NOTE ON BETA ELEMENTS IN HOMOTOPY, AND AN APPLICATION TO THE PRIME THREE CASE

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Abstract. Let $S^0_{(p)}$ denote the sphere spectrum localized at an odd prime p. Then we have the first beta element $\beta_1 \in \pi_{2p^2-2p-2}(S^0_{(p)})$, whose cofiber we denote by W. We also consider a generalized Smith-Toda spectrum V_r characterized by $BP_*(V_r) = BP_*/(p, v_1^r)$. In this note, we show that an element of $\pi_*(V_r \wedge W)$ gives rise to a beta element of homotopy groups of spheres. As an application, we show the existence of β_{9t+3} at the prime three to complete a conjecture of Ravenel's: $\beta_s \in \pi_{16s-6}(S^0_{(3)})$ exists if and only if s is not congruent to 4, 7 or 8 mod 9.

1. Introduction and statements of results

Let BP denote the Brown-Peterson spectrum at an odd prime number p. Then, we have the Hopf algebroid BP_*BP over BP_*, where

$$BP_* = \pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \dots] \quad \text{and} \quad BP_*BP = BP_*(BP) = BP_*[t_1, t_2, \dots]$$

for $v_i \in BP_{2p^i-2}$ and $t_i \in BP_{2p^i-2}BP$. It gives rise to the Adams-Novikov spectral sequence converging to homotopy groups $\pi_*(X)$ of a connective spectrum X with E_2-term

$$E_2^{s,t}(X) = \operatorname{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*(X)).$$

We consider the sphere spectrum S^0, the modulo p Moore spectrum M and a cofiber V_r of the map $\alpha^r : \Sigma^{rq}M \to M$ for a positive integer r and the Adams map $\alpha : \Sigma^q M \to M$, so that

$$S^0 \xrightarrow{p} S^0 \xrightarrow{i} M \xrightarrow{j} S^1 \quad \text{and} \quad \Sigma^{rq}M \xrightarrow{\alpha^r} M \xrightarrow{i_r} V_r \xrightarrow{j_r} \Sigma^{rq+1}M \tag{1.1}$$

are cofiber sequences. Here $q = 2p-2$ as usual. Suppose that $v_1^t v_2^s \in E_2^{0,(sp+s+t)q}(V_r)$ for integers $r > 0$, $s > 0$ and $t \geq 0$. Then, we define the beta element $\beta_{s/r-t}$ in the E_2-term by

$$\beta_{s/r-t} = \delta\delta_r(v_1^t v_2^s) \in E_2^{2,(sp+s+t-r)q}(S^0) \tag{1.2}$$

for the connecting homomorphisms $\delta_r \colon E_2^{s,t}(V_r) \to E_2^{s+1,t-rq}(M)$ and $\delta \colon E_2^{s,t}(M) \to E_2^{s+1,t}(S^0)$ associated to the cofiber sequences (1.1). We note that if $r - t = r' - t'$, then $\beta_{s/r-t} = \beta_{s/r'-t'}$, and abbreviate $\beta_{s/1}$ to β_s. In this paper, we study which of these elements survives to the homotopy groups $\pi_*(S^0)$ of spheres. For a prime greater than three, the following beta elements survive:

β_s for $s > 0$ by L. Smith [9],
$\beta_{tp/r}$ for $t > 0$ and $r \leq p$ with $(t, r) \neq (1, p)$ by S. Oka [4],[5], and
$\beta_{tp^2/r}$ for $t > 0$ and $r \leq 2p$ by S. Oka [6].

At the prime three, β_s survives if $s < 4$ by Toda [10], and does not if $s = 4$, and does if $s = 5$ by Oka [2]. Ravenel then conjectured that β_s survives in the spectral sequence if and only if $s \equiv 0, 1, 2, 3, 5, 6 \bmod 9$. In [8], we proved the 'only if' part. For the survivors, we have

β_s for $s > 0$ with $s \equiv 0, 1, 2, 5, 6 \bmod 9$ by M. Behrens and S. Pemmaraju [1].

We note that the element $\beta_1 \in \pi_{pq-2}(S^0)$ is given by

$$\beta_1 = jj_1[\beta i_1]i$$

for the maps i, j, i_1 and j_1 are the maps given in (1.1). Here, $[\beta i_1]$ denotes βi_1 for the self-map $\beta \in [V(1), V(1)]_{(p+1)q}$ $(V(1) = V_1)$ constructed by Smith [9] at a prime greater than three, and the generator of the homotopy group $[M, V(1)]_{16}$ given in [11] at the prime three. We define W by the cofiber sequence

$$S^{pq-2} \xrightarrow{\beta_1} S^0 \xrightarrow{\iota} W \xrightarrow{\kappa} S^{pq-1}. \tag{1.3}$$

Then we have our main theorem:

Theorem 1.4. *Suppose that there is an element $B_s \in \pi_{s(p+1)q}(V_r \wedge W)$ detected by $v_2^s \in E_2^{0,s(p+1)q}(V_r \wedge W)$. Then, the beta element $\beta_{s/l}$ for $0 < l \le r-2$ survives to $\pi_{(sp+s-l)q-2}(S^0)$.*

As an example, we have

Lemma 1.5. *At an odd prime, there exists $B_{tp} \in \pi_{tp(p+1)q}(V_p \wedge W)$ for $t > 0$ detected by $v_2^{tp} \in E_2^{0,tp(p+1)q}(V_p \wedge W)$.*

Corollary 1.6. *The beta elements $\beta_{tp/l}$ for $t > 0$ and $0 < l \le p-2$ survives.*

This corollary shows that β_{3t} survives at the prime three, and completes a proof of the conjecture.

2. Proofs of results

Applying the BP_*-homology to the first cofiber sequence of (1.1), we obtain

$$BP_*(M) \ = \ BP_*/(p).$$

We observe the E_2-term $E_2^*(X)$ of the Adams-Novikov spectral sequence as the cohomology of the reduced cobar complex $\widetilde{\Omega}^*_{BP_*BP}BP_*(X)$. Then, we have a vanishing line for a (-1)-connected spectrum X:

$$E_2^{s,t}(X) = 0 \quad \text{if } t < sq. \tag{2.1}$$

The structure maps of the Hopf algebroid BP_*BP act on generators by

$$\begin{array}{rcll} \eta_R(v_1) & = & v_1 + pt_1 & \\ \eta_R(v_2) & \equiv & v_2 + v_1 t_1^p - v_1^p t_1 & \bmod (p), \quad \text{and} \\ \Delta(t_1) & = & t_1 \otimes 1 + 1 \otimes t_1 & \end{array} \tag{2.2}$$

(*cf.* [7]). By this, we see that $v_1 \in E_2^{0,q}(M)$. Since $E_2^{sq+1,(s+1)q}(M) = 0$ for $s > 0$ by (2.1), v_1 is a permanent cycle. The Adams map α in (1.1) is given by $\alpha = m(M \wedge v_1)$ for the multiplication m of M, and so it induces v_1-multiplication on the BP_*-homology. It follows that

$$BP_*(V_r) \ = \ BP_*/(p, v_1^r)$$

for $r>0$, and we see that

$$v_1 \in E_2^{0,q}(V_r) \quad \text{and} \quad v_2^p \in E_2^{0,(p^2+p)q}(V_p) \tag{2.3}$$

by (2.2). For the later use, we notice that

$$\alpha i = m(i\wedge v_1) = v_1. \tag{2.4}$$

From [12], we read off

Lemma 2.5. *Suppose that $t-s<(p^2+p+1)q$. In this range, the Adams-Novikov E_2-term $E_2^{*,*}(M)$ is a subquotient of $\mathbb{Z}/p[v_1,v_2]\otimes\{h_0,h_1,h_2,g_0,k_0,k_0h_0,h_0h_2\}\otimes P(b_0,b_1,b_{20})$. Here the bi-degrees of the generators are:*

$$\begin{array}{lll} |h_i| = (1,p^iq)\ (i=0,1,2), & |g_0| = (2,(p+2)q), & |k_0| = (2,(2p+1)q), \\ |b_0| = (2,pq), & |b_1| = (2,p^2q), \quad \textit{and} & |b_{20}| = (2,(p^2+p)q). \end{array}$$

Proof. We have short exact sequences $0\longrightarrow BP_*/(p)\xrightarrow{v_1}BP_*/(p)\longrightarrow BP_*/(p,v_1)\longrightarrow 0$ and $0\longrightarrow BP_*/(p,v_1)\xrightarrow{v_2}BP_*/(p,v_1)\longrightarrow BP_*/(p,v_1,v_2)\longrightarrow 0$, which give rise to Bockstein spectral sequences converging to the Adams-Novikov E_2-terms $E_2^*(M)$ and $E_2^*(V_1)$ with E_1-terms $E_2^*(V_1)$ and $\mathrm{Ext}^*_{BP_*BP}(BP_*,BP_*/(p,v_1,v_2))$, respectively. In our range, we have $\mathrm{Ext}^*_{BP_*BP}(BP_*,BP_*/(p,v_1,v_2)) = \mathrm{Ext}^*_{\mathcal{P}}(\mathbb{Z}/p,\mathbb{Z}/p)$ for the subalgebra $\mathcal{P}$ of the Steenrod algebra generated by the reduced power operations. Thus, $E_2^*(M)$ is a subquotient of $\mathbb{Z}/p[v_1,v_2]\otimes\mathrm{Ext}^*_{\mathcal{P}}(\mathbb{Z}/p,\mathbb{Z}/p)$. We now read off the structure of $\mathrm{Ext}^*_{\mathcal{P}}(\mathbb{Z}/p,\mathbb{Z}/p)$ from [12]. $\square$

Corollary 2.6. *In our range, we have a vanishing line: $E_2^{2s+\varepsilon,tq}(V)=0$ for $V=M,V_r$, if $t<ps+\varepsilon$. Here, $\varepsilon=0,1$.*

Lemma 2.7. *Let $\delta\colon E_2^s(M)\to E_2^{s+1}(S^0)$ be the connecting homomorphism associated with the first cofiber sequence in (1.1). Then, it is a derivation and*

$$\delta(v_1)=h_0,\quad \delta(h_2)=-b_1 \quad \textit{and} \quad \delta(b_0)=0.$$

Proof. Note that h_i and b_i are represented by cocycles $t_1^{p^i}$ and $\sum_{k=1}^{p-1}\frac{1}{p}\binom{p}{k}t_1^{p-k}\otimes t_1^k$ of the cobar complex. By (2.2), we see that the differential d of the cobar complex acts on v_1 and $t_1^{p^i}$ as $d(v_1)=pt_1$ and $d(t_1^{p^i})=-pb_{i-1}$ for $i>0$. The lemma now follows from the definition of the connecting homomorphism. $\square$

The cofiber sequence (1.3) induces a split short exact sequence

$$0\longrightarrow E_2^{s.t}(V)\xrightarrow{\iota_*}E_2^{s,t}(V\wedge W)\xrightarrow{\kappa_*}E_2^{s,t-pq+1}(V)\longrightarrow 0$$

of E_2-terms for $V=M$ and V_r, and so

$$E_2^*(V\wedge W) \;=\; E_2(V)\oplus gE_2(V),$$

where g denotes a generator of degree $pq-1$ such that $\kappa_*(xg)=x$. Since E_3-term is a homology of E_2-terms and $d_2(g)=\beta_1$ for the element β_1 in (1.2), we have the long exact sequence

$$E_3^{s,t}(M)\xrightarrow{\partial}E_3^{s+2,t+pq}(M)\xrightarrow{\iota_*}E_3^{s+2,t+pq}(M\wedge W)\xrightarrow{\kappa_*}E_3^{s+2,t+1}(M) \tag{2.8}$$

with the connecting homomorphism ∂ given by $\partial(x)=x\beta_1$.

Lemma 2.9. *The element $v_2^p\in E_2^0(V_p\wedge W)$ in (2.3) survives to an element $B_p\in\pi_{p(p+1)q}(V_p\wedge W)$.*

Proof. Consider the cofiber sequence (1.1) with $r = p$. In the Adams-Novikov spectral sequence for computing $\pi_*(S^0)$, we have the Toda differential $d_{q+1}(b_1) = h_0b_0^p \in E_2^{q+3,(p^2+1)q}(S^0)$ up to nonzero scalar. By Lemma 2.5, $E_2^{q+2,(p^2+1)q}(M)$ is a subquotient of $\{v_1b_0^p\}$. Since $\delta(v_1b_0^p) = h_0b_0^p$, we see $d_{q+1}(h_2) = v_1b_0^p \in E_2^{q+2,(p^2+1)q}(M)$ up to nonzero scalar by Lemma 2.7. Note that $\beta_1 = b_0$. In the exact sequence (2.8), $v_1b_0^p = \partial(v_1b_0^{p-1})$, and so $d_{q+1}(\iota_*(h_2)) = 0$ in $E_3^{q+2,(p^2+1)q}(M \wedge W)$. Besides, Corollary 2.6 shows that $E_2^{sq+2,(p^2+s)q}(M) = 0$ for $s > 1$, and we see that $\iota_*(h_2) \in E_2^{1,p^2q}(M \wedge W)$ is a permanent cycle, which detects an element $\beta'_{p/p} \in \pi_{p^2q-1}(M \wedge W)$. Send it by α^p in (1.1). The element $\alpha^p\beta'_{p/p} \in \pi_{(p^2+p)q-1}(M \wedge W)$ is detected by an element of $E_2^{q+1,(p^2+p+1)q}(M \wedge W)$, since the E_2-term $E_2^{sq+1,p(p+1)q+sq}(M \wedge W) = E_2^{sq+1,p(p+1)q+sq}(M)$ for $s > 1$ is zero by Corollary 2.6. The E_2-term $E_2^{q+1,(p^2+p+1)q}(M)$ for $s = 1$ is a subquotient of

$$h_0b_0b_1\ (p=3),\ v_1^{p-1}v_2h_0b_0^{p-1},\ v_1^{2p}h_0b_0^{p-1},\ v_2h_1b_0^{p-1},\ v_1^{p+1}h_1b_0^{p-1}, v_1^{p-1}k_0h_0b_0^{p-2}$$

by Lemma 2.5, and so the E_3-term $E_3^{q+1,(p^2+p+1)q}(M \wedge W) = 0$ by (2.8). Therefore, $\alpha^p\beta'_{p/p} = 0$ and $\beta'_{p/p}$ is pulled back to an element B_p under the map j_p. □

We call a spectrum R a ring spectrum if there exist a multiplication $\mu\colon R \wedge R \to R$ and a unit $\iota\colon S^0 \to R$ such that $\mu(\iota \wedge R) = 1_R = \mu(R \wedge \iota)\colon R \to R$. By [3, Ex. 2.9] and [3, Ex. 5.7], we have

(2.10) *W and V_r for $r > 1$ are ring spectra.*

In particular, the spectrum $R_r = V_r \wedge W$ for $r > 1$ is a ring spectrum with multiplication $m_r = (\mu_r \wedge \mu_W)(V_r \wedge T \wedge W)\colon R_r \wedge R_r = V_r \wedge W \wedge V_r \wedge W \to V_r \wedge V_r \wedge W \wedge W \to V_r \wedge W = R_r$, where T denotes the switching map and μ_r and μ_W are the multiplications of V_r and W, respectively.

Proof of Lemma 1.5. Since $R_p = V_p \wedge W$ is a ring spectrum, we obtain a self-map $[\beta^p]\colon R_p \xrightarrow{R_p \wedge B_p} R_p \wedge R_p \xrightarrow{m_p} R_p$ inducing v_2^p on BP_*-homology. Now put $B_{tp} = [\beta^p]^{t-1}B_p$ to see the lemma. □

We consider the element $i_r\alpha^2 i \in \pi_{2q}(V_r) \cong \pi_{2q}(M) = \mathbb{Z}/p\{\alpha^2 i\}$ for $r > 2$ and for the maps in (1.1), which is detected by the element $v_1^2 \in E_2^0(V_r)$ by (2.4).

Lemma 2.11. *Let $r > 2$. There exists an element $\eta_r \in \pi_{(p+2)q-1}(V_r \wedge W)$ such that $\kappa_*(\eta_r) = i_r\alpha^2 i \in \pi_{2q}(V_r)$. Besides, it is detected by $v_1^2 g \in E_2^0(V_r \wedge W) = E_2^0(V_r) \oplus gE_2^0(V_r)$.*

Proof. Put $\delta = ij$ for the maps i, j in (1.1), and we have Yamamoto's relation $\alpha^2\delta = 2\alpha\delta\alpha - \delta\alpha^2 \in [M, M]_{2q-1}$ (*cf.* [11]). We compute

$$\alpha^2 i\beta_1 \ = \ \alpha^2\delta j_1[\beta i_1]i \ = \ (\delta\alpha^2 + \alpha\delta\alpha)j_1[\beta i_1]i \ = \ 0,$$

since $\alpha j_1 = 0$ by (1.1). It follows that $i_r\alpha^2 i \in \pi_{2q}(V_r)$ is pulled back to an element $\eta_r \in \pi_{(p+2)q-1}(V_r \wedge W)$ as desired. Since $i_r\alpha^2 i$ is detected by $v_1^2 \in E_2^0(V_r)$, η_r is detected by the element $v_1^2 g \in E_2^0(V_r \wedge W) = E_2^0(V_r) \oplus gE_2^0(V_r)$. □

Proof of Theorem 1.4. Consider the product $B_s\eta_r \in \pi_*(V_r \wedge W)$ for the element η_r in Lemma 2.11. Then, it is detected by $v_1^2v_2^sg$, since η_r induces a BP_*BP-comodule map $(\eta_r)_*\colon BP_*(V_r) \to BP_*(V_r \wedge W)$ such that $(\eta_r)_*(x) = v_1^2xg$ and B_s is detected by v_2^s. The map $\kappa_*\colon E_2^0(V_r \wedge W) \to E_2^0(V_r)$ assigns $v_1^2v_2^sg$ to $v_1^2v_2^s$, which is a permanent cycle detected by $\kappa(B_s\eta_r)$. Put now $\beta_{s/l} = jj_{l+2}a^{r,l+2}\kappa(B_s\eta_r) = j\alpha^{r-2-l}j_r\kappa(B_s\eta_r) \in \pi_*(S^0)$ for $l < r-2$, and we see the theorem by the Geometric Boundary Theorem (*cf.* [7]). Here, $a^{r,k}$ denotes a map in the cofiber sequence $V_{r-k} \longrightarrow V_r \xrightarrow{a^{r,k}} V_k$ obtained from applying the 3×3 Lemma to the cofiber sequences of (1.1) for $r-k$, r and k. We note that it satisfies $j_ka^{r,k} = \alpha^{r-k}j_r$. $\square$

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