homomorphisms corresponding λ_k and λ'_k. Then,

$$\delta_k\delta'_k(x) = b_{k-1}x \in E_2^{s+2,t+p^kq}(X_{k-1}\wedge M) \tag{2.6}$$

for $x\in E_2^{s,t}(X_{k-1}\wedge M)$. We state here a relation in the E_2-term $E_2^*(M)$:

Lemma 2.7. *In the Adams-Novikov E_2-term $E_2^2(M)$, $v_1^pb_1^2 = 0$.*

Proof. We define the element $y_{20}\in\Omega^2$ by $d(t_2^p) = -t_1^p\otimes t_1^{p^2} + v_1^py_1 + py_{20}$ in the cobar complex $\Omega^2_{BP_*(BP)}BP_*/(p^2)$. Since t_1 is primitive, $d(t_1^{p^{j+1}}) = -py_j \in \Omega^2_{BP_*(BP)}BP_*/(p^2)$, and so we obtain

$$d(y_j) = 0 \quad \text{and} \quad d(y_{20}) = -y_0\otimes t_1^{p^2} + t_1^p\otimes y_1 \in \Omega^3. \tag{2.8}$$

We also see that $d(\Delta(t)y) = y\otimes t - t\otimes y$ for cocycles $t\in\Omega^1$ and $y\in\Omega^2$. Then, the cocycle $v_1^py_1\otimes y_1\in\Omega^4$ cobounds the cochain $t_2^p\otimes y_1 - t_1^p\otimes\Delta(t_1^{p^2})y_1 + y_{20}\otimes t_1^{p^2} + \frac{1}{2}y_0\otimes t_1^{2p^2}\in\Omega^3$. $\square$

Lemma 2.9. *The Adams-Novikov E_2-term $E_2^{s,t}(X_2 \wedge M)$ for $t < (p^3 + p^2)q$ is a subquotient of the direct sum of the modules K_i for $0 \le i \le 3$ given by*

$$\begin{array}{rcl} K_0 & = & h_2 k(2)_*[v_1]/(v_1^p) \otimes E(h_{20}) \\ K_1 & = & k(1)_*[v_2]/(v_2^p) \otimes E(h_{20}, h_3, b_2) \\ K_2 & = & h_{21}\,(k(2)_* \oplus h_2 k(3)_*) \\ K_3 & = & b_{20}\,(k(2)_* \oplus h_2 k(3)_*) \otimes E(h_{20}, h_{21}) \otimes P(b_{20}) \end{array}$$

Proof. We begin with $E_2^*(X_4 \wedge M) = E_2^*(T(1) \wedge M)$. The spectral sequence (2.5) collapses for $k = 4$, and $E_2^*(X_3 \wedge M) = E_2^*(T(1) \wedge M) \otimes E(h_3) = \bigoplus_{i=1}^3 \widetilde{K}_i$ for

$$\begin{array}{rcl} \widetilde{K}_1 & = & k(1)_*[v_2] \otimes E(h_{20}, h_3) \\ \widetilde{K}_2 & = & h_{21} k(2)_*[v_3] \\ \widetilde{K}_3 & = & b_{20} k(2)_*[v_3] \otimes E(h_{20}, h_{21}) \otimes P(b_{20}) \end{array}$$

by (2.3). In the spectral sequence (2.5) for $k = 3$, we see that $d_1(v_2^p) = v_1^p h_2$, $d_1(v_3^s h_{21}) = s v_2 v_3^{s-1} h_{21} h_2$ and $d_1(v_3^s b_{20}) = s v_2 v_3^{s-1} b_{20} h_2$ up to sign. Under these relations, the d_1-homology is computed to be:

$$\begin{array}{rcl} H^*(k(1)_*[v_2] \otimes E(h_2) : d_1) & = & k(1)_*[v_2]/(v_2^p) \oplus h_2 k(2)_*[v_1]/(v_1^p) \quad \text{and} \\ H^*(k(2)_*[v_3] \otimes E(h_2) : d_1) & = & k(2)_* \oplus h_2 k(3)_*. \end{array}$$

The module $\bigoplus_{i=0}^3 K_i$ is the homology of the complex $E_2^*(X_3 \wedge M) \otimes E(h_2, b_2)$ with the differential given by the above d_1, whose subquotient is the E_2-term ${}^{SD}E_2^*$ of the spectral sequence (2.5) for $k = 3$. □

By the small descent spectral sequence (2.5) for $k = 2$, the module $E_2^{s,tq}(X \wedge M)$ is a subquotient of $\bigoplus_{i=0}^3 L_i^*$ for

$$L_i^* = K_i^* \otimes E(h_1) \otimes P(b_1). \tag{2.10}$$

These modules have the vanishing lines:

$$\begin{array}{rcl} L_0^{2s+e,tq} & = & 0 \quad \text{if } t < sp^2 + p + e(p+1) \text{ and } s > 0. \\ L_1^{2s+1+e,tq} & = & 0 \quad \text{if } t < sp^2 + p + e(p+1). \\ L_2^{2s+e,tq} & = & 0 \quad \text{if } t < sp^2 + 2p + e(p+1) \text{ and } s > 0. \\ L_3^{2s+1+e,tq} & = & 0 \quad \text{if } t < sp^2 + 2p + e(p+1) \text{ and } s > 0. \end{array}$$

Therefore, we have the following

Lemma 2.11. $E_2^{2s+1+e,tq}(X \wedge M) = 0$ *if* $t < sp^2 + p + e(p+1)$.

Lemma 2.12. *Every element of $L_1^{2s+1+e,tq}$ is a multiple of $v_1^p b_1^2$ if $s \ge 2$ and $t \ge (s+1)p^2 + 2p + e(p^2 - p - 1)$.*

Proof. By the assumption $s \ge 2$ and $t < p^3 + p^2$, there is no element of $L_1^{2s+1+e,tq}$ originating from $k(1)_*[v_2]/(v_2^p) \otimes E(h_{20}) \otimes \mathbb{Z}/p\{h_3, b_2\}$. So every element of it has the form $v_1^a v_2^b h_{20}^e h_1 b_1^s$ or $v_1^a v_2^b h_{20}^{1-e} b_1^{s+e}$. The condition on s and t shows $a \ge p$, since $b < p$. □

Corollary 2.13. *For an element $x \in E_2^{2s+1+e,tq}(X \wedge M)$ for $s \ge 2$ and $t \ge (s+1)p^2 + 2p - 1 + e(p^2 - p - 1)$, $v_1 x$ is detected by an element of $v_1 L_0^*$. In particular, $v_1^p x = 0$.*

Proof. By Lemma 2.9, we see that $v_1 x$ belongs to the submodule of $E_2^{s,tq}(X \wedge M)$ originating from $v_1 L_0^* \oplus v_1 L_1^*$. Every element originating from $v_1 L_1^*$ satisfying the condition is zero by Lemmas 2.7 and 2.12. We see $v_1^p x = 0$, since $v_1^p L_0^* = 0$. □

Lemma 2.14. *Put $u = p^3 + p^2 - 2p + 2$ and $c = \max\{1, p-8\}$. Then, for $x \in E_2^{q+1,(u+1)q}(X \wedge M)$, $v_1^c x = 0$.*

Proof. By Corollary 2.13, $v_1 x$ is detected by a linear combination of elements $v_1^9 v_2^{2p-6} h_2 h_{20} h_1 b_1^{p-2}$ and $v_1^7 v_2^{p-3} h_2 b_1^{p-1}$ by degree reason. This is zero in L_0^* if $p \le 7$. Otherwise, $v_1^{p-8}x = v_1^{p-7}(v_1 x)$ is a multiple of $v_1^p b_1^2$. Thus $v_1^{p-8}x$ dies in $E_2^{q+1,(u+p-7)q}(X \wedge M)$. □

3. The vanishing line of the Adams-Novikov E_3-term of $W \wedge M$

Consider the cofiber sequence (1.5). Then, it induces a short exact sequence $0 \to E_2^{s,t}(M) \to E_2^{s,t}(W \wedge M) \to E_2^{s,t+1-pq}(M) \to 0$ of E_2-terms, and the cohomologies E_3^* of the complexes (E_2^*, d_2) yield the long exact sequence

$$\begin{aligned} 0 \longrightarrow E_3^{s,(t+p)q-1}(W \wedge M) &\xrightarrow{(j_W)_*} E_2^{s,tq}(M) \xrightarrow{b_0} \\ E_2^{s+2,(t+p)q}(M) &\xrightarrow{(i_W)_*} E_3^{s+2,(t+p)q}(W \wedge M) \longrightarrow 0 \end{aligned} \tag{3.1}$$

Lemma 3.2. *The map $b_0 \colon E_2^{2s+1+e,tq}(M) \to E_2^{2s+3+e,(t+p)q}(M)$ is epimorphic if $t \le sp^2 + p + 1 + e(p^2 - p - 1)$.*

Proof. We apply $E_2^*(-)$ to the diagram (2.4) for $k = 1$. By (2.6), the map b_0 is a composite of the connecting homomorphisms δ_1 and δ_1' corresponding to λ_1 and λ_1'. The condition on the integers imply inequalities $t+p \le sp^2+2p+1+e(p^2-p-1) < (s+1)p^2+p+e(p+1)$ and $t+p-1 \le sp^2+2p+e(p^2-p-1) < \begin{cases} sp^2+2p+1 & e=0 \\ (s+1)p^2+p & e=1 \end{cases}$.
It follows that $E_2^{2s+3+e,(t+p)q}(X \wedge M) = 0$ and $E_2^{2s+2+e,(t+p-1)q}(X \wedge M) = 0$ by Lemma 2.11, and so the connecting homomorphisms are epimorphisms. □

Apply this to the exact sequence (3.1), and we obtain the following

Corollary 3.3. $E_3^{2s+1+e,tq}(W \wedge M) = 0$ *if* $t \le (s-1)p^2 + 2p + 1 + e(p^2 - p - 1)$.

Remark. In the same way as the proof of Lemma 3.2, we show that the map $b_0 \colon E_2^{2s+1+e,tq}(M) \to E_2^{2s+3+e,(t+p)q}(M)$ is monomorphic if $t < sp^2 + 1 + e(p+1)$. It follows that the map $b_0 \colon E_2^{2s+e,tq}(M) \to E_2^{2s+2+e,(t+p)q}(M)$ is an isomorphism if $t < sp^2 + 1 + e(p+1)$.

Lemma 3.4. *Put $u = p^3 + p^2 - 2p + 2$ and suppose that $\xi \in \pi_{uq-1}(W \wedge M)$ is detected by an element of $E_2^{q+1,(u+1)q}(W \wedge M)$. Then, $\alpha^{2p-2}\xi = 0$.*

Proof. We here also work on the diagram (2.4) for $k = 1$ applied $E_2^*(-)$. Consider an element $x \in E_2^{q+1,(u+1)q}(M)$, which is isomorphic to $E_2^{q+1,(u+1)q}(W \wedge M)$. Then, $v_1^c(\iota_1)_*(x) = 0 \in E_2^{q+1,(u+c+1)q}(X \wedge M)$ by Lemma 2.14, and there is an element $x_1 \in E_2^{q,(u+c)q}(\overline{X} \wedge M)$ such that $(\delta_1)_*(x_1) = v_1^c x$. We also see that $v_1^p(\iota_1')_*(x_1) = 0$ by Corollary 2.13, and obtain an element x_2 such that $(\delta_1')_*(x_2) = v_1^p x_1$. It follows that $v_1^{p+c}x = v_1^p(\delta_1)_*(x_1) = (\delta_1)_*(\delta_1')_*(x_2) = b_0 x_2$ by (2.6), and $v_1^{p+c}x = 0 \in E_3^{q+1,(u+p+c+1)q}(W \wedge M)$ by (3.1).

If $s > 0$, then the E_3-term $E_3^{(s+1)q+1,(u+s+1)q}(W \wedge M)$ is zero by Corollary 3.3, since $u + s + 1 = p^3 + p^2 - 2p + 3 + s < (s+1)(p-1)p^2 + p$. Therefore, the relation $v_1^{p+c}x = 0$ in the E_3-term holds in the homotopy. Since $p + c \le 2p - 2$, the lemma follows. □

4. $\beta'_{p^2/p^2-1} \in \pi_{(p^3+1)q-1}(W \wedge M)$

Apply the Adams-Novikov E_2-term $E_2^*(-)$ to the diagram (2.4) for $k = 2$, and we have a spectral sequence ${}^{SD}E_1 = E_2^*(X_2) \otimes E(h_1) \otimes P(b_1) \Rightarrow E_2^*(X)$. Since $E_2^{2s+e,tq}(X_2) = 0$ if $t < s(p^2+p) + e(p+1)$, we obtain the vanishing line:

Lemma 4.1. $E_2^{2s+1+e,tq}(X) = 0$ *if* $t < sp^2 + p + e(p+1)$.

In the same manner as the proof of Corollary 3.3, we obtain

Corollary 4.2. $E_3^{2s+e,tq}(W) = 0$ *if* $t < (s-1)p^2 + (1+e)(p+1)$.

The E_2-terms $E_2^{s,t}(X)$ for $t < (p^3+p)q$ are determined by Ravenel [6, ABC Th. 7.5.1]. In particular, we read off the following:

(4.3)
1) $E_2^{rq+1,(p^3+1+r)q}(X) = 0$ for $r > 0$.
2) $E_2^{q+2,(p^3+1)q}(X) = 0$.

Lemma 4.4. $\beta'_{p^2/p^2-1} \in \pi_{(p^3+1)q-1}(W \wedge M)$ *exists.*

Proof. By (4.3) 1), the element β_{p^2/p^2-1} of $E_2^{2,(p^3+1)q}(X)$ survives to an element of the homotopy group $\pi_{(p^3+1)q-2}(X)$. Apply the 3×3 Lemma on the cofiber sequences in (1.5) and (2.4), and we obtain the cofiber sequence

$$W \xrightarrow{\eta} X \xrightarrow{d_1} \Sigma^q X \xrightarrow{\overline{\eta}} \Sigma W,$$

where $\kappa_1\eta = \lambda'_1 j_W$, $d_1 = \iota'_1\kappa_1$ and $\kappa'_1 = j_W\overline{\eta}$. Since $\beta_{p^2/p^2-1} \in E_2^{2,(p^3+1)q}(W) = E_2^{2,(p^3+1)q}(S^0)$, the induced map $(d_1)_*\colon \pi_*(X) \to \pi_*(X)$ assigns β_{p^2/p^2-1} to an element detected by $E_2^{q+2,(p^3+1)q}(X)$, which is zero by (4.3) 2). Thus $\beta_{p^2/p^2-1} \in \pi_*(X)$ is pulled back to $\beta_{p^2/p^2-1} \in \pi_*(W)$.

Since $E_2^{s,tq}(W) = E_2^{s,tq}(S^0)$, $p\beta_{p^2/p^2-1} = 0$ in the E_2-term $E_2^{2,(p^3-1)q}(W)$. It follows that $p\beta_{p^2/p^2-1}$ is detected by an element of $E_2^{rq+2,(p^3-1+r)q}(W)$ for $r > 0$. By Corollary 4.2, the module $E_2^{rq+2,(p^3+1+r)q}(W)$ is zero, and the relation $p\beta_{p^2/p^2-1} = 0$ holds in homotopy. Hence, $\beta_{p^2/p^2-1} \in \pi_*(W)$ is pulled back to $\beta'_{p^2/p^2-1} \in \pi_*(W \wedge M)$ under the induced map $j_*\colon \pi_*(W \wedge M) \to \pi_*(W)$ from j in (1.1). □

Lemma 4.5. $\alpha^{p^2-1}\beta'_{p^2/p^2-1} = 0 \in \pi_{(p^3+p^2)q-1}(W \wedge M)$.

Proof. Oka [3] constructed the beta element $\beta'_{p^2/2p-2} \in \pi_{uq-1}(M)$ such that $\alpha^{2p-2}\beta'_{p^2/2p-2} = 0$ in homotopy and $\beta'_{p^2/2p-2} = v_1^{p^2-2p+1}\beta'_{p^2/p^2-1}$ in the E_2-term. Here, $u = p^3 + p^2 - 2p + 2$ as above. Consider an element $\xi = \alpha^{p^2-2p+1}\beta'_{p^2/p^2-1} - (i_W)_*(\beta'_{p^2/2p-2}) \in \pi_{uq-1}(W \wedge M)$. Then, it satisfies the condition of Lemma 3.4, and we have $\alpha^{p^2-1}\beta'_{p^2/p^2-1} = \alpha^{2p-2}(\xi + (i_W)_*(\beta'_{p^2/2p-2})) = 0$ as desired. □

Proof of Theorem 1.7. Consider the second cofiber sequence (1.1) for $a = p^2 - 1$. Then, by Lemma 4.5, we have an element $v \in \pi_*(W \wedge V_{p^2-1})$ such that $(j_{p^2-1})_*(v) = \beta'_{p^2/p^2-1}$. Since v is detected by an element of $E_2^0(W \wedge V_{p^2-1})$, we see that $v = v_2^{p^2}$ by degree reason. □

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