NOTES ON AN ALGEBRAIC STABLE HOMOTOPY CATEGORY

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Abstract. Ohkawa showed that the collection of Bousfield classes of the stable homotopy category of spectra is a set ([8]). Let $\mathcal{C}$ be an algebraic stable homotopy category in a sense of Hovey, Palmieri and Strickland [6]. We here show that Bousfield classes of $\mathcal{C}$ form a set by introducing a homology theory based on the generators of $\mathcal{C}$, in a similar manner as Dwyer and Palmieri did in [3]. We also consider a relation between Bousfield classes of finite objects and supports of them on a collection of objects.

1. Introduction

In the stable homotopy category $\mathcal{S}$ of spectra, the Bousfield class $\langle E\rangle$ of a spectrum E is the collection of spectra X with $E\wedge X=0$. Ohkawa [8] showed that the Bousfield classes of $\mathcal{S}$ form a set (*cf.* [3]). Then, several authors generalized it to categories with some structure ([2], [4], [7], [9]). In this paper, we consider an algebraic stable homotopy category $\mathcal{C}$ in the sense of [6], which is a triangulated closed symmetric monoidal category $(\mathcal{C},\wedge,S,F(-,-),\Sigma)$ with a set $\mathcal{G}$ of small objects of $\mathcal{C}$ such that $\mathrm{loc}\langle\mathcal{G}\rangle=\mathcal{C}$, satisfying that $\mathcal{C}$ admits arbitrary coproducts and that every cohomology functor on $\mathcal{C}$ is representable. Here, $\mathrm{loc}\langle\mathcal{G}\rangle$ denotes the smallest localizing subcategory containing $\mathcal{G}$, and we call an object A *small* if $[A,\bigvee_\alpha X_\alpha]_*=\bigoplus_\alpha[A,X_\alpha]_*$, where $\bigvee_\alpha X_\alpha$ denotes the coproduct of $\{X_\alpha\}$ in $\mathcal{C}$. For examples of an algebraic stable homotopy category, see [6, 1.2. Examples]. The Bousfield class $\langle E\rangle$ of E in an algebraic stable homotopy category $\mathcal{C}$ is the collection $\{X\in\mathcal{C}\mid E\wedge X=0\}$. Let $\mathfrak{a}$ denote the cardinal number of the set $\bigoplus_{F,F'\in\mathrm{thick}\langle\mathcal{G}\rangle}[F,F']_*$. Here, $\mathrm{thick}\langle\mathcal{G}\rangle$ denotes the smallest thick subcategory of $\mathcal{C}$ containing $\mathcal{G}$, whose objects we call $\mathcal{G}$*-finite.* Then, we have an analogous theorem to Ohkawa's:

Theorem 1.1. *Let $\mathcal{C}$ be an algebraic stable homotopy category. Then the Bousfield classes $\mathbb{B}(\mathcal{C})$ of $\mathcal{C}$ form a set, whose cardinal number is not greater than $2^{2^{\mathfrak{a}}}$.*

This follows from Lemma 2.4 and Corollary 2.6. We note that $\mathbb{B}(\mathcal{C})$ is a partially ordered set by setting $\langle E\rangle\geq\langle F\rangle$ if $\langle E\rangle\subset\langle F\rangle$. Consider a subset $\mathbb{DL}(\mathcal{C})$ of $\mathbb{B}(\mathcal{C})$ consisting of elements $x\in\mathbb{B}(\mathcal{C})$ satisfying $x\wedge x=x$. We call a non-zero element $a\in\mathbb{DL}(\mathcal{C})$ an *atom* if for any element $x\in\mathbb{B}(\mathcal{C})$, $a\wedge x=a$ or $a\wedge x=0$. Consider the set $\mathbb{A}(\mathcal{C})$ of atoms of $\mathbb{B}(\mathcal{C})$, and let $\mathfrak{b}$ be the cardinal number of $\mathbb{A}(\mathcal{C})$. Then,

Proposition 1.2. *The cardinal number of $\mathbb{B}(\mathcal{C})$ is not less than $2^{\mathfrak{b}}$.*

Here, we show this by use of a surjection $\mathrm{supp}\colon\mathbb{B}(\mathcal{C})\to 2^{\mathbb{A}(\mathcal{C})}$ defined by

$$\mathrm{supp}(b)=\{a\in\mathbb{A}(\mathcal{C})\mid a\wedge b\neq 0\}. \tag{1.3}$$

In the stable homotopy category $\mathcal{S}_{(p)}$ of p-local spectra, finite spectra are classified by their types. A finite spectrum X has *type* n if $K(n)_*(X)\neq 0$ and $K(m)_*(X)=0$

for $m < n$. Here, $K(n) \in \mathcal{S}_{(p)}$ denotes the n-th Morava K-theory. It is well known that if E and F are finite spectra, then E and F have the same type if and only if $\langle E\rangle = \langle F\rangle$. We generalize this to an algebraic stable homotopy category. We say that $\mathbb{A}(\mathcal{C})$ *detects ring objects* if for any non-zero ring object R, there is an atom $a \in \mathbb{A}(\mathcal{C})$ such that $\langle R\rangle \wedge a \neq 0$.

Proposition 1.4. *Suppose that $\mathbb{A}(\mathcal{C})$ detects ring objects. Let E and F be $\mathcal{G}$-finite objects. Then, $\langle E\rangle = \langle F\rangle$ if and only if* $\mathrm{supp}(\langle E\rangle) = \mathrm{supp}(\langle F\rangle)$.

We prove this in section three.

2. Ohkawa Theorem

Let $\mathcal{C}$ denote an algebraic stable homotopy category with a set $\mathcal{G}$ of generators. We call a subcategory $\mathcal{D}$ *thick* if it is closed under cofibrations and retracts, and denote by $\mathrm{thick}\langle\mathcal{G}\rangle$ the smallest thick subcategory containing $\mathcal{G}$.

For $E \in \mathcal{C}$, put

$$E^{\mathcal{G}}_*(X) = \bigoplus_{G\in\mathcal{G}} [G, E \wedge X]_*. \tag{2.1}$$

Since $\mathcal{G} = \{S\}$ in the stable homotopy category of spectra, $E^{\mathcal{G}}_*(X) = [S, E\wedge X]_* = \pi_*(E \wedge X)$ is the homology theory represented by E in the usual sense. Here, the homology theory in this paper means the homology functor defined in [6, Def. 1.1.3].

Lemma 2.2. 1) *$E^{\mathcal{G}}_*(-)$ is a homology theory.*

2) ([6, Lemma 1.4.5 (b)]) *If $E^{\mathcal{G}}_*(X) = 0$, then $E \wedge X = 0$.*

For an object $X \in \mathcal{C}$, let $\Lambda(X)$ denote the category whose objects are morphisms $u\colon Z \to X$ of $\mathcal{C}$ for $Z \in \mathrm{thick}\langle\mathcal{G}\rangle$ and whose morphisms between objects $u\colon Z \to X$ and $u'\colon Z' \to X$ are morphisms $Z \xrightarrow{v} Z'$ of $\mathcal{C}$ such that $u'v = u$. Then, we read off the following from [6, Cor. 2.3.11]:

Lemma 2.3. *For any objects E and X of $\mathcal{C}$, $E^{\mathcal{G}}_*(X) = \operatorname*{colim}_{\Lambda(X)} E^{\mathcal{G}}_*(X_\alpha)$, where $\{X_\alpha \to X\}$ is the set of objects of $\Lambda(X)$.*

Consider the following subset of $A(X) = \bigoplus_{F\in\mathrm{thick}\langle\mathcal{G}\rangle}[X, F]_*$:

$$\mathrm{ann}^E_X(x) = \{f \in [X, F]_* \mid F \in \mathrm{thick}\langle\mathcal{G}\rangle,\ E^{\mathcal{G}}_*(f)(x) = 0\} \subset A(X)$$

for $E \in \mathcal{C}$ and $x \in E^{\mathcal{G}}_*(X)$. Then the Ohkawa class of $E \in \mathcal{C}$ is the set

$$\langle\!\langle E\rangle\!\rangle = \{\mathrm{ann}^E_F(x) \mid F \in \mathrm{thick}\langle\mathcal{G}\rangle,\ x \in E^{\mathcal{G}}_*(F)\} \subset 2^{\bigoplus_{F\in\mathrm{thick}\langle\mathcal{G}\rangle} A(F)}.$$

Put

$$\mathbb{O} = \{\langle\!\langle E\rangle\!\rangle \mid E \in \mathcal{C}\}.$$

Lemma 2.4. *$\mathbb{O}$ is a set whose cardinal number is not greater than $2^{2^{\mathfrak{a}}}$, where $\mathfrak{a}$ denotes the cardinal number of $\bigoplus_{F\in\mathrm{thick}\langle\mathcal{G}\rangle} A(F) = \bigoplus_{F,F'\in\mathrm{thick}\langle\mathcal{G}\rangle}[F, F']_*$.*

For an object $E \in \mathcal{C}$, the Bousfield class of E is the collection

$$\langle E\rangle = \{X \in \mathcal{C} \mid E \wedge X = 0\}.$$

We denote the collection of all Bousfield classes of $\mathcal{C}$ by $\mathbb{B}$: $\mathbb{B} = \{\langle E\rangle \mid E \in \mathcal{C}\}$. We define a partial ordering on $\mathbb{B}$ and $\mathbb{O}$ as follows:

- $\langle E\rangle \geq \langle F\rangle$ if $E \wedge X = 0$ implies that $F \wedge X = 0$, and

- $\langle\!\langle E\rangle\!\rangle \geq \langle\!\langle F\rangle\!\rangle$ if for any $\mathrm{ann}^F_A(x) \in \langle\!\langle F\rangle\!\rangle$, there exists $y \in E^{\mathcal{G}}_*(A)$ such that $\mathrm{ann}^F_A(x) = \mathrm{ann}^E_A(y)$.

Then we have a similar lemma as [3, Lemma 1.7]:

Lemma 2.5. *If $\langle\!\langle E\rangle\!\rangle \geq \langle\!\langle F\rangle\!\rangle$, then $\langle E\rangle \geq \langle F\rangle$.*

Proof. Suppose that $\langle\!\langle E\rangle\!\rangle \geq \langle\!\langle F\rangle\!\rangle$ and let X be an object such that $E \wedge X = 0$. Note that $F^{\mathcal{G}}_*(X) = \operatorname*{colim}_{\Lambda(X)} F^{\mathcal{G}}_*(X_\alpha)$ by Lemma 2.3. Take an element $x \in F^{\mathcal{G}}_*(X_\alpha)$. By hypothesis, for $\mathrm{ann}^F_{X_\alpha}(x) \in \langle\!\langle F\rangle\!\rangle$, there is an element $y \in E^{\mathcal{G}}_*(X_\alpha)$ such that $\mathrm{ann}^F_{X_\alpha}(x) = \mathrm{ann}^E_{X_\alpha}(y)$. Since $E \wedge X = 0$, we have $0 = E^{\mathcal{G}}_*(X)$, which equals $\operatorname*{colim}_{\Lambda(X)} E^{\mathcal{G}}_*(X_\alpha)$ by Lemma 2.3. It follows that there is a morphism $f_{\alpha\beta}\colon X_\alpha \to X_\beta \in \Lambda(X)$ for an object $f_\beta\colon X_\beta \to X \in \Lambda(X)$ such that $f_{\alpha\beta} \in \mathrm{ann}^E_{X_\alpha}(y) = \mathrm{ann}^F_{X_\alpha}(x)$. Therefore, $F^{\mathcal{G}}_*(f_{\alpha\beta})(x) = 0$, and so $F^{\mathcal{G}}_*(f_\alpha)(x) = F^{\mathcal{G}}_*(f_\beta)F^{\mathcal{G}}_*(f_{\alpha\beta})(x) = 0 \in F^{\mathcal{G}}_*(X)$. Since X_α and x are both arbitrary, we see that $F^{\mathcal{G}}_*(X) = 0$, and hence $F \wedge X = 0$ by Lemma 2.2. □

Corollary 2.6. *The map $f\colon \mathbb{O} \to \mathbb{B}$ defined by $f(\langle\!\langle E\rangle\!\rangle) = \langle E\rangle$ is well-defined. Furthermore, it is an order-preserving surjection.*

Let $\mathbb{DL}$ denote the subset of $\mathbb{B}$ consisting of elements x such that $x \wedge x = x$. Here, the pairing '$\wedge$' is inherited from $\mathcal{C}$, that is, if $x = \langle X\rangle$ and $y = \langle Y\rangle$ for objects X and $Y \in \mathcal{C}$, then $x \wedge y = \langle X \wedge Y\rangle$. We notice that '$\wedge$' is not always a meet in the lattice $\mathbb{B}$. The set $\mathbb{DL}$ is an ordered set bounded below. We call a non-zero element x of $\mathbb{DL}$ an *atom* if $x \wedge y = x$ or $x \wedge y = 0$ for any $y \in \mathbb{B}$. Let $\mathbb{A}$ denote the subset of $\mathbb{DL}$ consisting of atoms. Note that if both of x and y are atoms, then $x \wedge y = x$ if $x = y$ and $x \wedge y = 0$ otherwise. Consider the mapping $\mathrm{supp}\colon \mathbb{B} \to 2^{\mathbb{A}}$ defined by (1.3). We also consider the ordering on $2^{\mathbb{A}}$ by inclusion.

Lemma 2.7. *The mapping* supp *is an order-preserving surjection.*

Proof. We see that supp is a surjection, since for a subset $S \subset \mathbb{A}$, we have $s = \bigvee_{a\in S} a \in \mathbb{B}$ satisfying $\mathrm{supp}(s) = S$. Suppose that $e = \langle E\rangle \geq \langle F\rangle = f$. For an element $a = \langle A\rangle \notin \mathrm{supp}(e)$, $A \wedge E = 0$, and so $A \wedge F = 0$. Thus, $a \notin \mathrm{supp}(f)$, and $\mathrm{supp}(f) \subset \mathrm{supp}(e)$. □

Corollary 2.8. *The cardinal number of $\mathbb{B}$ is not less than $2^{\mathfrak{b}}$ for the cardinal number $\mathfrak{b}$ of $\mathbb{A}$.*

Remark 2.9. For the stable homotopy category $\mathcal{S}_{(p)}$ of p-local spectra, the role of $\mathbb{A}$ is played by $\{\langle K(n)\rangle \mid n \in \mathbb{N} \cup \{\infty\}\}$, whose cardinal number is $\aleph_0$. Here, $K(n)$ denotes the n-th Morava K-theory if $n < \infty$, and the mod p Eilenberg-Mac Lane spectrum if $n = \infty$.

3. Bousfield classes and supports on $\mathcal{G}$-finite objects

In this section, we apply a thick subcategory theorem for the set $\mathbb{A}$ of atoms used in the previous section. Let $\mathbb{B}$ denote the set of Bousfield classes of a fixed algebraic stable homotopy category $\mathcal{C}$.

We call an object R a *ring object* if R admits an associative multiplication $\mu\colon R \wedge R \to R$ and a unit $\eta\colon S \to R$. Consider the following condition on the category $\mathcal{C}$:

(3.1) For any ring object $R \neq 0$, $\langle R\rangle \wedge \mathbb{A}^\vee \neq 0$ for $\mathbb{A}^\vee = \bigvee_{a\in\mathbb{A}} a$.

In this case, we say that $\mathbb{A}$ *detects ring objects*.

Remark 3.2. In the stable homotopy category $\mathcal{S}_{(p)}$ of p-local spectra, the nilpotence theorem [5, Th. 3 i)] of Hopkins and Smith says that an element α of a homotopy group of a ring spectrum R is nilpotent if and only if $K(n)_*(\alpha)$ is nilpotent for all $0 \le n \le \infty$. It follows that the set $\{\langle K(n)\rangle \mid n \in \mathbb{N} \cup \{\infty\}\} \subset \mathbb{A}$ detects ring objects.

We here call an object F *$\mathcal{G}$-finite* if $F \in \mathrm{thick}\langle\mathcal{G}\rangle$, that is, F belongs to the thick subcategory generated by $\mathcal{G}$, and a thick subcategory $\mathcal{D}$ a *$\mathcal{G}$-ideal* if $X \wedge G \in \mathcal{D}$ for any $X \in \mathcal{D}$ and $G \in \mathcal{G}$. We see that, under (3.1), the set $\mathbb{A}$ of atoms satisfies the conditions of [6, Th. 5.2.2], and so we have the following:

Proposition 3.3 ([6, Th. 5.2.2])**.** *Suppose that the condition* (3.1) *holds. Then, every $\mathcal{G}$-ideal $\mathcal{D}$ of small objects (= $\mathcal{G}$-finite objects) is expressed by*

$$\mathcal{D} = \{X \in \mathrm{thick}\langle\mathcal{G}\rangle \mid \mathrm{supp}(\langle X\rangle) \subset \mathrm{supp}(\mathcal{D})\}.$$

Here $\mathrm{supp}(\mathcal{D}) = \bigcup_{X\in\mathcal{D}} \mathrm{supp}(\langle X\rangle)$.

Corollary 3.4. *Under the condition* (3.1)*, the class of $\mathcal{G}$-ideals of small objects is a set whose cardinal number is not greater than $2^{\mathfrak{b}}$.*

For an object E, consider the subcategories

$$\begin{array}{rcl} \mathcal{T}_E &=& \{X \in \mathrm{thick}\langle\mathcal{G}\rangle \mid \mathrm{supp}(\langle X\rangle) \subset \mathrm{supp}(\langle E\rangle)\} \quad \text{and} \\ \mathcal{T}_E^B &=& \{X \in \mathrm{thick}\langle\mathcal{G}\rangle \mid \langle X\rangle \le \langle E\rangle\}. \end{array}$$

Lemma 3.5. *Both of $\mathcal{T}_E$ and $\mathcal{T}_E^B$ are $\mathcal{G}$-ideals and $\mathcal{T}_E^B \subset \mathcal{T}_E$.*

Proof. The last statement follows from Lemma 2.7. By [6, Th. 2.1.3 (a)], it suffices to show that both of the categories are thick. If $X \vee Y \in \mathcal{T}_E$, then $\mathrm{supp}(\langle X\rangle) \subset \mathrm{supp}(\langle X \vee Y\rangle) \subset \mathrm{supp}(\langle E\rangle)$, and so $X \in \mathcal{T}_E$. Suppose that $X, Y \in \mathcal{T}_E$, and $X \to Y \to Z$ is a cofiber sequence. If $\langle A\rangle \notin \mathrm{supp}(\langle E\rangle)$, then $\langle A\rangle \notin \mathrm{supp}(\langle X\rangle)$ and $\langle A\rangle \notin \mathrm{supp}(\langle Y\rangle)$, which implies that $A \wedge X = 0 = A \wedge Y$. It follows that $A \wedge Z = 0$. Therefore, $\mathrm{supp}(\langle Z\rangle) \subset \mathrm{supp}(\langle E\rangle)$. Thus, $\mathcal{T}_E$ is thick. For $\mathcal{T}_E^B$, a similar argument works. ☐

Corollary 3.6. *Let E be a $\mathcal{G}$-finite object. Then, $\mathcal{T}_E = \mathcal{T}_E^B$.*

Proof. By Proposition 3.3 and Lemma 3.5, $\mathcal{T}_E^B = \{X \in \mathrm{thick}\langle\mathcal{G}\rangle \mid \mathrm{supp}(\langle X\rangle) \subset \mathrm{supp}(\mathcal{T}_E^B)\}$. For $X \in \mathcal{T}_E^B$, $\mathrm{supp}(\langle X\rangle) \subset \mathrm{supp}(\langle E\rangle)$ by Lemma 2.7. Since E is $\mathcal{G}$-finite, we see that $\mathrm{supp}(\mathcal{T}_E^B) = \mathrm{supp}(\langle E\rangle)$. ☐

Corollary 3.7. *Let X and Y be $\mathcal{G}$-finite objects. Then, $\langle X\rangle = \langle Y\rangle$ if and only if* $\mathrm{supp}(\langle X\rangle) = \mathrm{supp}(\langle Y\rangle)$.

Proof. The 'only if' part follows from Lemma 2.7. Suppose that $\mathrm{supp}(\langle X\rangle) = \mathrm{supp}(\langle Y\rangle)$. Then, $\mathcal{T}_X = \mathcal{T}_Y$, and so $\mathcal{T}_X^B = \mathcal{T}_Y^B$ by Corollary 3.6. Noticing that $X \in \mathcal{T}_X^B$, we see the 'if' part. ☐

References

1. A. K. Bousfield, The Boolean algebra of spectra, Comment. Math. Helv. **54** (1979), 368–377.
2. C. Casacuberta, J. J. Gutiérrez and J. Rosický, A generalization of Ohkawa's theorem, Comp. Math. **150** (2014), 893–902.
3. W. G. Dwyer and J. H. Palmieri, Ohkawa's theorem: there is a set of Bousfield classes. Proc. Amer. Math. Soc., **129(3)** (2001), 881–886.
4. W. G. Dwyer and J. H. Palmieri, The Bousfield lattice for truncated polynomial algebras, Homology Homotopy Appl. **10** (2008), 413–436.

5. M. J. Hopkins and J. H. Smith, Nilpotence and stable homotopy theory II, Ann. of Math. **148** (1998), 1–49.
6. M. Hovey, J. H. Palmieri, and N. P. Strickland, Axiomatic stable homotopy theory, Mem. Amer. Math. Soc. **128/610** (1997).
7. S. B. Iyengar and H. Krause, The Bousfield lattice of a triangulated category and stratification, Math. Z. **273** (2013), 1215–1241.
8. T. Ohkawa, The injective hull of homotopy types with respect to generalized homology functors, Hiroshima Math. J. **19** (1989), 631–639.
9. G. Stevenson, An extension of Dwyer's and Palmieri's proof of Ohkawa's theorem on Bousfield classes, preprint, 2011.
`https://www.math.uni-bielefeld.de/~gstevens/Ohkawa_thm_paper.pdf`

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