The existence of β_{9t+3} in stable homotopy of spheres at the prime three

Katsumi Shimomura

Abstract. Let β_s be the generator of the second line of the E_2-term of the Adams-Novikov spectral sequence converging to the stable homotopy groups $\pi_*(S^0)$ of spheres at the prime three. Ravenel conjectured that the generator β_s survives to a homotopy element if and only if $s \equiv 0, 1, 2, 3, 5, 6 \bmod 9$. In [9], we proved the 'only if' part. In [1], Behrens and Pemmaraju showed that β_s survives to a homotopy element if $s \equiv 0, 1, 2, 5, 6 \bmod 9$. In this paper, we show the existence of a self-map $\beta\colon \Sigma^{144}V_r \to V_r$ for $r < 9$, that induces v_2^9 on BP_*-homology. Here V_r denotes the spectrum characterized by the BP_*-homology $BP_*(V_r) = BP_*/(3, v_1^r)$. Oka [5] showed that the 'if' part follows from the existence of the self-map β on V_3. Therefore, in particular, we obtain $\beta_{9t+3} \in \pi_{144t+42}(S^0)$. The self-maps show the existence of other members $\beta_{9t/r} \in \pi_{144t-4r-2}(S^0)$ for $t > 0$ and $0 < r < 9$, $\beta_{3t/2} \in \pi_{48t-10}(S^0)$ for $t > 0$ and $\beta_{9t+6/3} \in \pi_{144t+82}(S^0)$ for $t \geq 0$ of the beta family in $\pi_*(S^0)$.

1. Introduction

Let S^0 denote the sphere spectrum localized at a prime p, and let $V(n)$ for $n \geq 0$ denote the Smith-Toda spectrum defined by the BP_*-homology $BP_*(V(n)) = BP_*/(p, v_1, \dots, v_n)$. Here, BP denotes the Brown-Peterson spectrum with coefficient ring $BP_* = \boldsymbol{Z}_{(p)}[v_1, v_2, \dots]$. Note that $V(-1) = S^0$, and $V(0)$ is the mod p Moore spectrum M. It is shown that if $n < 4$, then $V(n)$ exists if and only if $p > 2n$ (*cf.* [11], [15], [7]). The spectra $V(0) = M$ and $V(1)$ for $p \geq 3$ lie in the cofiber sequences

$$S^0 \xrightarrow{3} S^0 \xrightarrow{i} M \xrightarrow{j} S^1 \quad \text{and} \quad \Sigma^{2p-2}M \xrightarrow{\alpha} M \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{2p-1}M, \tag{1.1}$$

where α denotes the Adams map such that $BP_*(\alpha) = v_1$. For the prime $p > 3$, L. Smith [11] defined the β-element as $\beta_s = jj_1\beta^s i_1 i$ for $s > 0$ in the homotopy groups $\pi_*(S^0)$ by constructing the self-map $\beta\colon \Sigma^{2p^2-2}V(1) \to V(1)$. We notice that the cofiber of β is $V(2)$. Hereafter, we assume that the prime p is three. Then, Toda [15] showed the non-existence of the Smith-Toda spectrum $V(2)$, which indicates the non-existence of the self-map β. Thus, there seems no way to define the β-family in the homotopy groups $\pi_*(S^0)$ different from the case where the prime p is greater than three. Consider the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(X)$ of a spectrum X with E_2-term $E_2^{*,*}(X) = \mathrm{Ext}_{BP_*BP}^{*,*}(BP_*, BP_*(X))$ for the Hopf algebroid (BP_*, BP_*BP) associated to BP. Then Miller, Ravenel and Wilson [2] defined a β-element β_s for $s > 0$ in the E_2-term $E_2^{2,16s-4}(S^0)$ as $\delta\delta'(v_2^s)$ for $v_2^s \in E_2^{0,16s}(V(1))$, where $\delta : E_2^{1,16s-4}(M) \to E_2^{2,16s-4}(S^0)$ and $\delta'\colon E_2^{0,16s}(V(1)) \to E_2^{1,16s-4}(M)$ are the connecting homomorphisms associated to the cofiber sequences in (1.1). Toda [12] constructed the homotopy element β_s detected by $\beta_s \in E_2^{2,*}(S^0)$ for $s < 4$ and Oka [3] showed that $\beta_4 \in E_2^{2,*}(S^0)$ is not a permanent cycle and β_5 is, and Ravenel conjectured that β_s is a permanent cycle of the spectral sequence if and only if $s \equiv 0, 1, 2, 3, 5, 6 \bmod 9$. In [9], we proved the 'only if' part. Behrens and Pemmaraju showed in [1] the 'if' part except for β_{9t+3} by constructing the self-map $[\beta^9]\colon \Sigma^{144}V(1) \to V(1)$ that induces v_2^9 on BP_*-homology. Let V_r denote a spectrum with BP_*-homology $BP_*/(3, v_1^r)$, which lies in the cofiber sequence

$$\Sigma^{4r}M \xrightarrow{\alpha^r} M \xrightarrow{i_r} V_r \xrightarrow{j_r} \Sigma^{4r+1}M. \tag{1.2}$$

Note that $V_1 = V(1)$ and that V_r for $r > 1$ is a ring spectrum by Oka [4], while $V(1)$ is not by Toda [15]. Oka showed in [5] that the 'if' part of the conjecture follows from the existence of a similar self-map $[\beta^9]\colon \Sigma^{144}V_3 \to V_3$ that induces v_2^9 on BP_*-homology.

We study such a self-map in this paper. For this sake, we consider the element $x_{106} = \beta_{9/9} \pm \beta_7 \in \pi_{106}(S^0)$ given by Ravenel [7]. Since the order of x_{106} is three, we have an element $x'_{106} \in \pi_{107}(M)$ such that $j_*(x'_{106}) = x_{106}$, and define $\beta'_{9/r} = \alpha^{9-r}x'_{106} \in \pi_{143-4r}(M)$ for $0 < r < 9$. By the self-map $[\beta^9]$ given in [1], we define the β-element $\beta'_9 = j_1[\beta^9]i_1 i$, and $\alpha\beta'_9 = 0 \in \pi_{143}(M)$; nevertheless, the relation $\alpha\beta'_{9/1} = 0 \in \pi_{143}(M)$ is not trivial. The following is our key lemma.

Lemma 1.3. *$\alpha^9 x'_{106} = 0 \in \pi_*(M)$, and so $\alpha^r\beta'_{9/r} = 0 \in \pi_*(M)$ for $0 < r < 9$.*

This implies that $\beta'_{9/r}$ is pulled back to $v_2^9 \in \pi_{144}(V_r)$ under the map $j_{r*}\colon \pi_{144}(V_r) \to \pi_{143-4r}(M)$. Since V_r is a ring spectrum if $r > 1$, the element v_2^9 yields the self-map.

Theorem 1.4. *There exists the self-map* $[\beta^9]\colon \Sigma^{144}V_r \to V_r$ *for* $1 < r < 9$ *that induces* v_2^9 *on* BP_**-homology.*

Corollary 1.5. (*cf.* Oka [5]) *If* $s \equiv 0,1,2,3,5,6 \bmod 9$*, then* $\beta_s \in E_2^{2,16s-4}(S^0)$ *is a permanent cycle.*

In order to define β_{9t+3} and β_{9t+6} in $\pi_*(S^0)$ from the self-map, Oka showed the existence of the homotopy elements $v_1^2v_2^3$ of $\pi_{56}(V_3)$ and $v_1v_2^6$ of $\pi_{100}(V_2)$ in [5, Lemmas 3 and 4].

Lemma 1.6. *There exist elements* $v_1^2v_2^3$ *and* $v_1v_2^6 \in \pi_*(V_4)$ *that induces* $v_1^2v_2^3$ *and* $v_1v_2^6$ *on* BP_**-homology.*

This follows from Lemmas 3.17 and 3.18. In the same manner as Oka did in [5], we obtain

Corollary 1.7. *Let* t *be a positive integer. Then there exist essential homotopy elements* $\beta_{9t/r} \in \pi_{144t-4r-2}(S^0)$ *for* $0 < r < 9$*,* $\beta_{3t/r} \in \pi_{48t-4r-2}(S^0)$ *for* $r = 1,2$ *and* $\beta_{9t-3/3} \in \pi_{144t-62}(S^0)$ *of order three. Besides, we have* $\widetilde{\beta}_{9t/9} \in \pi_{144t-38}(S^0)$ *such that* $\langle \alpha_1, 3, \widetilde{\beta}_{9t/9}\rangle = \beta_{9t/8}$*.*

Here, $\alpha_r \in \pi_{4r-1}(S^0)$ for $r > 0$ denotes the α-element defined by $\alpha_r = j\alpha^r i$ for the maps in (1.1). These α- and β-elements satisfy the Toda bracket relations $\langle \alpha_k, 3, \beta_{9t/r}\rangle = \beta_{9t/r-k}$ for $0 < k < r$, $\langle \alpha_1, 3, \beta_{3t/2}\rangle = \beta_{3t}$ and $\langle \alpha_2, 3, \beta_{9t+6/3}\rangle = \langle \alpha_1, 3, \beta_{9t+6/2}\rangle = \beta_{9t+6}$ in the homotopy groups $\pi_*(S^0)$ by definition.

Proposition 1.8. *Let* t *be a non-negative integer. In* $\pi_*(S^0)$*,* $\langle \alpha_r, 3, \beta_{9t/r}\rangle = 0$*,* $\langle \alpha_1, 3, \beta_3\rangle = 0$*,* $\langle \alpha_2, 3, \beta_{9t+3/2}\rangle = \beta_{9t+2}\beta_1^2$ $(t > 0)$*,* $\langle \alpha_3, 3, \beta_{9t+6/3}\rangle = \beta_{9t+5}\beta_1^2$ *and* $\langle \alpha_3, 3, \beta_{9t+3/2}\rangle = \beta_{9t+1}\beta_1^4$ *up to sign.*

The cofiber of the self-map of Theorem 1.4 yields the spectrum $M(1,r,9)$.

Corollary 1.9. *There exists a spectrum* $M(1,r,9)$ *such that* $BP_*(M(1,r,9)) = BP_*/(3, v_1^r, v_2^9)$ *for* $1 < r < 9$*.*

Furthermore, the element x'_{106} itself is pulled back to an element detected by $v_2^9 \pm v_1^8v_2^7$ by Lemma 1.3, which induces the self-map (see Corollary 5.1).

Proposition 1.10. *There exists a spectrum* $M(1,9,9)$ *such that* $BP_*(M(1,9,9)) = BP_*/(3, v_1^9, v_2^9 \pm v_1^8v_2^7)$*.*

This paper is organized as follows: In the next section, we introduce the β-elements in the E_2-terms of the Adams-Novikov spectral sequence, and then we determine some Adams-Novikov E_2-terms by Ravenel's small descent spectral sequence. In section 3, we show Lemma 3.13 on the differential $d_5(v_2^3)$, which plays the crucial role to show Lemma 1.6 and Proposition 1.8. Section four is devoted to study the homotopy group $\pi_{143}(M)$. By use of this, we prove Lemma 1.3 in the last section. We also introduce β-elements in the homotopy groups $\pi_*(S^0)$ and prove Corollary 1.7 and Proposition 1.8.

The author would like to thank the referee for not only reminding him that $d_5(v_2^3) \neq 0$ in the Adams-Novikov spectral sequence for $\pi_*(V_1 \cup_{\beta_1^2} \Sigma^{21}V_1)$, but also pointing him out that original version of Lemma 5.4 is ambiguous.

2. The β-elements in the E_2-term of the Adams-Novikov spectral sequence and the small descent spectral sequence

Let BP denote the Brown-Peterson spectrum at the prime three. Then it defines the Hopf algebroid $(BP_*, BP_*BP) = (\pi_*(BP), BP_*(BP)) = (\boldsymbol{Z}_{(3)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots])$. The internal degrees of the generators are $|v_n| = 2\times 3^n - 2 = |t_n|$. The structure maps of it behave on generators as follows:

$$\begin{aligned}
&\eta_R(v_1) = v_1 + 3t_1, \quad \eta_R(v_2) \equiv v_2 + v_1t_1^3 - v_1^3t_1 \mod (3),\\
&\eta_R(v_3) \equiv v_3 + v_2t_1^9 - v_2^3t_1 + v_1t_2^3 \mod (3, v_1^2)\\
&\Delta(t_1) = t_1\otimes 1 + 1\otimes t_1, \quad \Delta(t_2) = t_2\otimes 1 + t_1\otimes t_1^3 + 1\otimes t_2 + v_1b_{10} \quad \text{and}\\
&\Delta(t_3) \equiv t_3\otimes 1 + t_1\otimes t_2^3 + t_2\otimes t_1^9 + 1\otimes t_3 + v_2b_{11} + v_1b_{20} \mod (3, v_1^2),
\end{aligned} \tag{2.1}$$

where b_{1k} for $k \geq 0$ and b_{20} is defined by

$$d(t_1^{3^{k+1}}) = 3b_{1k} \quad \text{and} \quad d(t_2^3) = -t_1^3\otimes t_1^9 - v_1^3b_{11} + 3b_{20}$$

in the cobar complex $\Omega^*_{BP_*(BP)}BP_*$. This implies

$$b_{1k} \equiv -t_1^{3^k}\otimes t_1^{2\times 3^k} - t_1^{2\times 3^k}\otimes t_1^{3^k} \quad \text{and} \quad d(b_{20}) \equiv b_{10}\otimes t_1^9 - t_1^3\otimes b_{11} \mod (3). \tag{2.2}$$

It gives rise to the Adams-Novikov spectral sequence $E_2^{s,t}(X) \Rightarrow \pi_{t-s}(X)$ with $E_2^{s,t}(X) = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*(X))$. Consider the spectra M and V_r for $r > 0$ defined by the cofiber sequences (1.1) and (1.2). Then they induces the long exact sequences

$$\begin{aligned}
&\cdots \longrightarrow E_2^s(S^0) \xrightarrow{3} E_2^s(S^0) \xrightarrow{i_*} E_2^s(M) \xrightarrow{\delta} E_2^{s+1}(S^0) \longrightarrow \cdots, \quad \text{and}\\
&\cdots \longrightarrow E_2^s(M) \underset{\alpha^r{}_*}{\longrightarrow} E_2^s(M) \underset{i_{r*}}{\longrightarrow} E_2^s(V_r) \underset{\delta_r}{\longrightarrow} E_2^{s+1}(M) \longrightarrow \cdots
\end{aligned} \tag{2.3}$$

of the Adams-Novikov E_2-terms. By (2.1), we see that for $t > 0$,

(2.4) $$v_2^t \in E_2^{0,16t}(V_1), \quad v_2^{3t} \in E_2^{0,48t}(V_r) \quad (0 < r \le 3), \quad \text{and} \quad v_2^{9t} \in E_2^{0,144t}(V_r) \quad (0 < r \le 9).$$

We define the β-elements $\beta'_{t/r}$ in $E_2^1(M)$ (resp. $\beta_{t/r}$ in $E_2^2(S^0)$) with $\beta'_t = \beta'_{t/1}$ (resp. $\beta_t = \beta_{t/1}$) by

(2.5) $$\beta'_{t/r} = \delta_r(v_2^t) \quad (\text{resp. } \beta_{t/r} = \delta\delta_r(v_2^t)),$$

if $v_2^t \in E_2^0(V_r)$. By cochains of the cobar complex $\Omega^*_{BP_*BP}BP_*$, these β-elements are represented as

(2.6) $$\beta_1 = [b_{10}], \quad \beta_{3/3} = [b_{11} + \cdots], \quad \beta'_2 = [-v_2t_1^3 + v_1t_1^6 + v_1^2v_2t_1 + v_1^3t_1^4 + v_1^5t_1^2],$$

where $\cdots$ in the representative of $\beta_{3/3}$ denotes an element of the ideal $(3, v_1^5)$.

We call a spectrum R a ring spectrum if there exist a multiplication $\mu\colon R \wedge R \to R$ and a unit $\iota\colon S^0 \to R$ such that $\mu(\iota \wedge R) = 1_R = \mu(R \wedge \iota)\colon R \to R$. Note that the mod 3 Moore spectrum is not an associative ring spectrum by Toda [15, Lemma 6.2], and neither are V_r's. Though V_1 is not a ring spectrum [14], Oka showed in [4, Ex. 2.9] and [4, Cor. 2.6] the following theorem:

(2.7) (Oka) *V_r for $r > 1$ are ring spectra.*

In order to study the E_2-terms of the Adams-Novikov spectral sequence, we adopt Ravenel's small descent spectral sequence. Ravenel constructed spectra $T(m)$ and $T(m)_k$ for $m, k \ge 0$ such that $BP_*(T(m)) = BP_*[t_1, \ldots, t_m] \subset BP_*(BP)$ and $BP_*(T(m)_k) = BP_*(T(m))\{t_{m+1}^j : 0 \le j \le k\} \subset BP_*(T(m+1))$ in [7] and [8], which fit in the cofiber sequences

(2.8) $$\begin{array}{l} T(m)_{3^k-1} \xrightarrow{\iota_{m,k}} T(m)_{3^{k+1}-1} \xrightarrow{\kappa_{m,k}} \Sigma^{2\times 3^k(3^{m+1}-1)}T(m)_{2\times 3^k-1} \xrightarrow{\lambda_{m,k}} \Sigma T(m)_{3^k-1} \quad \text{and} \\ T(m)_{2\times 3^k-1} \xrightarrow[\iota'_{m,k}]{} T(m)_{3^{k+1}-1} \xrightarrow[\kappa'_{m,k}]{} \Sigma^{4\times 3^k(3^{m+1}-1)}T(m)_{3^k-1} \xrightarrow[\lambda'_{m,k}]{} \Sigma T(m)_{2\times 3^k-1} \end{array}$$

(see [7, (7.1.14), (7.1.15)]). These induce an exact couple that defines the algebraic (resp. topological) small descent spectral sequence

(2.9) $$\begin{array}{l} {}^AE_1^{*,*} = \Lambda(h_{m+1,k}) \otimes \boldsymbol{Z}/3[b_{m+1,k}] \otimes E_2^*(X \wedge T(m)_{3^{k+1}-1}) \Longrightarrow E_2^*(X \wedge T(m)_{3^k-1}) \\ (\text{resp.}\,{}^TE_1^{*,*} = \Lambda(h_{m+1,k}) \otimes \boldsymbol{Z}/3[b_{m+1,k}] \otimes \pi_*(X \wedge T(m)_{3^{k+1}-1}) \Longrightarrow \pi_*(X \wedge T(m)_{3^k-1})) \end{array}$$

for a spectrum X with $h_{m+1,k} \in {}^AE_1^{1,0}$ (resp. ${}^TE_1^{1,2\times 3^k(3^{m+1}-1)}$), $b_{m+1,k} \in {}^AE_1^{2,0}$ (resp. ${}^TE_1^{2,2\times 3^{k+1}(3^{m+1}-1)}$) and $d_r\colon {}^AE_r^{s,t} \to {}^AE_r^{s+r,t-r+1}$ (resp. $d_r\colon {}^TE_r^{s,t} \to {}^TE_r^{s+r,t+r-1}$) (*cf.* [7, Th. 7.1.13, Th. 7.1.16], see also [8, Th. 1.17, Th. 1.21]). Here, h_{ij} and b_{ij} are represented by a cochain of the cobar complex $\Omega^*_{BP_*(BP)}BP_*$ whose leading terms are $t_i^{3^j}$ and $-t_i^{3^j} \otimes t_i^{2\times 3^j} - t_i^{2\times 3^j} \otimes t_i^{3^j}$, respectively. Let s and t denote positive integers with $t - s < 144$, and consider the mod 3 Moore spectrum M. Then, we see that $E_2^*(M \wedge T(3)) = \boldsymbol{Z}/3[v_1, v_2, v_3]$, which is isomorphic to $E_2^*(M \wedge T(2)_2)$. The small descent spectral sequence ${}^AE_1 = \Lambda(h_{30}) \otimes \boldsymbol{Z}/3[v_1, v_2, v_3] \Rightarrow E_2^*(M \wedge T(2))$ for $m = 2$ and $k = 0$ collapses from the E_1-term. In our range, $E_2^*(M \wedge T(1)_8) = E_2^*(M \wedge T(2))$. The spectral sequence ${}^AE_1 = \Lambda(h_{21}, h_{30}) \otimes \boldsymbol{Z}/3[v_1, v_2, v_3] \Rightarrow E_2^*(M \wedge T(1)_2)$ for $m = 1$ and $k = 1$ has the differentials induced by the relation $d_1(v_3) = v_1h_{21}$ read off from (2.1). Then, we obtain $E_2^*(M \wedge T(1)_2) = (\boldsymbol{Z}/3[v_1, v_2] \oplus h_{21}\boldsymbol{Z}/3[v_2] \otimes \Lambda(v_3)) \otimes \Lambda(h_{30})$. In the spectral sequence ${}^AE_1 = \Lambda(h_{20}) \otimes \boldsymbol{Z}/3[b_{20}] \otimes E_2^*(M \wedge T(1)_2) \Rightarrow E_2^*(M \wedge T(1))$, the relation $d_1(h_{30}) = v_1b_{20}$ seen by (2.1) yields non-trivial differentials and

(2.10) $$E_2^{*,*}(M \wedge T(1)) = (\boldsymbol{Z}/3[v_1, v_2] \oplus b_{20}\boldsymbol{Z}/3[v_2, b_{20}] \oplus h_{21}\boldsymbol{Z}/3[v_2, b_{20}] \otimes \Lambda(v_3, h_{30})) \otimes \Lambda(h_{20}).$$

Put $X_k = T(0)_{3^k-1}$. Then, the spectral sequence (2.9) is rewritten as

(2.11) $${}^AE_1 = \Lambda(h_k) \otimes \boldsymbol{Z}/3[b_k] \otimes E_2^*(X \wedge X_{k+1}) \Longrightarrow E_2^*(X \wedge X_k)$$

for a spectrum X and $k \ge 0$. Here, h_k and b_k denotes the elements represented by the cocycles $t_1^{3^k}$ and b_{1k}, respectively.

Lemma 2.12. *The E_2-term $E_2^*(M \wedge X_2)$ with the internal degree less than 144 is isomorphic to the tensor product of $\Lambda(h_{20})$ and the direct sum*

$$\begin{array}{l} \boldsymbol{Z}/3[v_1, v_2]/(v_2^3) \otimes \Lambda(h_3) \oplus b_2\boldsymbol{Z}/3[v_1, v_2]/(v_1^6, v_2^3) \oplus h_2\boldsymbol{Z}/3[v_1, v_2]/(v_1^3, v_2^6) \\ \oplus \boldsymbol{Z}/3[v_2]\{h_{21}, b_{20}\} \otimes \Lambda(b_{20}) \oplus h_2h_{21}\Lambda(v_3, h_{30}) \oplus h_2b_{20}\Lambda(h_{21}, b_{20}). \end{array}$$

Proof. Noticing that $E_2^{*,*}(M \wedge X_4) = E_2^{*,*}(M \wedge T(1))$ in our range, we see that the spectral sequence (2.11) for $k = 3$ collapses and so $E_2^{*,*}(M \wedge X_3) = E_2^{*,*}(M \wedge T(1)) \otimes \Lambda(h_3)$. Consider the spectral sequence (2.11) for $k = 2$. Then, the differential $d_1(v_2^3) = v_1^3h_2$ and $d_1(v_2^6h_2) = v_1^6b_2$ act on the first summand of (2.10), and $d_1(v_3h_{21}) = v_2h_{21}h_2$ and $d_1(h_{21}h_{30} + v_3b_{20}) = v_2b_{20}h_2 + h_{21}h_{20}h_2$ act on the direct sum of the second and the third summands of (2.10). Observing the homology of each summand gives the lemma. q.e.d.

Since $|v_k| = 2 \times 3^k - 2 = |t_k|$ and $|b_{20}| = 48$, the lemma implies the vanishing line.

Corollary 2.13. *The E_2-term $E_2^{s,t}(M \wedge X_2) = 0$ if one of the following conditions holds: (1) $s > 5$, (2) $s = 5$, $t < 112$, (3) $s = 4$, $t < 96$, (4) $s = 3$, $t < 64$, (5) $s = 2$, $t < 48$, (6) $s = 1$, $t < 16$.*

Corollary 2.14. *The homotopy groups $\pi_*(M \wedge X_2)$ is isomorphic to the E_2-term.*

We notice that this is also shown by the topological version of the spectral sequence (2.11):

$$ {}^TE_1 = \Lambda(h_k) \otimes \boldsymbol{Z}/3[b_k] \otimes \pi_*(X \wedge X_{k+1}) \Longrightarrow \pi_*(X \wedge X_k). \tag{2.15}$$

Lemma 2.16. $E_2^{5,52}(V_3) \subset \boldsymbol{Z}/3\{i_{3*}\beta_2'\beta_1^2, i_{3*}i_*\alpha_1\beta_{3/3}\beta_1, v_1^2v_2h_0\beta_1^2\}$.

Proof. Consider the exact sequence

$$E_2^{s,t-12}(M \wedge X_2) \xrightarrow{v_1^3} E_2^{s,t}(M \wedge X_2) \xrightarrow{i_{3*}} E_2^{s,t}(V_3 \wedge X_2) \xrightarrow{j_{3*}} E_2^{s+1,t-12}(M \wedge X_2).$$

For the internal degree less than 53, $E_2^*(M \wedge X_2)$ is isomorphic to $\big(\boldsymbol{Z}/3[v_1, v_2]/(v_2^3) \oplus h_2\boldsymbol{Z}/3[v_1, v_2]/(v_1^3)\big) \otimes \Lambda(h_{20}) \oplus \boldsymbol{Z}/3\{h_{21}, b_{20}\}$ by Lemma 2.12. Since $j_{3*}(v_2^3) = h_2$, $E_2^*(V_3 \wedge X_2)$ is isomorphic to the direct sum of $v_2^3\Lambda(v_1)$ and the image of i_{3*} : $i_{3*}\big((\boldsymbol{Z}/3[v_1, v_2]/(v_1^3, v_2^3) \oplus h_2\boldsymbol{Z}/3[v_1, v_2]/(v_1^3)) \otimes \Lambda(h_{20}) \oplus \boldsymbol{Z}/3\{h_{21}, b_{20}\}\big)$. By the spectral sequences (2.11) for $k = 0$ and 1, we see that $E_2^{5,52}(V_3) \subset \big(E_2^{*,*}(V_3 \wedge X_2) \otimes \Lambda(h_0, h_1) \otimes \boldsymbol{Z}/3[b_0, b_1]\big)^{5,52} = \boldsymbol{Z}/3\{i_{3*}v_1^2v_2h_0b_0^2, i_{3*}v_2h_1b_0^2, i_{3*}h_0b_1b_0\}$. Since $b_0 = \beta_1$, $i_{3*}v_2h_1 = \beta_2'$ and $h_0b_1 = i_*\alpha_1\beta_{3/3}$, the lemma follows. q.e.d.

Lemma 2.17. *Each element of $E_2^{5,136}(M)$ is killed by v_1^3.*

Proof. Consider the spectral sequences (2.11) for $k = 1, 2$. Then, v_1^3 killed elements of $E_2^{5,136}(M)$ originated from the summands of $E_2^*(X_2 \wedge M)$ other than the first summand $A = \boldsymbol{Z}/3[v_1, v_2]/(v_2^3) \otimes \Lambda(h_{20}, h_3, b_2)$. Put $K = \{x \in E_2^{5,136}(M) : v_1^3x = 0\}$. Then, $E_2^{5,136}(M)/K \subset (A \otimes \Lambda(h_0, h_1) \otimes \boldsymbol{Z}/3[b_0, b_1])^{5,136}$ by the spectral sequence (2.11) for $k = 0, 1$. We consider the complex $A \otimes \Lambda(h_1) \otimes \boldsymbol{Z}/3[b_1]$ with differential given by $d_1(v_2) = v_1h_1$ and $d_1(v_2^2h_1) = v_1^2b_1$. Then, the cohomology of it is $\big(\boldsymbol{Z}/3[v_1] \oplus b_1\boldsymbol{Z}/3[b_1] \otimes \Lambda(v_1) \oplus h_1\Lambda(v_2) \otimes \boldsymbol{Z}/3[b_1]\big) \otimes \Lambda(h_{20}, h_3, b_2)$. Similarly, consider the complex $\boldsymbol{Z}/3[v_1] \otimes \Lambda(h_0, h_{20}, h_3, b_2) \otimes \boldsymbol{Z}/3[b_0]$ with differentials given by $d_2(v_1h_{20}) = v_1^2b_0$. Then its cohomology is $(\boldsymbol{Z}/3[v_1] \oplus \boldsymbol{Z}/3\{h_{20}, b_0, v_1b_0\} \otimes \boldsymbol{Z}/3[b_0]) \otimes \Lambda(h_0, h_3, b_2)$, and $E_2^{5,136}(M)/K = 0$ as desired. q.e.d.

Lemma 2.18. *In the E_2-term $E_2^3(V_1)$, $h_1b_1^2 = \pm v_2^3h_1b_0^2$.*

Proof. Consider elements of $E_2^2(V_1)$ defined by the Massey products: $b_n = \langle h_n, h_n, h_n\rangle$, $g_n = \langle h_n, h_n, h_{n+1}\rangle$, $k_n = \langle h_n, h_{n+1}, h_{n+1}\rangle$ and $a_n = \langle h_n, h_{n+1}, h_{n+2}\rangle$. Then these satisfies $b_nh_{n+1} = h_ng_n$, $h_ng_{n+1} = k_nh_{n+2}$ and $g_nh_{n+2} = h_na_n$ by the juggling theorem [7, Th. A1.4.6]. Furthermore, the differentials $d(b_{20})$, $d(t_3)$ and $d(v_2)$ of the cobar complex $\Omega_{BP_*(BP)}BP_*/(3, v_1)$ gives us the relations $h_1b_1 = b_0h_2$ (by (2.2)), $a_0 = v_2b_1$ (by (2.1)) and $v_2h_2 = v_2^3h_0$ (by (2.1)), respectively. Now the lemma follows from the computation $h_1b_1^2 = h_2b_1b_0 = h_1g_1b_0 = g_1h_0g_0 = k_0g_0h_2 = k_0h_0a_0 = v_2g_0h_1b_1 = v_2g_0h_2b_0 = v_2^3h_0g_0b_0 = v_2^3h_1b_0^2$ in $E_2^3(V_1)$. q.e.d.

Lemma 2.19. *Let $x \in E_2^{4,96}(M)$ be the element that detects $ix_{92} \in \pi_{92}(M)$. Then, $b_0^4x \neq 0 \in E_2^{12,144}(M)$.*

Proof. The element $b_0x \in E_2^{6,108}(M)$ is essential, since $x_{92}\beta_1$ is the generator of $\pi_{102}(S^0)$ of order three in [7, Table A.3.4]. In the spectral sequence (2.11) for $k = 0$, a killer of b_0^4x sits in the direct sum of

$$E_2^{11,144}(M \wedge X_1), E_2^{10,140}(M \wedge X_1), E_2^{9,132}(M \wedge X_1), E_2^{8,128}(M \wedge X_1), E_2^{7,120}(M \wedge X_1) \text{ and } E_2^{6,116}(M \wedge X_1).$$

Since $E_2^{s,t}(M \wedge X_1) \subset E_2^{*,*}(M \wedge X_2) \otimes \Lambda(h_1) \otimes \boldsymbol{Z}/3[b_1]$, we see that the above E_2-terms are zero except for $E_2^{7,120}(M \wedge X_1) \subset \boldsymbol{Z}/3\{h_1b_1^3\}$ and $E_2^{6,116}(M \wedge X_1) \subset \boldsymbol{Z}/3\{v_2h_{20}h_1b_1^2, v_1^4h_{20}h_1b_1^2, v_1^2b_1^3\}$ by Lemma 2.12 and Corollary 2.13. We see that $d_1(v_1^3v_2h_{20}b_1^2) = v_1^4h_{20}h_1b_1^2$ and $d_1(v_2^2h_1b_1^2) = v_1^2b_1^3$ in the spectral sequence (2.11) for $k = 1$. We also see that $d_r(h_1b_1^3) = 0$ and $d_1(v_2h_1h_{20}b_1^2) = v_1v_2b_1^3h_0$ in the spectral sequence (2.11) for $k = 0$, and nothing kills the element b_0^4x in the spectral sequence (2.11) for $k = 0$. q.e.d.

3. The Adams-Novikov differential on v_2^3

In this section we compute the Adams-Novikov differential on $v_2^3 \in E_2^{0,48}(V_3)$ by use of some relations in $[M,M]_*$ given in [14, (6.5), Th. 6.8]:

$$(3.1)\quad \begin{array}{rl} \text{(i)} & \delta\delta = 0 = \alpha\beta_{(1)} = \beta_{(1)}\alpha \\ \text{(ii)} & \alpha^2\delta = -\delta\alpha^2 - \alpha\delta\alpha, \text{ and so } \alpha\delta\alpha\delta = \delta\alpha\delta\alpha \text{ and } \alpha^3\delta = \delta\alpha^3 \\ \text{(iii)} & \alpha\delta\beta_{(1)} = \beta_{(1)}\delta\alpha \\ \text{(iv)} & \beta_{(1)}\beta_{(1)} = \delta\alpha\delta\beta_{(1)}\delta\beta_{(1)}\delta \\ \text{(v)} & \alpha\beta_{(2)} = \beta_{(2)}\alpha = \beta_{(1)}\delta\beta_{(1)}\delta\beta_{(1)} \end{array}$$

Here, α denotes the Adams map as before, $\delta = ij$, $\beta_{(1)} = j_1[\beta i_1] = [j_1\beta]i_1$ and $\beta_{(2)} = [j_1\beta][\beta i_1]$, in which i, j, i_1, j_1 are maps in (1.1) and $[\beta i_1]$ and $[j_1\beta]$ are the elements introduced in [14] to define the β-elements $\beta_k = j\beta_{(k)}i$ in $\pi_*(S^0)$ for $k = 1, 2$. For the later use, we also introduce elements:

$$(3.2)\qquad \varepsilon = \langle\alpha_1, \alpha_1, \beta_1^3\rangle = j\beta_{(1)}\beta_{(2)}i \in \pi_{37}(S^0) \quad\text{and}\quad \varepsilon' = \beta_{(1)}\beta_{(2)}i \in \pi_{38}(M).$$

Note that ε (resp. ε') is detected by $h_0b_1 \in E_2^{3,40}(S^0)$ (resp. $\alpha i b_1 \in {}^TE_1{}^{0,38}(M \wedge X_1)$). Before computing the differential, we show the following well known lemma which is shown easily from the above relations:

Lemma 3.3. $\beta_{(1)}i\beta_1^5 = 0$.

Proof. This follows from the computation: $\beta_{(1)}i\beta_1^5 = \beta_{(1)}\delta\beta_{(1)}\delta\beta_{(1)}\delta\beta_{(1)}\delta\beta_{(1)}\delta\beta_{(1)}i = \beta_{(2)}\alpha\delta\alpha\beta_{(2)}i = 0$, since $\beta_{(2)}\alpha^2 = 0 = \alpha^2\beta_{(2)}$. q.e.d.

Since β_3 generates $\pi_{42}(S^0)$ and is of order three, we have an element $\beta_{(3)} \in [M,M]_{43}$ such that $j\beta_{(3)}i = \beta_3$. We also consider the operation $\theta\colon [X,Y]_* \to [X,Y]_{*+1}$ given in [14, p.209]. Toda [14, (2.10), (3.7)] shows that for any $\xi \in [M,M]_t$,

$$(3.4)\qquad \xi\alpha - \alpha\xi = \alpha\delta\theta(\xi) - \delta\theta(\xi)\alpha = -\theta(\xi)\delta\alpha + (-1)^{t+1}\alpha\theta(\xi)\delta.$$

It is shown in [14, (2.7)] that $\xi\delta - (-1)^t\delta\xi + \delta\theta(\xi)\delta = (j\xi i) \wedge 1_M$ for $\xi \in [M,M]_t$. By [14, Th. 6.4, Th. 6.8], we see that $\beta_{(s)}\delta + \delta\beta_{(s)} = \beta_s \wedge 1_M$ for $s = 1, 2$, and so

$$(3.5)\qquad (\beta_{(s)}\delta + \delta\beta_{(s)})\xi = \xi(\beta_{(s)}\delta + \delta\beta_{(s)}) \quad (s = 1, 2) \quad \text{for any } \xi \in [M,M]_* \ (\textit{cf.}\ [14, (3.8)']).$$

Proposition 3.6. *In $[M,M]_{47}$, we have the following relations up to sign:*

$$\begin{aligned} \alpha\beta_{(3)} &= \beta_{(1)}\delta\beta_{(1)}\delta\beta_{(2)} - \beta_{(1)}\delta\beta_{(2)}\delta\beta_{(1)} + \delta\beta_{(1)}\beta_{(2)}\delta\beta_{(1)} \\ &= -\beta_{(1)}\beta_{(2)}\delta\beta_{(1)}\delta + \delta\beta_{(1)}\beta_{(2)}\delta\beta_{(1)}. \end{aligned}$$

Proof. From [7, Table A3.4], we read off the homotopy group $[M,M]_{47} = \boldsymbol{Z}/3\{\beta_{(1)}\delta\beta_{(1)}\delta\beta_{(2)}, \delta\beta_{(1)}\beta_{(2)}\delta\beta_{(1)}, \beta_{(1)}\beta_{(2)}\delta\beta_{(1)}\delta\}$ (see (3.2)). By (3.1) and (3.5),

$$(3.7)\qquad \begin{aligned} \beta_{(1)}\beta_{(2)}\delta\beta_{(1)}\delta &= \beta_{(1)}(\beta_{(2)}\delta + \delta\beta_{(2)})\beta_{(1)}\delta - \beta_{(1)}\delta\beta_{(2)}\beta_{(1)}\delta = -\beta_{(1)}\delta\beta_{(2)}\beta_{(1)}\delta \\ &= -\beta_{(1)}\delta\beta_{(2)}(\beta_{(1)}\delta + \delta\beta_{(1)}) + \beta_{(1)}\delta\beta_{(2)}\delta\beta_{(1)} = -\beta_{(1)}\delta\beta_{(1)}\delta\beta_{(2)} + \beta_{(1)}\delta\beta_{(2)}\delta\beta_{(1)}. \end{aligned}$$

Then, we put

$$(3.8)\qquad \alpha\beta_{(3)} = a\beta_{(1)}\delta\beta_{(1)}\delta\beta_{(2)} + b\beta_{(1)}\delta\beta_{(2)}\delta\beta_{(1)} + c\delta\beta_{(1)}\beta_{(2)}\delta\beta_{(1)}$$

for some $a, b, c \in \boldsymbol{Z}/3$. Since $\theta(\beta_{(3)}) \in [M,M]_{44} = \boldsymbol{Z}/3\{\alpha^{11}, ix_{45}j\}$, we put

$$\theta(\beta_{(3)}) = m\alpha^{11} + nix_{45}j$$

for $m, n \in \boldsymbol{Z}/3$, and see that $\delta\theta(\beta_{(3)}) = m\delta\alpha^{11}$. By (3.1), (3.4) and (3.8), $\alpha\beta_{(3)}\alpha - \alpha^2\beta_{(3)} = \alpha(\alpha\delta\theta(\beta_{(3)}) - \delta\theta(\beta_{(3)})\alpha) = m(\alpha^2\delta\alpha^{11} - \alpha\delta\alpha^{12}) = m(-\delta\alpha^{13} + \alpha^{13}\delta)$,

$$\alpha\beta_{(3)}\alpha = a(\beta_{(1)}\delta)^4\beta_{(1)} \quad\text{and}\quad \alpha^2\beta_{(3)} = c(\beta_{(1)}\delta)^4\beta_{(1)}.$$

Since $[M,M]_{51} = \boldsymbol{Z}/3\{\delta\alpha^{13}, \alpha^{13}\delta, (\beta_{(1)}\delta)^4\beta_{(1)}, \delta\beta_{(2)}\delta\beta_{(2)}\delta\}$, we see that $a = c$ and $m = 0$. On the other hand,

$$\begin{aligned} (a+b)\alpha_1\beta_1^2\beta_2^2 &= \alpha_1\beta_2(a\beta_1^2\beta_2 + b\beta_1^2\beta_2) \\ &= \alpha_1\beta_2 j(a\beta_{(1)}\delta\beta_{(1)}\delta\beta_{(2)} + b\beta_{(1)}\delta\beta_{(2)}\delta\beta_{(1)} + c\delta\beta_{(1)}\beta_{(2)}\delta\beta_{(1)})i \\ &= \alpha_1\beta_2 j\alpha\beta_{(3)}i \quad (\text{by (3.8)}) \quad = \beta_2\alpha_1 j\alpha\beta_{(3)}i = j\beta_{(2)}\delta\alpha\delta\alpha\beta_{(3)}i \\ &= j\beta_{(2)}\alpha\delta\alpha\delta\beta_{(3)}i \quad (\text{by (3.1)(ii)}) \quad = j(\beta_{(1)}\delta)^3\alpha\delta\beta_{(3)}i \\ &= \beta_1^3\alpha_1\beta_3 = 0 \quad (\text{since } \alpha_1\beta_1^3 \in \pi_{33}(S^0) = 0) \end{aligned}$$

in $\pi_{75}(S^0) = \boldsymbol{Z}/3\{\alpha_{19}\} \oplus \boldsymbol{Z}/9\{x_{75}\}$, where $3x_{75} = \alpha_1\beta_1^2\beta_2^2$. It follows that $b = -a$. If $a = 0$, then $\beta_{(3)}i \in \pi_{43}(M)$ is pulled back to $v_2^3 \in \pi_*(V_1)$ under the map j_{1*} and so $v_2^3 \in E_2^{0,48}(L_2V_1)$ is a permanent cycle, which contradicts to [9, Prop. 8.4]. Here, L_2 denotes the Bousfield-Ravenel localization functor with respect to $v_2^{-1}BP$. Therefore, $a \neq 0$.

The second equation follows from by (3.7). q.e.d.

Remark. In this proof, we also show that $\delta\theta(\beta_{(3)}) = 0 = \theta(\beta_{(3)})\delta$, and so $\beta_{(3)}\delta + \delta\beta_{(3)} = \beta_3 \wedge 1_M$.

Note that $\pi_*(S^0)$ is a commutative ring and $i\varepsilon = \delta\beta_{(1)}\beta_{(2)}i$ by (3.2).

Corollary 3.9. *$\alpha\beta'_3 = i\varepsilon\beta_1 \in \pi_{43}(M)$ up to sign.*

Corollary 3.10. *In the Adams-Novikov spectral sequence for $\pi_*(V_1)$,*

$$d_5(v_2^3) = \pm i_{1*}i(\alpha_1\beta_{3/3}\beta_1) \in E_5^{5,52}(V_1).$$

Proof. Since $j_{1*}(v_2^3) = \beta'_3 \in E_2^{1,44}(M)$ is a permanent cycle in the Adams-Novikov spectral sequence, $i_{1*}\alpha_*(\beta'_3) = i_{1*}(\alpha\beta'_3)$ must be killed by v_2^3, and the corollary follows from Corollary 3.9, since $\varepsilon \in \pi_{37}(S^0)$ is detected by $\alpha_1\beta_{3/3} \in E_2^{3,40}(S^0)$. q.e.d.

Lemma 3.11. *For the maps δ and δ_3 in (2.3), we have $\delta\delta_3(v_1^2v_2h_0) = h_0b_0$ in $E_2^*(S^0)$.*

Proof. Note that v_2h_0 in $E_2^1(V_2)$ is represented by $v_2t_1 - v_1t_2 + v_1t_1^4$. Then a routine computation with (2.1) shows $\delta_3(v_1^2v_2h_0) = v_1b_0$, whose δ-image is h_0b_0, since $\delta(v_1) = h_0$ and $\delta(b_0) = 0$ by (2.1). q.e.d.

Recall [13] the Toda differential

(3.12) $$d_5(\beta_{3/3}) = \pm\alpha_1\beta_1^3 = \pm h_0b_0^3 \in E_2^{5,38}(S^0).$$

Lemma 3.13. *For $v_2^3 \in E_2^{0,48}(V_3)$ in (2.4), $d_5(v_2^3) = i_{3*}i_*(\alpha_1\beta_{3/3}\beta_1) \pm v_1^2v_2h_0\beta_1^2 \in E_2^{5,52}(V_3)$ up to sign.*

Proof. By Lemma 2.16, we put

(3.14) $$d_5(v_2^3) = ai_{3*}(\beta'_2\beta_1^2) + bi_{3*}i_*(\alpha_1\beta_{3/3}\beta_1) + cv_1^2v_2h_0\beta_1^2 \in E_2^{5,52}(V_3)$$

for integers $a, b, c \in \boldsymbol{Z}/3$. Consider the cofiber sequence

(3.15) $$\Sigma^4V_2 \xrightarrow{\widetilde{\alpha}} V_3 \xrightarrow{\varphi_{31}} V_1 \xrightarrow{\delta_{12}} \Sigma^5V_2$$

obtained by Verdier's axiom from the cofiber sequences of (1.2). Send the equation (3.14) to $E_2^5(V_1)$ under φ_{31}, and we see that $a = 0$ and $b = \pm1$ by Corollary 3.10.

Next send (3.14) to $E_2^*(S^0)$ under the maps δ_3 and δ in (2.3). Then we obtain $d_5(\beta_{3/3}) = ch_0b_0^3$ in $E_2^*(S^0)$ by Lemma 3.11, and the Toda differential (3.12) shows that $c = \pm1$ as desired. q.e.d.

Lemma 3.16. *$d_5(v_2^{9t+3}) = \pm\beta'_{9t+2}b_0^2 \in E_2^{5,144t+52}(V_1)$ for $t > 0$ and $d_5(v_2^{9t+6}) = \pm\beta'_{9t+5}b_0^2 \in E_2^{5,144t+100}(V_1)$ for $t \geq 0$.*

Proof. In this proof, we compute everything up to sign. By Lemma 3.13, $d_5(v_2^{9t+3}) \equiv v_2^{9t}h_0b_1b_0 + v_1^2v_2^{9t+1}h_0b_0^2 \in E_2^{5,144t+52}(V_3)$, and so $d_5(v_2^{9t+3}) \equiv v_2^{9t}h_0b_1b_0 \in E_2^{5,144t+52}(V_1)$. The cochain $v_2^{9t-3}v_3b_{10}b_{11}$ yields the relation $v_2^{9t}h_0b_0b_1 = v_2^{9t-2}h_2b_0b_1$ of homology, which equals $v_2^{9t-2}h_1b_1^2$ by $d(v_2^{9t-2}b_{20}b_1)$ with (2.2). Then the first relation follows from Lemma 2.18, since $\beta'_{9t+2} = v_2^{9t+1}h_1$.

The second one follows similarly from $d_5(v_2^{9t+6}) \equiv -v_2^{9t+3}h_0b_1b_0 \in E_2^{5,144t+100}(V_1)$. q.e.d.

Lemma 3.17. *$d_9(v_1v_2^3) = \pm h_1b_0^4$ in $E_9^{9,60}(V_3)$, and $v_1^2v_2^3$ is a permanent cycle in $E_r^{0,56}(V_4)$.*

Proof. Consider the cofiber sequence (3.15). Lemma 3.13 shows that $d_5(v_2^3) = \pm h_0b_0b_1$ in $E_2^{5,52}(V_2)$. Let g_0 be the element defined by the Massey product $\langle h_0, h_0, h_1\rangle$, which contains an element represented by the cochain $t_1 \otimes t_2 - t_1^2 \otimes t_1^3$. Then, $d(t_1 \otimes t_2 - t_1^2 \otimes t_1^3) = v_1t_1 \otimes b_{10}$ in the cobar complex $\Omega^3BP_*/(3)$, which shows that $\delta_{12*}(g_0) = h_0b_0$ for the connecting homomorphism δ_{12*}. It follows that $d_5(v_2^3) = \delta_{12*}(g_0b_1)$. Furthermore, $d_5(g_0b_1) = g_0d_5(b_1) = \pm g_0h_0b_0^3 = \pm h_1b_0^4$, since b_1 belongs to the E_2-term $E_2^2(S^0)$ and $g_0 \in E_2^2(V_1)$ is a permanent cycle. Therefore, $d_9(v_1v_2^3) = d_9(\widetilde{\alpha}_*(v_2^3)) = \pm h_1b_0^4 \in E_2^{9,60}(V_3)$.

Send the relation to V_4 under the map $\widetilde{\alpha}\colon V_3 \to V_4$ obtained in the same manner as the one in (3.15), and we have $d_9(v_1^2v_2^3) = \pm v_1h_1b_0^4 = 0$, since $v_1h_1 = v_1^3h_0$ in the E_2-term by $d(v_2)$ and $v_1^3h_0b_0^3 = 0$ in the E_9-term by the Toda differential. The elements $\alpha_1 = h_0 \in E_2^{1,4}(S^0)$ and $\beta_1 = b_0 \in E_2^{2,12}(S^0)$ gives the vanishing line: $E_2^{s,t}(X) = 0$ if $t < 6s - 2$ for a connected spectrum X. It follows that $E_2^{s,s+55}(V_4) = 0$ for $s > 9$, and $v_1^2v_2^3$ is a permanent cycle. q.e.d.

In [5, Lemma 3], Oka showed the existence of the element $[\widetilde{\alpha}\beta^6] \in \pi_{100}(V_2)$ detected by $v_1v_2^6 \in E_2^{0,100}(V_2)$.

Lemma 3.18. *The element $v_1v_2^6 \in E_2^{0,100}(V_4)$ is a permanent cycle, which detects an element $[\widetilde{\alpha}\beta^6] \in \pi_{100}(V_4)$.*

Proof. From Ravenel's table [7, Table A.3.4], we read off the existence of $\beta'_{6/3} \in \pi_{83}(M)$. Then $\alpha^2\beta'_{6/3}$ belongs to $\pi_{91}(M)$, which is a $\boldsymbol{Z}/3$-module generated by β'_6, $i\beta_1\gamma_2$, $i\beta_1x_{81}$ and $i\alpha_{23}$ by the table. We may put $\alpha^2\beta'_{6/3} = \beta'_6 + ai\beta_1\gamma_2 + bi\beta_1x_{81}$ for some $a, b \in \boldsymbol{Z}/3$, since β'_6 and $i\alpha_{23}$ are detected by elements of $E_2^{1,92}(M)$, and $\alpha^2\beta'_{6/3} = \beta'_6$ in the E_2-term $E_2^{1,92}(M)$. On the other hand, Oka's result shows that $\alpha^2\beta'_6 = \alpha^2 j_{2*}(v_1v_2^6) = 0$. Besides, $\alpha^2 i\beta_1 = \alpha^2\delta j_1[\beta i_1]i = -(\delta\alpha^2 + \alpha\delta\alpha)j_1[\beta i_1]i = 0$ by (3.1). It follows that $\alpha^4\beta'_{6/3} = \alpha^2(\beta'_6 + ai\beta_1\gamma_2 + bi\beta_1x_{81}) = 0$, and $\beta'_{6/3}$ is pulled back to $v_1v_2^6 \in \pi_{100}(V_4)$ under the map $j_{4*}\colon \pi_{100}(V_4) \longrightarrow \pi_{83}(M)$. q.e.d.

4. On the homotopy group $\pi_{143}(M)$

For proving Lemma 4.5 below, we read off the homotopy groups $\pi_k(M \wedge X_1)$ from Ravenel's table [7, Table A.3.4]. Here M and X_1 are the mod 3 Moore spectrum and Ravenel's spectrum considered in section two. Put $\overline{X_1} = T(0)_1$, and we have the cofiber sequence

(4.1) $$S^3 \xrightarrow{\alpha_1} S^0 \xrightarrow{\iota_1} \overline{X_1} \xrightarrow{\kappa_1} S^4.$$

We read off the homotopy groups $\pi_*(M)$ from Ravenel's table [7, Table A.3.4] and we obtain the homotopy groups $\pi_*(M \wedge \overline{X_1})$ as in the following tables. In the tables, $\xi' \in \pi_{s+1}(M)$ for an element $\xi \in \pi_s(S^0)$ denotes an element such that $j_*(\xi') = \xi$. We further notice the relations

(4.2) $$\alpha_{1*}(\alpha^{3k}i) = \alpha^{3k}\delta\alpha i = i\alpha_{3k+1}, \quad \alpha_{1*}(\alpha^{3k+1}i) = \alpha^{3k+1}\delta\alpha i = -\alpha^{3k}(\delta\alpha^2 + \alpha^2\delta)i = i\alpha_{3k+2},$$

by (3.1) and $\alpha_{1*}(\alpha^{3k-1}i) = i\alpha_{3k/\nu+2}$ by $v_1^{3k-1}h_0 = i_*(\alpha_{3k/\nu+2})$ in the E_2-term for $\nu = \max\{n : 3^n|k\}$.

dimension k	$\pi_{k-8}(M)$	$\pi_{k-3}(M)$	$\pi_k(M)$	$\pi_{k-4}(M)$	$\pi_{k-1}(M)$	$\pi_k(M \wedge \overline{X_1})$
106	0	$i\alpha_{26}$ $\beta'_{6/3}\beta_1^2$ $x'_{92}\beta_1$	ix_{106}	$i\beta_{6/3}\beta_1^2, ix_{92}\beta_1,$ $x'_{81}\beta_1^2, \gamma'_2\beta_1^2$	$\gamma'_2\beta_1^2\alpha_1$	$\iota_1 ix_{106}, i\beta_{9/9},$ $h_{20}\gamma_2\beta_1,$ $\langle\beta_{(1)}, \beta_{(1)}, \beta'_{6/3}\rangle$
96	$\alpha^{22}i$	$ix_{93},$ $x'_{92},$ $\beta'_{6/3}\beta_1$	$\alpha^{24}i,$ $\beta'_{6/3}\beta_1\alpha_1$	$\alpha^{23}i, ix_{92},$ $i\beta_1\beta_{6/3},$ $\gamma'_2\beta_1,$ β'_1x_{81}	$i\alpha_{24/2},$ $i\alpha_1\beta_1\beta_{6/3},$ $\gamma'_2\beta_1\alpha_1,$ $\beta'_1x_{81}\alpha_1$	$h_{20}\gamma_2,$ $\iota_1\alpha^{24}i$
86	$i\alpha_1x_{75}$ $= ix_{68}\beta_1$	$i\alpha_{21/2},$ $\beta'_{6/3}$	$\beta'_1x_{75},$ $i\beta_{6/2},$ $\beta'_{6/3}\alpha_1$	$\gamma'_2,$ $x'_{81},$ $i\beta_{6/3}$	$\gamma'_2\alpha_1,$ $x'_{81}\alpha_1,$ $i\alpha_1\beta_{6/3},$ $i\beta_1x_{75}$	$\iota_1\beta'_1x_{75},$ $\iota_1 i\beta_{6/2}$
76	$\alpha^{17}i,$ ix_{68}	$\beta'_2\beta_2\beta_1^2$	$\alpha^{19}i,$ $\beta'_2\beta_2\beta_1^2\alpha_1$	$\alpha^{18}i,$ $i\beta_1^2\beta_2^2$	$i\alpha_{19},$ $ix_{75},$ β'_5	$\iota_1\alpha^{19}i,$ $\overline{\iota_1x'_{75}}$

Here, we use the relations $\kappa_{1*}(i\beta_{9/9}) = i\beta_{6/3}\beta_1^2$, $\kappa_{1*}(\langle\beta_{(1)}, \beta_{(1)}, \beta'_{6/3}\rangle) = x'_{81}\beta_1^2$ and $\kappa_{1*}(h_{20}\gamma_2) = ix_{92}$ given in [7, Th. 7.5.3] in the dimensions 106 and 96. In the dimension 76, $\overline{\iota_1x'_{75}} \in \pi_{76}(M \wedge \overline{X_1})$ denotes an element detected by $\iota_{1*}(x'_{75}) \in E_2^{4,80}(M \wedge \overline{X_1})$ for $x'_{75} \in E_2^{4,80}(M)$ such that $\delta(x'_{75}) = x_{75} \in E_2^{5,80}(S^0)$, where δ is the connecting homomorphism in (2.3). The element $x'_{75} \in E_2^{4,80}(M)$ supports the Adams-Novikov differential $d_5(x'_{75}) = i\alpha_1\beta_1^2\beta_2^2$, and $\kappa_{1*}(\overline{\iota_1x'_{75}}) = i\beta_1^2\beta_2^2$.

dimension k	$\pi_{k-8}(M)$	$\pi_{k-5}(M)$	$\pi_{k-7}(M)$	$\pi_{k-4}(M)$	$\pi_{k-4}(M \wedge \overline{X_1})$	$\pi_k(M)$
99	$i\alpha_{23}, \beta'_6,$ $i\gamma_2\beta_1,$ $ix_{81}\beta_1$	$\beta'_6\alpha_1,$ $i\alpha_1\beta_1\gamma_2,$ $i\beta_5\beta_1^2$	$\alpha^{23}i, ix_{92}$ $i\beta_1\beta_{6/3},$ $\gamma'_2\beta_1, x'_{81}\beta_1,$	$i\alpha_{24/2},$ $i\alpha_1\beta_1\beta_{6/3},$ $\gamma'_2\beta_1\alpha_1, \beta'_5\beta_1^2$	$\overline{\alpha_{23}}$	$i\alpha_{25},$ ix_{99}
89	$i\gamma_2,$ ix_{81}	$\alpha^{21}i,$ $i\alpha_1\gamma_2,$ $i\alpha_1x_{81}$	$i\beta_{6/3},$ $\gamma'_2,$ x'_{81}	$i\alpha_1\beta_{6/3}$ $\gamma'_2\alpha_1,$ $x'_{81}\alpha_1,$ $i\beta_1x_{75}$	$\iota_1 i\beta_1x_{75}$	0
79	$i\alpha_{18/3}$	$i\beta_5$	$\alpha^{18}i,$ $i\beta_1^2\beta_2^2$	$i\alpha_{19},$ $ix_{75},$ β'_5	$\iota_1 ix_{75},$ $\iota_1\beta'_5,$ $\overline{\alpha_{18/3}}$	$x'_{75}\alpha_1,$ $i\alpha_{20}$

Here $x_{99} = \langle \alpha_1, \alpha_1, x_{92} \rangle$ and $x_{81}\alpha_1 = \beta_5\beta_1$, and $\overline{\alpha}_s$ denotes an element such that $\kappa_1(\overline{\alpha}_s) = i\alpha_s$.

Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & \pi_{k-7}(M) & = & \pi_{k-7}(M) & & \\
 & & \downarrow{\scriptstyle \lambda'} & & \downarrow{\scriptstyle \alpha_1} & & \\
\pi_k(M) & \xrightarrow{\iota_1} & \pi_k(M \wedge \overline{X_1}) & \xrightarrow{\kappa_1} & \pi_{k-4}(M) & \xrightarrow{\alpha_1} & \pi_{k-1}(M) \\
\| & & \downarrow{\scriptstyle \iota'} & & \downarrow{\scriptstyle \iota_1} & & \| \\
\pi_k(M) & \xrightarrow{\iota} & \pi_k(M \wedge X_1) & \xrightarrow{\kappa} & \pi_{k-4}(M \wedge \overline{X_1}) & \xrightarrow{\lambda} & \pi_{k-1}(M) \\
 & & \downarrow{\scriptstyle \kappa'} & & \downarrow{\scriptstyle \kappa_1} & & \\
 & & \pi_{k-8}(M) & = & \pi_{k-8}(M) & &
\end{array}$$

induced from the cofiber sequences in (2.8) and (4.1), in which ι, ι', κ and κ' denote $\iota_{0,0}$, $\iota'_{0,0}$, $\kappa_{0,0}$ and $\kappa'_{0,0}$ in (2.8). We notice that $\lambda\lambda' = \beta_1 \wedge M$ and $\lambda'\lambda = \beta_1 \wedge M \wedge \overline{X_1}$. Then we obtain the following lemma from the above tables:

Lemma 4.3. *The homotopy groups $\pi_k(M \wedge X_1)$ are as follows:*

1. $\pi_{106}(M \wedge X_1) = \boldsymbol{Z}/3\{\iota i x_{106}, \iota' i\beta_{9/9}, \iota'\langle \beta_{(1)}, \beta_{(1)}, \beta'_{6/3}\rangle\}$.
2. $\pi_{99}(M \wedge X_1) = \boldsymbol{Z}/3\{\iota i x_{99}, \overline{\alpha_{23}}\}$. *Here,* $x_{99} = \langle \alpha_1, \alpha_1, x_{92}\rangle$.
3. $\pi_{96}(M \wedge X_1) = \boldsymbol{Z}/3\{\iota\alpha^{24} i, \iota' h_{20}\gamma_2\}$.
4. $\pi_{89}(M \wedge X_1) = \boldsymbol{Z}/3\{\iota'\eta_3\}$. *Here,* $\kappa_{1*}(\eta_3) = i\beta_1 x_{75}$.
5. $\pi_{86}(M \wedge X_1) = \boldsymbol{Z}/3\{\iota\beta'_1 x_{75}, \iota i\beta_{6/2}\}$
6. $\pi_{79}(M \wedge X_1) = \boldsymbol{Z}/3\{\iota' u'_2, \overline{\alpha_{18/3}}\}$
7. $\pi_{76}(M \wedge X_1) = \boldsymbol{Z}/3\{\iota\alpha^{19} i, \overline{\iota x'_{75}}\}$

Here, $\overline{\alpha_s}$ denotes an element such that $\kappa'_*(\overline{\alpha_s}) = i\alpha_s$ for the projection $\kappa' \colon M \wedge X_1 \to M$ to the top cells, and $\overline{\iota x'_{75}} \in \pi_{76}(M \wedge X_1)$ denotes an element detected by $\iota_*(x'_{75}) \in E_2^{4,80}(M \wedge X_1)$ for $x'_{75} \in E_2^{4,80}(M)$.

Proof. From [7, Th. 7.5.3], we read off the relations

$$\kappa_*(\iota'\eta_3) = \iota_1 i\beta_1 x_{75}, \quad \kappa_*(u_2) = \beta_5 \quad \text{and} \quad \kappa'_*(b_{20}\beta_2) = x_{68}.$$

In dimensions 99, 96, 79 and 76, we use the relation (4.2). Furthermore, in dimension 106, $\lambda'(ix_{99}) = h_{20}\gamma_2\beta_1$, since $\kappa_{1*}(h_{20}\gamma_2\beta_1) = ix_{92}\beta_1 = i\alpha_1 x_{99} = \kappa_1\lambda' i x_{99}$, and so $\iota'(h_{20}\gamma_2\beta_1) = \iota'\lambda'(ix_{99}) = 0$. The element $\lambda'(i) \in \pi_7(M \wedge \overline{X_1})$ satisfies $\kappa_1\lambda'(i) = i\alpha_1$ and $\lambda\lambda'(i) = i\beta_1$. This together with $\beta_1 x_{75} \neq 0 \in \pi_*(S^0)$, we see that $i\alpha_1 x_{75}$ is not pulled back to $\pi_{86}(M)$ under the map κ'. In dimension 79, note that $\lambda_*(\iota_1 i x_{75}) = ix_{75}\alpha_1 = i\beta_1 x_{68} \neq 0$. We also see that ix_{68} is not pull back to $\pi_{76}(M \wedge X_1)$ under κ', since $\lambda' i x_{68} \neq 0$ by $\lambda\lambda' i x_{68} = i\beta_1 x_{68} \neq 0$. q.e.d.

If $\xi \in \pi_*(X)$ for a spectrum X is detected by an element $x \in E_2^s(X)$, then we write filt $\xi = s$. Let H_t^s denote the subgroup of $\pi_t(M \wedge X_1)$ generated by the elements with filt $\xi \geq s$.

Lemma 4.4. *The subgroups H_t^s of the homotopy groups $\pi_t(M \wedge X_1)$ are as follows:*

1. $H^8_{140} = 0$.
2. $H^7_{133} \subset \boldsymbol{Z}/3\{b_1\iota i x_{99}\}$.
3. $H^6_{130} \subset \boldsymbol{Z}/3\{b_1\iota' h_{20}\gamma_2\}$.
4. $H^5_{123} \subset \boldsymbol{Z}/3\{v_2 h_{20} b_{20}^2, b_1\iota'\eta_3\}$.
5. $H^4_{120} \subset \boldsymbol{Z}/3\{b_1\iota\beta'_1 x_{75}, b_1\iota i\beta_{6/2}\}$.
6. $H^3_{113} = \pi_{113}(M \wedge X_1) \subset \boldsymbol{Z}/3\{b_1\iota' u'_2\}$.
7. $H^2_{110} = \pi_{110}(M \wedge X_1) \subset \boldsymbol{Z}/3\{v_1 b_2, b_1\overline{\iota x'_{75}}\}$.

Proof. Observe the spectral sequence (2.15) for $k = 1$, and we have a spectral sequence

$$\pi_t(M \wedge X_2) \oplus h_1\pi_{t-11}(M \wedge X_2) \oplus b_1\pi_{t-34}(M \wedge X_1) \Longrightarrow \pi_t(M \wedge X_1).$$

In particular,

$$K_t^s \oplus h_1 K_{t-11}^{s-1} \oplus b_1 H_{t-34}^{s-2} \Longrightarrow H_t^s.$$

Here, K_t^s denotes the subgroup of $\pi_t(M \wedge X_2)$ generated by the elements with filt $\xi \geq s$. By Corollary 2.13, we see that $H^8_{140} \subset b_1 H^6_{106}$, $H^7_{133} \subset b_1 H^5_{99}$ and $H^6_{130} \subset b_1 H^4_{96}$. Since $H^6_{106} = 0$, $H^5_{99} = \boldsymbol{Z}/3\{\iota i x_{99}\}$ and

$H^4_{96} = \boldsymbol{Z}/3\{\iota' h_{20}\gamma_2\}$ by Lemma 4.3, we obtain the first three inclusions of the lemma. By Lemma 2.12, the above spectral sequences for the other homotopy groups are:

$$\boldsymbol{Z}/3\{v_2h_{20}b_{20}^2\} \oplus b_1H^3_{89} \Longrightarrow H^5_{123}, \quad \boldsymbol{Z}/3\{v_2h_1h_{21}b_{20}, v_2^3h_1h_{20}b_{20}\} \oplus b_1H^2_{86} \Longrightarrow H^4_{120},$$
$$\boldsymbol{Z}/3\{v_1v_2^3h_1h_2h_{20}\} \oplus b_1\pi_{79}(M\wedge X_1) \Longrightarrow \pi_{113}(M\wedge X_1) \quad \text{and}$$
$$\boldsymbol{Z}/3\{v_1b_2, v_2^3h_{20}h_{21}, v_2^4b_{20}, v_1^{13}v_2^2h_1h_{20}, v_1^{17}v_2h_1h_{20}, v_1^{21}h_1h_{20}, v_2^4h_1h_2\} \oplus b_1\pi_{76}(M\wedge X_1) \Longrightarrow \pi_{110}(M\wedge X_1).$$

Let g_1 denote an element represented by the cocycle $t_1^3\otimes t_2^3 - t_1^6\otimes t_1^9 + v_1^3b_{20}$ in the cobar complex. Then $v_2h_1h_{21}b_{20}$ is replaced by $v_2b_{20}g_1$. Since $d_3(v_2b_{20} - h_1h_{30}) = v_2h_1b_1$, $d_3(v_2h_1h_{21}b_{20}) = d_3(v_2b_{20}g_1) = v_2h_1b_1g_1 \equiv v_2h_2b_1^2$ mod (v_1^3). Indeed, $g_1 = \langle h_1, h_1, h_2\rangle$ mod (v_1^3). Thus, the first element of the second spectral sequence dies. For the second element, we see the essential differential $d_2(v_2^3h_1h_{20}b_{20}) = v_1^3h_2h_1h_{20}b_{20} = -v_1^2v_2h_2h_{20}h_1b_1$, since $v_1^3b_{20} = -h_{20}h_2 - v_1^2v_2b_1$. In the third spectral sequence, $d_1(v_2^4h_2h_{20}) = v_1v_2^3h_1h_2h_{20}$, and in the last spectral sequence, $d_1(v_2^3h_{20}h_{21}) = v_2^3h_{20}h_1h_2$, $d_2(v_2^4b_{20} + v_2^3h_{20}h_{21}) = v_1^2v_2^3h_2b_1$, $d_1(v_1^{13}v_2^2h_1h_{20}) = v_1^{15}h_{20}b_1$, $d_1(v_1^{16}v_2^2h_{20}) = -v_1^{17}v_2h_1h_{20}$, $d_1(v_1^{20}v_2h_{20}) = v_1^{21}h_1h_{20}$ and $d_1(v_2^4h_{21}) = v_2^4h_1h_2$. Therefore,

$$H^5_{123} \subset \boldsymbol{Z}/3\{v_2h_{20}b_{20}^2\} \oplus b_1H^3_{89}, \quad H^4_{120} \subset b_1H^2_{86}, \quad \pi_{113}(M\wedge X_1) \subset b_1\pi_{79}(M\wedge X_1) \quad \text{and}$$
$$\pi_{110}(M\wedge X_1) \subset \boldsymbol{Z}/3\{v_1b_2\} \oplus b_1\pi_{76}(M\wedge X_1).$$

We observe that $\iota\alpha^{19}ib_1 = \iota\alpha^{18}\varepsilon'\alpha_1 = \iota\alpha^{18}\beta_{(1)}\beta_{(2)}i\alpha_1 = 0$ by (3.1) and (3.2). We further see that $\overline{\alpha_{18/3}}b_1 = v_1^{15}b_1\widetilde{v_2h_0} = 0$, since $v_1^6b_1 = v_1^3h_1h_2 = 0$. Here, $\overline{\alpha_{18/3}} = v_1^{15}\widetilde{v_2h_0}$ for an element $\widetilde{v_2h_0}$ by the definition of $\overline{\alpha_{18/3}}$. The lemma now follows from Lemma 4.3. q.e.d.

Lemma 4.5. *Each essential element of $\pi_{143}(M)$ has the Adams-Novikov filtration less than nine.*

Proof. Let G^s_t denotes the subgroup of $\pi_t(M)$ consisting of elements ξ with filt $\xi \ge s$. As above, we have another spectral sequence

$$\beta_1^4\pi_{103}(M) \oplus \bigoplus_{\varepsilon+2s\le 7, \varepsilon=0,1, s\ge 0} \alpha_1^\varepsilon\beta_1^sH^{9-2s-\varepsilon}_{143-10s-3\varepsilon} \Longrightarrow G^9_{143}$$

arising from the spectral sequence (2.15) for $k = 0$. Indeed, comparing with (2.11) for $k = 0$, we see that the above spectral sequence detects G^9_{143}. From Ravenel's table [7], we read off the homotopy group $\pi_{103}(M)$ is $\boldsymbol{Z}/3$-module generated by $i\alpha_{26}$, $\beta_{(1)}i\beta_1\beta_{6/3}$ and $\gamma_{(2)}\beta_{(1)}i\beta_1$, where $\gamma_{(2)}$ denotes an element such that $j\gamma_{(2)}i = \gamma_2$. Note that Ravenel wrote x_{92} for $j\gamma_{(2)}\beta_{(1)}i$. It is well known that $\beta_{(1)}i\beta_1^5 = 0$ and so $\beta_1^6 = j\beta_{(1)}i\beta_1^5 = 0$ (see Lemma 3.3). Besides, $\alpha_{26}\beta_1 = j\alpha^{26}\delta\beta_{(1)}i = 0$ by (3.1). Thus, the first summand of the E_1-term is zero.

Next we evaluate $\pi_{113}(M)$. Similarly consider a part of the above spectral sequence:

$$\beta_1\pi_{103}(M) \oplus \alpha_1\pi_{110}(M\wedge X_1) \oplus \pi_{113}(M\wedge X_1) \Longrightarrow \pi_{113}(M).$$

Then, the generators v_1b_2 and $b_1\overline{\iota x'_{75}} \in \pi_{110}(M\wedge X_1)$ are pulled back to $\pi_{110}(M\wedge \overline{X_1})$, and we have $\lambda_*(v_1b_2) = v_1h_0b_2$ and $\lambda_*(b_1\overline{\iota_1x'_{75}}) = \varepsilon' x_{75}$ in $\pi_{113}(M)$, since $\lambda_*(\overline{\iota_1x'_{75}}) = \alpha ix_{75}$. The element h_0b_2 is represented by $\langle\alpha_1, \alpha_1, \beta^3_{3/3}\rangle$, since $d_5(b_2) = h_0b_1^3$ in the Adams-Novikov spectral sequence [6], and $\alpha_1 = h_0$ and $\beta_{3/3} = b_1$ in the E_2-term. Turn to the generators of $\pi_{113}(M\wedge X_1)$. $d_1(\iota' u'_2b_1) = \iota'\kappa_*(\iota' u'_2b_1) = \iota\beta'_5b_1$, which is detected by $v_1^2v_2^2h_2 \in \pi_{75}(M\wedge X_2)$ in the spectral sequence (2.15) for $k = 2$. No generator of $\pi_{99}(M\wedge X_2)$ and $\pi_{110}(M\wedge X_2)$ hits any of β'_5 and $\lambda_{0,1_*}\beta'_5$ for the boundary $\lambda_{0,1_*}\colon \pi_{75}(M\wedge X_1) \to \pi_{98}(M\wedge \overline{X_2})$ induced from the map in (2.8). Indeed, the relevant generators of $\pi_*(M\wedge X_2)$ are $v_2^4h_2$, v_1b_2, $v_2^4b_{20}$ and $v_2^3h_{20}h_{21}$, and none of the differentials on them hits $v_1^2v_2h_2 \in \pi_{75}(M\wedge X_2)$ in (2.15) for $k = 2$. Therefore, $d_1(\iota' u'_2b_1) = \iota\beta'_5b_1 \ne 0$. These argument shows that $\pi_{113}(M) \subset \boldsymbol{Z}/3\{i\langle\alpha_2, \alpha_1, \beta_1^2\beta_{6/3}\rangle, \varepsilon' x_{75}, \beta_{(1)}i\beta_1^2\beta_{6/3}, \gamma_{(2)}\beta_{(1)}i\beta_1^2\}$. Here $v_1h_0b_2$ represents $\alpha i\langle\alpha_1, \alpha_1, \beta^3_{3/3}\rangle = i\langle\alpha_2, \alpha_1, \beta_1^2\beta_{6/3}\rangle$.

We next consider

$$\beta_1\pi_{113}(M) \oplus \alpha_1\pi_{120}(M\wedge X_1) \oplus H^5_{123} \Longrightarrow G^5_{123}.$$

In the spectral sequence, $d_1(v_2h_{20}b_{20}^2) = h_0h_1h_{21}b_{20}h_{20} + \dots$ and $v_2h_{20}b_{20}^2$ dies in G^5_{123}. The localization map $E_2^*(M) \to E_2^*(L_2M)$ assigns η_3 to $\beta_6\zeta_1$ by [7, (7.5.7)], and we have the Adams-Novikov differential $d_5(\beta_6b_1\zeta_2) = \beta_6h_0b_0^2\zeta_2 \ne 0$ by [9, Prop. 9.9, Cor. 10.4]. It follows that $b_1\eta_3 \in \pi_{123}(M\wedge X_1)$ also dies in the above spectral sequence. Noticing that $b_1\iota x = \iota'\iota_1b_1x$ and $\lambda\iota_1b_1x = \alpha_1b_1x = \varepsilon x$, we obtain

$$G^5_{123} \subset \boldsymbol{Z}/3\{\iota\beta'_1\varepsilon x_{75}, \iota i\beta_{6/2}\varepsilon, \langle i\alpha_2, \alpha_1, \beta_1^3\beta_{6/3}\rangle, \varepsilon'\beta_1x_{75}, \beta_{(1)}i\beta_1^3\beta_{6/3}, \gamma_{(2)}\beta_{(1)}i\beta_1^3\}.$$

Here, $\langle i\alpha_2, \alpha_1, \beta_1^3\beta_{6/3}\rangle = i\alpha\varepsilon'\beta_{6/3}$. Since $\varepsilon'\beta_1^2 = 0$ and $\beta_{(1)}\delta\beta_1^5 = 0$, we see $\beta_1^2G^5_{123} = 0$.

The element ιix_{99} is detected by $\langle h_0, h_0, x_{92}\rangle$. Since $d_5(b_1) = h_0b_0^3$ by the Toda differential, $d_5(ix_{99}b_1) = ix_{99}h_0b_0^3 = -ih_0\langle h_0, h_0, x_{92}\rangle b_0^3 = ix_{92}b_0^4 \ne 0$ by Lemma 2.19, and $ix_{99}\beta_{3/3}$ dies in the Adams-Novikov spectral sequence for computing the homotopy group $\pi_{133}(M)$. Besides, by observing $\pi_{99}(M)$, we see that $\lambda_*(h_{20}\gamma_2) = aix_{99}$ for some $a \in \boldsymbol{Z}/3$ if $h_{20}\gamma_2$ is a permanent cycle in (2.15) for $k = 0$. Indeed, there are two generators in $\pi_{99}(M)$ and the other generator has the filtration degree one. Therefore, $\lambda_*(b_1h_{20}\gamma_2) = aix_{99}\beta_{3/3}$, which is shown above to be zero, and we obtain $G^9_{143} = 0$ as desired. q.e.d.

5. The β-elements in stable homotopy

The cofiber sequences in (1.2) that define V_r induce the cofiber sequence that lies in the commutative diagram

$$\begin{array}{ccccc} & & M & = & M \\ & & \downarrow{\scriptstyle i_9} & & \downarrow{\scriptstyle i_r} \\ V_{9-r} & \xrightarrow{\widetilde{\alpha^r}} & V_9 & \xrightarrow{\varphi_{9r}} & V_r. \end{array}$$

Proof of Lemma 1.3. Put $\xi = \alpha^9 x'_{106} \in \pi_{143}(M)$. By virtue of Lemma 4.5, we assume that the Adams-Novikov filtration of ξ is five. If there exists an element $\chi \in \pi_{107}(M)$ of filtration degree five such that $\alpha^9\chi = \xi$, then $x'_{106} - \chi$ is pulled back to $v_2^9 \pm v_1^8 v_2^7$. Therefore, we replace x'_{106} with $x'_{106} - \chi$, and obtain $\xi = 0$. If no such element exists, then $i_{9*}(\xi) = d_5(v_2^9 \pm v_1^8 v_2^7) \in E_2^{5,148}(V_9)$. Since $i_{3*}(\xi) = (\varphi_{93} i_9)_*(\xi) = \varphi_{93*}(d_5(v_2^9 \pm v_1^8 v_2^7)) = d_5(\varphi_{93*}(v_2^9 \pm v_1^8 v_2^7)) = d_5(v_2^9) = 3v_2^6 d_5(v_2^3) = 0$ in $E_2^{5,148}(V_3)$, we have an element $\chi \in E_2^{5,136}(M)$ such that $v_1^3\chi = \xi$. By Lemma 2.17, χ is killed by v_1^3, and so $\xi = v_1^3\chi = 0$. q.e.d.

Corollary 5.1. *The element $v_2^9 \pm v_1^8 v_2^7 \in E_2^{0,144}(V_9)$ is a permanent cycle in the Adams-Novikov spectral sequence converging to $\pi_*(V_9)$. Besides, it yields a self-map $\Sigma^{144}V_9 \to V_9$ that induces $v_2^9 \pm v_1^8 v_2^7$ on BP_*-homology.*

Corollary 5.2. $d_5(v_2^9) = \pm v_1^8 v_2^5 h_1 b_0^2 \in E_2^{5,148}(V_9)$.

Proof. Suppose that $d_5(v_2^9) = \xi$. By Lemma 3.16, $d_5(v_2^6) = \pm\beta'_5\beta_1^2$, and so $d_5(v_1^8 v_2^7) = v_1^8 v_2 d_5(v_2^6) = \pm v_1^8 v_2^5 h_1 b_0^2$ in $E_5^5(V_9)$. Then, $d_5(v_2^9 \pm v_1^8 v_2^7) = \xi \pm v_1^8 v_2^5 h_1 b_0^2$, which is zero. q.e.d.

We define the β-elements in the homotopy groups not in the E_2-term. For $s = 1, 2$, we state the definition in section three. The β-elements $\beta_{(s)} \in [M, M]_*$ for $s \equiv 0, 1, 2, 5, 6 \bmod 9$ are given as the composites

$$\beta_{(9t)} = j_1[\beta^9]i_1, \quad \beta_{(9t+1)} = j_1[\beta^9]^t[\beta i_1], \quad \beta_{(9t+2)} = [j_1\beta][\beta^9]^t[\beta i_1],$$
$$\beta_{(9t+5)} = j_1[\beta^9]^t[\beta^5 i_1] \quad \text{and} \quad \beta_{(9t+6)} = [j_1\beta][\beta^9]^t[\beta^5 i_1]$$

for the self-map $[\beta^9]\colon \Sigma^{144}V_1 \to V_1$ given in [1]. Here $[x]$ denotes an imaginary element used in [14]. Assuming the existence of the self-map $[\beta^9]\colon \Sigma^{144}V_2 \to V_2$, Oka also gave another definition:

$$\beta_{(9t+s)} = j_2[\beta^9]^t[\widetilde{\alpha}\beta^s]i_2$$

by use of the homotopy elements $[\widetilde{\alpha}\beta^s] \in [V_2, V_2]_{16s+4}$ for $s = 0, 1, 2, 5, 6$ in [5, Lemma 3]. Similarly, we define the β-elements $\beta_{(9t/r)}$ for $0 < r < 9$, $\beta_{(9t+3/r)}$ for $r = 1, 2$ and $\beta_{(9t+6/r)}$ for $r = 1, 2, 3$ with $\beta_{(3s/1)} = \beta_{(3s)}$ as

(5.3) $$\beta_{(9t/r)} = j_r[\beta^9]^t i_r, \quad \beta_{(9t+3/r)} = j_{r+2}[\beta^9]^t[\widetilde{\alpha^2}\beta^3]i_{r+2} \quad \text{and} \quad \beta_{(9t+6/r)} = j_{r+1}[\beta^9]^t[\widetilde{\alpha}\beta^6]i_{r+1}.$$

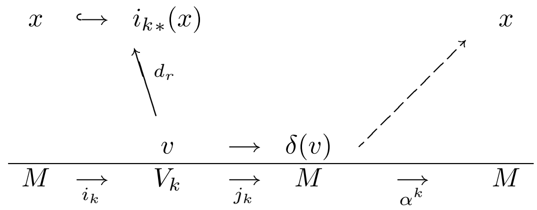
Here, the elements $[\widetilde{\alpha^2}\beta^3]$ and $[\widetilde{\alpha}\beta^6]$ in $[V_r, V_r]_*$ for $r = 2, 3, 4$ denote the self-maps induced from the homotopy elements $v_1^2 v_2^3$ and $v_1 v_2^6$ in Lemma 1.6. We also write $\beta'_s = \beta_{(s)} i \in \pi_*(M)$ and $\beta_s = j\beta'_s \in \pi_*(S^0)$. Note that $\beta'_{9/r} = \alpha^{9-r}x'_{106}$ and $\beta_{9/r} = \langle \alpha_{9-r}, 3, x_{106}\rangle$.

Proof of Corollary 1.7. By the geometric boundary theorem [7], we see that each β-element in (2.5) detects the corresponding element in (5.3). Let $B\colon \Sigma^{144}V_9 \to V_9$ denote the self-map given in Corollary 5.1. Then, the latter half follows from defining the β-elements by $\widetilde{\beta}_{9t/9} = jj_9B^t i_9 i$. q.e.d.

By a diagram chasing on the Adams-Novikov resolutions over the cofiber sequence $M \xrightarrow{i_k} V_1 \xrightarrow{j_k} \Sigma^{4k+1}M \xrightarrow{\alpha^k} \Sigma M$ in (1.2), we obtain the following lemma:

Lemma 5.4. *Suppose that $d_r(v) = i_{k*}(x) \in E_r^r(V_k)$ for an element $v \in E_r^0(V_k)$ and an element $x \in E_r^r(M)$, which detects an essential homotopy element $\xi \in \pi_*(M)$ and that $\delta(v) \in E_2^1(M)$ detects a homotopy element ζ. Then, $\alpha^k\zeta = \xi$.*

The basic idea of the proof is described by the chart:



The same argument as [10] using $\beta_{9t+1}\beta_5 = [\beta_{9t+5}\beta_1']\zeta_2 \in \eta(\widetilde{GZ})$ instead of $\beta_{9t+1}\beta_2 = [\beta_{9t+2}\beta_1']\zeta_2 \in \eta(\widetilde{GZ})$ in the proof of [10, Th. A, Cor. B] shows the following

Lemma 5.5. *$\beta_{9t+1}\beta_5\beta_1^j \neq 0 \in \pi_*(S^0)$ if $j < 2$ and $\beta_{9t+5}\beta_1^2 \neq 0 \in \pi_*(S^0)$.*

There are some examples:

- $d_5(v_2^{9t+3}) = \pm i_{1*}(\beta'_{9t+2}\beta_1^2) \in E_5^5(V_1)$ for $t > 0$ by Lemma 3.16, and $\beta'_{9t+2}\beta_1^2$ is an essential homotopy element by [10, Cor. B].
- $d_5(v_2^{9t+6}) = \pm i_{1*}(\beta'_{9t+5}\beta_1^2) \in E_5^5(V_1)$ for $t \geq 0$ by Lemma 3.16, and $\beta'_{9t+5}\beta_1^2$ is an essential homotopy element by Lemma 5.5.
- $d_9(v_1v_2^{9t+3}) = \pm v_2^{9t}h_1b_0^4 = \pm\beta'_{9t+1}\beta_1^4$ in $E_9^9(V_2)$ by Lemma 3.17, and $\beta'_{9t+1}\beta_1^4$ is an essential permanent cycle, since so is $\beta_{9t+1}\beta_1^4$ by [10, Th. A].

Corollary 3.9 and these examples with Lemma 5.4 implies

Theorem 5.6. *Let t be non-negative integers. Then $\alpha^2\beta'_{3/2} = \alpha\beta'_3 = i\varepsilon\beta_1$, $\alpha^2\beta'_{9(t+1)+3/2} = \alpha\beta'_{9(t+1)+3} = \beta'_{9(t+1)+2}\beta_1^2$, $\alpha^3\beta'_{9t+6/3} = \alpha^2\beta'_{9t+6/2} = \alpha\beta'_{9t+6} = \beta'_{9t+5}\beta_1^2$ and $\alpha^3\beta'_{9t+3/2} = \alpha^2\beta'_{9t+3} = \beta'_{9t+1}\beta_1^4$. In particular, $\alpha\beta'_{9(t+1)+2}\beta_1^2 = \beta'_{9(t+1)+1}\beta_1^4$. Here, every equality is up to sign.*

Proof of Proposition 1.8. The relation $\alpha^k\beta' = \xi \in \pi_*(M)$ for β' such that $j\beta' = \beta$ implies $\langle\alpha_k, 3, \beta\rangle = j\xi \in \pi_*(S^0)$ by the definition of the Toda bracket. Therefore, Lemma 1.3 implies the first relation $\langle\alpha_r, 3, \beta_{9t/r}\rangle = 0$. Furthermore, we read off from Theorem 5.6 that $\langle\alpha_2, 3, \beta_{3/2}\rangle = \langle\alpha_1, 3, \beta_3\rangle = 0$, and that for $t \geq 0$, $\langle\alpha_2, 3, \beta_{9(t+1)+3/2}\rangle = \langle\alpha_1, 3, \beta_{9(t+1)+3}\rangle = \beta_{9(t+1)+2}\beta_1^2$, $\langle\alpha_3, 3, \beta_{9t+6/3}\rangle = \langle\alpha_2, 3, \beta_{9t+6/2}\rangle = \langle\alpha_1, 3, \beta_{9t+6}\rangle = \beta_{9t+5}\beta_1^2$ and $\langle\alpha_3, 3, \beta_{9t+3/2}\rangle = \langle\alpha_2, 3, \beta_{9t+3}\rangle = \beta_{9t+1}\beta_1^4$ in the homotopy groups $\pi_*(S^0)$ up to sign. q.e.d.

References

[1] M. Behrens and S. Pemmaraju, On the existence of the self map v_2^9 on the Smith-Toda complex $V(1)$ at the prime 3, Contemp. Math. **346** (2004), 9–49.

[2] H. R. Miller, D. C. Ravenel, and W. S. Wilson, Periodic phenomena in Adams-Novikov spectral sequence, Ann. of Math. **106** (1977), 469–516.

[3] S. Oka, The homotopy groups of spheres II, Hiroshima Math. J. **2** (1972), 99-161.

[4] S. Oka, Ring spectra with few cells, Japan J. Math. **5** (1979), 81–100.

[5] S. Oka, Note on the β-family in stable homotopy of spheres at the prime 3, Mem. Fac. Sci. Kyushu Univ. **35** (1981), 367–373.

[6] D. C. Ravenel, The nonexistence of odd primary Arf invariant elements in stable homotopy, Math. Proc. Cambridge Philos. Soc. bf 83 (1987), 429–433.

[7] D. C. Ravenel, Complex Cobordism and Stable Homotopy Groups of Spheres, Second edition, AMS Chelsea Publishing, Providence, 2004.

[8] D. C. Ravenel, The method of infinite descent in stable homotopy theory I, Contemp. Math. **293** (2002), 251–284.

[9] K. Shimomura, The homotopy groups of the L_2-localized Toda-Smith spectrum $V(1)$ at the prime 3, Trans. Amer. Math. Soc. **349** (1997), 1821–1850.

[10] K. Shimomura, On the action of β_1 in the stable homotopy of spheres at the prime 3, Hiroshima Math. J. **30** (2000), 345–362.

[11] L. Smith, On realizing complex bordism modules, IV, Applications to the stable homotopy groups of spheres, Amer. J. Math. **99** (1971), 418–436.

[12] H. Toda, p-primary components of homotopy groups IV, Mem. Coll. Sci. Univ. Kyoto Ser. A, **32** (1959), 288–332.

[13] H. Toda, An important relation in homotopy groups of spheres, Proc. Japan Acad. **43** (1967), 839–942.

[14] H. Toda, Algebra of stable homotopy of $\boldsymbol{Z}_p$-spaces and applications, J. Math. Kyoto Univ., **11** (1971), 197–251.

[15] H. Toda, On spectra realizing exterior parts of the Steenrod algebra, Topology **10** (1971), 53–65.

Department of Mathematics, Faculty of Science, Kochi University, Kochi, 780-8520, Japan

E-mail address: katsumi@math.kochi-u.ac.jp