On the chromatic $\mathrm{Ext}^0(M^1_{n-1})$ on $\Gamma(m+1)$ for an odd prime

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Abstract. Let M^1_{n-1} denote the cokernel of the localization map $BP_*/I \to v_{n-1}^{-1}BP_*/I$, where I denotes the ideal of BP_* generated by v_i's for $0 \le i \le n-2$. The chromatic $\mathrm{Ext}^0(M^1_{n-1})$ on $\Gamma(m+1)$, which we denote $\mathrm{Ex}^0(m,n)$, is isomorphic to the 0-th line of the E_2-term of the Adams-Novikov spectral sequence for computing the homotopy groups of a spectrum, whose BP_*-homology is $M^1_{n-1} \otimes_{BP_*} BP_*[t_1, t_2, \dots, t_m]$ for the generators t_i of BP_*BP. In [9], the homotopy groups of such a spectrum are determined for $m+1 \ge n(n-1)$ by computing $\mathrm{Ex}^*(m,n)$. The 0-th line $\mathrm{Ex}^0(m,3)$ is determined by Ichigi, Nakai and Ravenel [1]. Here, we determine the 0-th line $\mathrm{Ex}^0(m,n)$ under the condition: $(n-1)^2 \le m+1 < n(n-1)$ and $n \ge 4$.

1. Introduction

Let BP be the Brown-Peterson spectrum at an odd prime p, and the pair $(BP_*, BP_*BP) = (\mathbb{Z}_{(p)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots])$ the associated Hopf algebroid. Ravenel [5] constructed the spectra $T(m)$ for $m \ge 0$ as well as a map $T(m) \to BP$ that induces the inclusion $BP_*(T(m)) = BP_*[t_1, \dots, t_m] \subset BP_*BP$ of BP_*BP-comodules. The Smith-Toda spectrum $V(k)$ is characterized by the BP_*-homology: $BP_*(V(k)) = BP_*/(p, v_1, \dots, v_k)$. We consider a spectrum $V_m(k)$ such that $BP_*(V_m(k)) = BP_*/(p, v_1, \dots, v_k)[t_1, \dots, t_m]$. Furthermore, we consider the Bousfield localization functor $L_n\colon \mathcal{S} \to \mathcal{S}$ with respect to $v_n^{-1}BP$ on the stable homotopy category $\mathcal{S}$ of p-local spectra (see [4]). If $L_nV(k)$ exists, then $L_nV_m(k) = T(m) \wedge L_nV(k)$. We notice that $L_nV(n-1)$ exists if $n^2 + n < 2p$ (see [10]). We are interested in the homotopy groups of $L_nT(m)$. The homotopy groups are determined from those of $L_nV_m(k)$ by virtue of the Bockstein spectral sequences. We study the homotopy groups by the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(X)$ of a spectrum X with E_2-term $E_2^*(X) = \mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*(X))$. Our input is a result of Ravenel's:

(1.1)(*cf.* [5, Cor. 6.5.6]) *If $n < m+2$ and $n < 2(p-1)(m+1)/p$, then*

$$E_2^*(L_nV_m(n-1)) \;=\; E_m(n)_* \otimes E(h_{k,j} : m+1 \le k \le m+n,\ j \in \mathbb{Z}/n),$$

where

$$E_m(n)_* \;=\; v_n^{-1}\mathbb{Z}/p[v_n, v_{n+1}, \dots, v_{n+m}].$$

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In order to study the Adams-Novikov E_2-term $E_2^*(L_nV_m(n-2))$, we consider a spectrum $V_m(n-2)_\infty$ defined as a cofiber of the localization map $V_m(n-2) \xrightarrow{\eta} L_{n-1}V_m(n-2)$. Put

$$\begin{aligned}\mathrm{Ex}^*(m,n) &= \mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*(V_m(n-2)_\infty)) \\ &= \mathrm{Ext}^*_{BP_*BP}(BP_*, M^1_{n-1} \otimes_{BP_*} BP_*[t_1, t_2, \dots, t_m]),\end{aligned}$$

for

$$M^1_{n-1} = v_n^{-1}BP_*/(p, v_1, \dots, v_{n-2}, v_{n-1}^{p^\infty}).$$

Then the Adams-Novikov E_2-term $E_2^*(L_nV_m(n-2))$ is obtained from $\mathrm{Ex}^*(m,n)$ and $E_2^*(L_{n-1}V_m(n-2))$. Note that for $n=1$, $\mathrm{Ex}^*(0,1)$ is the Adams-Novikov E_2-term for $\pi_*(L_1S^0)$, whose structure is found in [4, Th.8.10] (see also [3]). $\mathrm{Ex}^*(0,2)$ is the E_2-term for $\pi_*(L_2V(0))$, which is determined by the second author ([6],[7],[8]), and $\mathrm{Ex}^*(m,2)$ is determined by Kamiya and the second author in [2]. For $m+1 \ge n(n-1)$, the second author also determined $\mathrm{Ex}^*(m,n)$ in [9]. In [1], Ichigi, Nakai and Ravenel determined $\mathrm{Ex}^0(m,3)$ for $m>1$. In this paper, we determine the Ext group $\mathrm{Ex}^0(m,n)$ for (m,n) with $(n-1)^2 \le m+1 < n(n-1)$ and $n \ge 4$.

One of our tool is the change of rings theorem (*cf.* [5]):

$$\mathrm{Ex}^*(m,n) = \mathrm{Ext}^*_{\Gamma(m+1)}\left(BP_*, v_n^{-1}BP_*/(p, v_1, \dots, v_{n-2}, v_{n-1}^{p^\infty})\right)$$

for the associated Hopf algebroid

$$(BP_*, \Gamma(m+1)) = (BP_*, BP_*BP/(t_1, \dots, t_m)) = (BP_*, BP_*[t_{m+1}, t_{m+2}, \dots]).$$

Let $D_m(n)_*$ denote the algebra

$$(1.2) \qquad D_m(n)_* = E_{m-1}(n)_*[v_{n-1}] = v_n^{-1}\mathbb{Z}/p[v_{n-1}, v_n, \dots, v_{m+n-1}].$$

Then, our main theorem is obtained from the following proposition which is proved in the next section.

PROPOSITION 1.3. *Suppose that*

1) *For each integer $k \ge 0$, there is an element $x_k \in v_n^{-1}BP_*$ such that $x_k \equiv v_{m+n}^{p^k} \bmod I(1)$ and*

$$d(x_k) \equiv v_{n-1}^{a_k} v_n^{a'_k} v_{m+n}^{a''_k} g_k \mod I(a_k+1)$$

for nonnegative integers a_k, a'_k and a''_k and $g_k \in \{t_{m+i}^{p^j} : 0 < i \le n, j \in \mathbb{Z}/n\}$. Here $d(x) = \eta_R(x) - x \in v_n^{-1}\Gamma(m+1)$, and $I(k)$ denotes the ideal of $v_n^{-1}BP_*$ generated by $p, v_1, \dots, v_{n-2}$ and v_{n-1}^k.

2) *The elements $v_{m+n}^{(s-1)p^k + a''_k} g_k$ for nonnegative integers s and k represent linearly independent generators over $E_{m-1}(n)_*$ in $E_2^1(L_nV_m(n-1))$.*

Then, $\mathrm{Ex}^0(m,n)$ is the direct sum of $(v_{n-1}^{-1}D_m(n)_*)/D_m(n)_*$ and $D_m(n)_*$-modules generated by $x_k^s/v_{n-1}^{a_k}$ isomorphic to $D_m(n)_*/(v_{n-1}^{a_k})$ for each integers k, s with $k \ge 0$ and $p \nmid s > 0$.

Consider the integers

$$a_{kn+j} = \begin{cases} p^{k(n-1)+j}\dfrac{p^{k+1}-1}{p-1} & k<n \\ p^j\left(\dfrac{p^{n(k+2-n)}-1}{p^{2n}-1}A + a_{n^2-2n}\right) & k-n \text{ is even} \ge 0 \\ p^j\left(p^n\dfrac{p^{n(k+1-n)}-1}{p^{2n}-1}A + a_{n^2-n}\right) & k-n \text{ is odd} \ge 1 \end{cases}$$

for $A = p^n a_{n^2-n} + p^{m+1} - p^{(n-1)^2}$. Then, there exist elements x_k satisfying the assumptions of Proposition 1.3 for the integers a_k, which we show in section three for $(n-1)^2 \le m+1 < n(n-1)$ and $n \ge 3$. Thus, we obtain our main theorem:

THEOREM 1.4. *Let $(n-1)^2 \le m+1 \le n(n-1)$ and $n \ge 3$. Then,*

$$E_2^0(m,n) = (v_{n-1}^{-1}D_m(n)_*)/D_m(n)_* \oplus \bigoplus_{\substack{k\ge 0\\ p\nmid s>0}} \big(D_m(n)_*/(v_{n-1}^{a_k})\big)\langle x_k^s/v_{n-1}^{a_k}\rangle.$$

Note that if $n=3$, then the result is the same as that in [1].

REMARK. The computation on section three shows that the structure of $\mathrm{Ex}^0(m,n)$ depends on k such that $k(n-1) \le m+1 < (k+1)(n-1)$. In [9], it is shown that $\mathrm{Ex}^*(m,n)$ has the same structure for $k \ge n$. In this paper, we consider the case $k = n-1$.

2. Preliminaries

Throughout the paper, we fix the positive integers m and $n \ge 4$ satisfying the condition

$$(n-1)^2 \le m+1 < n(n-1).$$

2.1. The structure map η_R. In $\Gamma(m+1) = BP_*[t_{m+1}, t_{m+2}, \dots]$, we compute the action of η_R on v_i's. We have the formulas of Hazewinkel's and Quillen's:

$$\begin{aligned} v_n &= p\ell_n - \textstyle\sum_{i=1}^{n-1} \ell_i v_{n-i}^{p^i} \in BP_*\otimes\mathbb{Q} = \mathbb{Q}[\ell_1,\ell_2,\dots], \\ \eta_R(\ell_n) &= \ell_n + \textstyle\sum_{i=1}^{n-m} \ell_{n-m-i} t_{m+i}^{p^{n-m-i}} \in \Gamma(m+1)\otimes\mathbb{Q}. \end{aligned}$$

In particular, mod $(\ell_1, \ell_2, \dots, \ell_{n-2})$,

$$v_i \equiv p\ell_i \quad (i < 2n-2) \quad \text{and} \quad v_k = p\ell_k - \textstyle\sum_{i=n-1}^{k-n+1} \ell_i v_{k-i}^{p^i} \quad (k > 2n-2).$$

LEMMA 2.1. *The right unit $\eta_R\colon : BP_* \to \Gamma(m+1)$ acts on generators v_i as follows :*

$$\begin{aligned} \eta_R(v_i) &= v_i \in \Gamma(m+1) \quad \text{for } i \le m, \\ \eta_R(v_{m+k}) &= v_{m+k} + pt_{m+k} \quad (0<k<n) \end{aligned}$$

$$
\begin{aligned}
\eta_R(v_{m+n+k}) \quad &\equiv \quad v_{m+n+k} + \sum_{j=0}^{k} \left(v_{n-1+j} t_{m+1+k-j}^{p^{n-1+j}} - v_{n-1+j}^{p^{m+1+k-j}} t_{m+1+k-j} \right) \\
&\qquad -\omega_k \quad \bmod I_{n-1} \quad (0 \le k \le n)
\end{aligned}
$$

where I_k denotes the ideal generated by $p, v_1, \dots, v_{k-1}$, and

$$
\omega_k = \begin{cases} 0 & k < n-1, \\ v_{n-1} w_{m+n,n-2} & k = n-1, \\ v_{n-1} w_{m+n+1,n-2} + v_n w_{m+n,n-1} & k = n \end{cases} \tag{2.2}
$$

for the elements $w_{m+n,i}$ and $w_{m+n+1,i}$ defined by

$$
\begin{aligned}
d(v_{m+n}^{p^{i+1}}) \quad &\equiv \quad v_{n-1}^{p^{i+1}} t_{m+1}^{p^{n+i}} - t_{m+1}^{p^{i+1}} v_{n-1}^{p^{m+i+2}} + p w_{m+n,i} \\
&\qquad \bmod (p^2, v_1, \dots, v_{n-2}), \quad \textit{and} \\
d(v_{m+n+1}^{p^{i+1}}) \quad &\equiv \quad v_{n-1}^{p^{i+1}} t_{m+2}^{p^{n+i}} + v_n^{p^{i+1}} t_{m+1}^{p^{n+i+1}} - t_{m+1}^{p^{i+1}} v_n^{p^{m+i+2}} - t_{m+2}^{p^{i+1}} v_{n-1}^{p^{m+i+3}} \\
&\qquad + p w_{m+n+1,i} \quad \bmod (p^2, v_1, \dots, v_{n-2}).
\end{aligned}
$$

PROOF. This follows from routine computation:

$$
\begin{aligned}
\eta_R(v_{m+k}) \quad &= \quad p(\ell_{m+k} + t_{m+k}) - \textstyle\sum_{i=n-1}^{m+k-n+1} \ell_i v_{m+k-i}^{p^i} \\
&= \quad v_{m+k} + p t_{m+k} \quad (0 < k < n) \\
\eta_R(v_{m+n+k}) \quad &\equiv \quad p(\ell_{m+n+k} + \textstyle\sum_{j=0}^{k} \ell_{n-1+j} t_{m+1+k-j}^{p^{n-1+j}} + t_{m+n+k}) \\
&\qquad - \textstyle\sum_{i=n-1}^{m} \ell_i v_{m+n+k-i}^{p^i} - \sum_{i=1}^{k+1} (\ell_{m+i} + t_{m+i}) v_{n+k-i}^{p^{m+i}} \\
&\equiv \quad v_{m+n+k} + \textstyle\sum_{j=0}^{k} v_{n-1+j} t_{m+1+k-j}^{p^{n-1+j}} - \sum_{i=1}^{k+1} t_{m+i} v_{n+k-i}^{p^{m+i}} \\
&\qquad \bmod I_{n-1} \quad (0 \le k < n-1) \\
\eta_R(v_{m+2n-1}) \quad &\equiv \quad p(\ell_{m+2n-1} + \textstyle\sum_{j=0}^{n-1} \ell_{n-1+j} t_{m+n-j}^{p^{n-1+j}} + t_{m+2n-1}) \\
&\qquad -\ell_{n-1}(v_{m+n}^{p^{n-1}} + v_{n-1}^{p^{n-1}} t_{m+1}^{p^{2n-2}} - v_{n-1}^{p^{m+n}} t_{m+1}^{p^{n-1}} + p w_{m+n,n-2}) \\
&\qquad - \textstyle\sum_{i=n}^{m} \ell_i v_{m+2n-1-i}^{p^i} - \sum_{i=1}^{n-1} (\ell_{m+i} + t_{m+i}) v_{2n-1-i}^{p^{m+i}} \\
&\qquad -(l_{m+n} + \ell_{n-1} t_{m+1}^{p^{n-1}} + t_{m+n}) v_{n-1}^{p^{m+n}} \\
&\equiv \quad v_{m+2n-1} + \textstyle\sum_{j=0}^{n-1} v_{n-1+j} t_{m+n-j}^{p^{n-1+j}} - \sum_{i=1}^{n} t_{m+i} v_{2n-1-i}^{p^{m+i}} \\
&\qquad -v_{n-1} w_{m+n,n-2} \quad \bmod I_{n-1} \\
\eta_R(v_{m+2n}) \quad &\equiv \quad p(m_{m+2n} + \textstyle\sum_{j=0}^{n} \ell_{n-1+j} t_{m+1+n-j}^{p^{n-1+j}} + t_{m+2n-1}) \\
&\qquad -\ell_{n-1}(v_{m+n+1}^{p^{n-1}} + v_{n-1}^{p^{n-1}} t_{m+2}^{p^{2n-2}} + v_n^{p^{n-1}} t_{m+1}^{p^{2n-1}} - v_{n-1}^{p^{m+n+1}} t_{m+2}^{p^{n-1}} \\
&\qquad -v_n^{p^{m+n}} t_{m+1}^{p^{n-1}} + p w_{m+n+1,n-2}) \\
&\qquad -\ell_n(v_{m+n}^{p^n} + v_{n-1}^{p^n} t_{m+1}^{p^{2n-1}} - v_{n-1}^{p^{m+n+1}} t_{m+1}^{p^n} + p w_{m+n,n-1}) \\
&\qquad - \textstyle\sum_{i=n}^{m} \ell_i v_{m+2n-i}^{p^i} - \sum_{i=1}^{n-1} (l_{m+i} + t_{m+i}) v_{2n-i}^{p^{m+i}} \\
&\qquad -(l_{m+n} + l_{n-1} t_{m+1}^{p^{n-1}} + t_{m+n}) v_n^{p^{m+n}} \\
&\qquad -(l_{m+n+1} + l_{n-1} t_{m+2}^{p^{n-1}} + l_n t_{m+1}^{p^n} + t_{m+n+1}) v_{n-1}^{p^{m+n+1}} \\
&\equiv \quad v_{m+2n} + \textstyle\sum_{j=0}^{n} v_{n-1+j} t_{m+1+n-j}^{p^{n-1+j}} - \sum_{i=1}^{n+1} t_{m+i} v_{2n-i}^{p^{m+i}} \\
&\qquad -v_{n-1} w_{m+n+1,n-2} - v_n w_{m+n,n-1} \quad \bmod I_{n-1}
\end{aligned}
$$

□

2.2. Proof of Proposition 1.3. The proof is based on [3, Remark 3.11], which states, in our case, that if the commutative diagram of two exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E^0 & \xrightarrow{1/u} & D^0 & \xrightarrow{u} & D^0 & \xrightarrow{\delta} & E^1 \\
 & & \| & & \downarrow f & & \downarrow f & & \| \\
0 & \longrightarrow & E^0 & \xrightarrow{1/u} & \mathrm{Ex}^0(m,n) & \xrightarrow{u} & \mathrm{Ex}^0(m,n) & \xrightarrow{\delta} & E^1
\end{array}$$

commutes, then f is an isomorphism. Hereafter, we set $u = v_{n-1}$.

Let D^0 be the module of the proposition, that is, the $D_m(n)_*$-modules generated by $1/u^j$ for $j > 0$ and x_k^s/u^{a_k} for each integers k, s with $k \ge 0$ and $p \nmid s > 0$. Then, $D^0 \subset M^1_{n-1} = v_n^{-1}BP_*/(p, v_1, \dots, v_{n-2}, v_{n-1}^\infty)$. Note that $\mathrm{Ex}^0(m,n)$ is the kernel of the map $d\colon M^1_{n-1} \to M^1_{n-1} \otimes_{BP_*} \Gamma(m+1)$ given by $d(x/u^j) = (\eta_R(x) - x)/u^j$. Thus, the first supposition shows that the every element x_k^s/u^{a_k} belongs to $\mathrm{Ex}^0(m,n)$. We also see that $1/u^j \in \mathrm{Ex}^0(m,n)$ by Lemma 2.1. Now we define the map f to be the inclusion. Then, the map $1/u\colon \mathrm{Ex}^0 \to D^0$ is well defined, since the elements v^s_{m+n}/u belong to D^0. Since D^0 is a $D_m(n)_*$-module, D^0 admits the self map u. The map $\delta\colon D^0 \to E^0$ is defined by the composite δf. These show the existence of the commutative diagram.

We will show that the upper sequence is exact. Since the diagram commutes and the lower sequence is exact, the sequence $0 \to E^0 \xrightarrow{1/u} D^0 \xrightarrow{u} D^0$ is exact and the composite δu is trivial. Suppose that $\delta(x) = 0$ for $x = \sum_{k\ge 0, p\nmid s>0} a_{k,s} x_k^s/u^{a_k} + \sum_j a_j/u^j$ for $a_{k,s}$, $a_j \in D_m(n)_*$. Then, by the first supposition,

$$\begin{aligned}
0 = \delta(x) &= \textstyle\sum_{k\ge 0, p\nmid s>0} a_{k,s}\delta(x_k^s/u^{a_k}) \\
&= \textstyle\sum_{k\ge 0, p\nmid s>0} s\varphi(a_{k,s}) v_n^{a_k'} v_{m+n}^{(s-1)p^k + a_k''} g_k.
\end{aligned}$$

for the map φ in the exact sequence $D_m(n)_* \xrightarrow{u} D_m(n)_* \xrightarrow{\varphi} E_{m-1}(n)_*$. Then the second assumption shows that $s\varphi(a_{k,s}) = 0 \in E_{m-1}(n)_*$ for every k, s, and hence $a_{k,s} = ub_{k,s}$ for an element $b_{k,s} \in D_m(n)_*$. It follows that $x = uy$ for $y = \sum_{k\ge 0, p\nmid s>0} b_{k,s}x_k^s/u^{a_k} + \sum_j a_j/u^{j+1}$, as desired.

3. The elements x_n

In the sequel, we put $v_n = 1$, and use the notation

$$\begin{gathered}
u = v_{n-1}, \quad w_k = v_{m+n+k}, \quad s_k = t_{m+k}, \\
I = I_{n-1} = (p, v_1, \dots, v_{n-2}) \quad \text{and} \quad I(k) = (p, v_1, \dots, v_{n-2}, v_{n-1}^k)
\end{gathered}$$

for the sake of simplicity. We also introduce integers:

$$\begin{aligned}
P &= p^{n-1}, \quad Q = p^{m+1} \\
e_P(k) &= \frac{P^k - 1}{P - 1}, \quad e(k) = \frac{p^k - 1}{p - 1}, \quad \text{and} \\
A_k &= P^k e(k+1)
\end{aligned}$$

Now we introduce elements X_k and X'_k for $k \le n$, which correspond to the elements x_{kn} and x_{kn+1}.

$$
\begin{aligned}
X_k &= \begin{cases} w_0 & k=0 \\ (X'_{k-1})^P - u^{(pP-1)A_{k-1}} X_{k-1} & \\ \qquad -u^{p(P-1)A_{k-1}+Q-P^{k-1}} X'_{k-1} & 0<k\le n \end{cases} \\
X'_k &= X_k^p - (-1)^k W_{k+1} \\
W_k &= u^{pA_{k-1}} \Bigg(w_k^{P^{k-1}} + \sum_{i=1}^{k-1} (-1)^{k+i} \Big(u^{-p^{i+1}P^i A_{k-1-i}} v_{n+i}^{P^{k-1}} X_{k-1-i}^{p^{i+1}P^i} \\
&\qquad -u^{-P^{i-1}A_{k-1-i}} v_{n+i}^{p^{k-i-1}P^{k-1}Q} X_{k-1-i}^{P^{i-1}} \Big) \Bigg).
\end{aligned} \tag{3.1}
$$

The element ω_k of (2.2) is a multiple of u, and we put:

$$
\begin{aligned}
\sigma_k &\equiv u^{-P^{k-1}} \omega_k^{P^{k-1}} \quad \text{and} \\
\sigma &= \sigma_{n-1} = w_{m+n,n-2}^{P^{n-2}} \\
&\equiv u^{(pP)^{n-2}} X_{n-2}^{p-1} s_1^{(pP)^{n-2}P} \quad \bmod I(2(pP)^{n-2}).
\end{aligned} \tag{3.2}
$$

Note that $\sigma_k = 0$ if $k < n-1$, and $= \sigma$ if $k = n-1$.

Lemma 3.3. *The differential $d\colon v_n^{-1}BP_* \to v_n^{-1}\Gamma(m+1)$ acts on X_k and X'_k for $k \ge 0$ as follows:*

$$
\begin{aligned}
d(X_k) &\equiv (-1)^k u^{A_k} \left(\left(s_{k+1}^{P^{k+1}} - \sigma_k^P \right) - u^{Q-P^k} \left(s_{k+1}^{P^k} - \sigma_k \right) \right), \\
&\qquad \bmod I(2Q + A_k - e_P(k+1)) \text{ for } k \le n-1, \\
d(X'_k) &\equiv (-1)^k u^{pA_k} \left(s_{k+1}^{P^k} - u^{P^k} \left(s_{k+2}^{P^{k+1}} - \sigma_{k+1} \right) \right) \\
&\qquad \bmod I(Q + pA_k - P^{k-1}) \text{ for } k \le n-2, \\
d(X'_{n-1}) &\equiv (-1)^{n+1} u^{pA_{n-1}} s_n^{P^{n-1}} \quad \bmod I(pA_{n-1} + P^{n-1}), \quad \text{and} \\
d(X_n) &\equiv (-1)^n u^{pPA_{n-1}+Q-P^{n-1}} \sigma \quad \bmod I(Q + pPA_{n-1}).
\end{aligned}
$$

Here, we set $P^{-1} = 0$.

Remark. In the case $m+1 \ge n(n-1)$, $d(X_n) \equiv (-1)^{n+1} u^{pPA_{n-1}} \sigma_{n-1}^P$ mod $I(pPA_{n-1} + Q - P^{n-1})$ in our notation.

Proof. Since

$$
d(X_0) = d(w_0) \equiv u s_1^{p^{n-1}} - u^{p^{m+1}} s_1 = u s_1^P - u^Q s_1 \quad \bmod I
$$

by Lemma 2.1, we have the first step of the induction.

Suppose that we have the congruences on $d(X_i)$ for $i \le k < n-1$. Since $n \ge 3$, we see that $k < m$ and $d(v_{n+i}) \equiv 0 \bmod I_{n-1}$ for $i \le k$. Then we compute by Lemma 2.1 as follows:

$$\begin{aligned}
d\left(X_k^p\right) &\equiv (-1)^k u^{pA_k}\left(s_{k+1}^{pP^{k+1}} - u^{pQ-pP^k} s_{k+1}^{pP^k}\right) \\
&\qquad \bmod I(2pQ + pA_k - pe_P(k+1)) \\
d\left(W_{k+1}\right) &\equiv u^{pA_k}\Big(d\left(w_{k+1}^{P^k}\right) \\
&\quad + \sum_{i=1}^{k}(-1)^{k+i+1} d\Big(u^{-p^{i+1}P^i A_{k-i}} v_{n+i}^{P^k} X_{k-i}^{p^{i+1}P^i} \\
&\qquad -u^{-P^{i-1}A_{k-i}} v_{n+i}^{p^{k-i}P^k Q} X_{k-i}^{P^{i-1}}\Big) \\
&\equiv u^{pA_k}\Bigg(u^{P^k} s_{k+2}^{P^{k+1}} + s_{k+1}^{pP^{k+1}} - s_{k+1}^{P^k} \\
&\quad + \sum_{i=1}^{k}\left(v_{n+i}^{P^k} s_{k+1-i}^{p^{i+1}P^{k+1}} - v_{n+i}^{p^{k-i}P^k Q} s_{k+1-i}^{P^k}\right) - \omega_{k+1}^{P^k} \\
&\quad - \sum_{i=1}^{k}\left(v_{n+i}^{P^k} s_{k+1-i}^{p^{i+1}P^{k+1}} - v_{n+i}^{p^{k-i}P^k Q} s_{k+1-i}^{P^k}\right)\Bigg) \\
&\equiv u^{pA_k}\left(u^{P^k} s_{k+2}^{P^{k+1}} + s_{k+1}^{pP^{k+1}} - s_{k+1}^{P^k} - u^{P^k}\sigma_{k+1}\right)
\end{aligned}$$

mod $I(pA_k + Q - P^{k-1})$, which yields the congruence on $d(X'_k)$.

$$\begin{aligned}
d((X'_k)^P) &\equiv (-1)^k u^{pPA_k}\left(s_{k+1}^{P^{k+1}} - u^{P^{k+1}}\left(s_{k+2}^{P^{k+2}} - \sigma_{k+1}^P\right)\right) \\
&\qquad \bmod I(PQ + pPA_k - P^k) \\
d(-u^{(pP-1)A_k} X_k) &\equiv (-1)^{k+1} u^{pPA_k}\left(s_{k+1}^{P^{k+1}} - u^{Q-P^k} s_{k+1}^{P^k}\right) \\
&\qquad \bmod I(2Q + pPA_k - e_P(k+1)) \\
d\left(-u^{p(P-1)A_k + Q - P^k} X'_k\right) &\equiv (-1)^{k+1} u^{pPA_k + Q - P^k}\left(s_{k+1}^{P^k} - u^{P^k}\left(s_{k+2}^{P^{k+1}} - \sigma_{k+1}\right)\right) \\
&\qquad \bmod I(2Q + pPA_k - P^k - P^{k-1}),
\end{aligned}$$

and we have

$$d\left(X_{k+1}\right) \equiv (-1)^{k+1} u^{pPA_k}\left(u^{P^{k+1}}\left(s_{k+2}^{P^{k+2}} - \sigma_{k+1}^P\right) - u^Q\left(s_{k+2}^{P^{k+1}} - \sigma_{k+1}\right)\right)$$

mod $I(2Q + pPA_k - e_P(k+1))$. Since $A_{k+1} = pPA_k + P^{k+1}$ and

$$pPA_k - e_P(k+1) = A_{k+1} - P^{k+1} - e_P(k+1) = A_{k+1} - e_P(k+2),$$

we obtain $d(X_{k+1})$. This completes the induction to obtain the congruences on $d(X'_{n-2})$ and $d(X_{n-1})$.

A similar computation as above shows the congruence on $d(X'_{n-1})$, where we take the ideal in the congruence so that $\omega_n^{P^{n-1}}$ is annihilated. For $d(X_n)$, using the congruences on $d(X_{n-1})$ and $d(X'_{n-1})$, we compute

$$
\begin{aligned}
d\left((X'_{n-1})^P\right) &\equiv (-1)^{n+1}u^{pPA_{n-1}}s_n^{P^n} \quad \bmod I(pPA_{n-1}+P^n) \\
d\left(-u^{(pP-1)A_{n-1}}X_{n-1}\right) & \\
&\equiv (-1)^n u^{pPA_{n-1}}\left(\left(s_n^{P^n}-\sigma^P\right)-u^{Q-P^{n-1}}\left(s_n^{P^{n-1}}-\sigma\right)\right) \\
&\qquad \bmod I(2Q+pPA_{n-1}-e_P(n)) \\
d\left(-u^{p(P-1)A_{n-1}+Q-P^{n-1}}X'_{n-1}\right) & \\
&\equiv (-1)^n u^{pPA_{n-1}+Q-P^{n-1}}s_n^{P^{n-1}} \quad \bmod I(pPA_{n-1}+Q)
\end{aligned}
$$

to obtain $d(X_n)$. □

Consider an element $Y = X_0^{(pP)^{n-2}} - X_{n-2}$, and we see that it is congruent to zero modulo $I((pP-1)(pP)^{n-3})$, and

$$
\begin{aligned}
d(X_{n-2}^{p-1}Y) &\equiv X_{n-2}^{p-1}\left(u^{(pP)^{n-2}}s_1^{(pP)^{n-2}P} - d(X_{n-2})\right) \\
&\equiv \sigma - X_{n-2}^{p-1}d(X_{n-2}) \quad \bmod I(2(pP)^{n-2}-(pP)^{n-3}).
\end{aligned} \tag{3.4}
$$

We introduce integers $e_2(k)$, A, A_k, c_k and C_k defined by

$$
\begin{aligned}
e_2(2k) &= \frac{(pP)^{2k}-1}{(pP)^2-1}, \\
A &= pPA_{n-1}+Q-P^{n-1}, \\
A_k &= \begin{cases} P^k e(k+1) & k<n \\ e_2(k+2-n)A + A_{n-2} & k-n \text{ is even} \geq 0 \\ (pP)e_2(k+1-n)A + A_{n-1} & k-n \text{ is odd} \geq 1 \end{cases} \\
c_k &= \begin{cases} (pP)e_2(k-1)A + (pP-1)A_{n-2} & k \text{ is odd} \geq 1, \\ (pP)^2 e_2(k-2)A + (pP-1)A_{n-1} & k \text{ is even} \geq 2, \end{cases}
\end{aligned}
$$

We replace X_n by

$$X_n - (-1)^n u^A X_{n-2}^{p-1}Y,$$

and define inductively the elements X_{n+k} for $k>0$ by

$$X_{n+k} = X_{n+k-1}^{pP} - (-1)^n u^{(pP)^k A} X_{n+k-3}^{(p-1)pP} Y_k. \tag{3.5}$$

for

$$Y_k = X_{n+k-3}^{pP} - X_{n+k-2}$$

Here we notice that $pP = p^n$.

LEMMA 3.6. *The element Y_k is a multiple of u^{c_k} for $k>0$.*

PROOF. By the definition (3.1), we see that Y_1 and Y_2 are divisible by $u^{(pP-1)A_{n-2}}$ and $u^{(pP-1)A_{n-1}}$, respectively. For $k>2$, we see that $c_k = (pP)^{k-2}A + c_{k-2}$ by (3.5), and verify the lemma by induction. □

We further introduce integers A''_k and elements G_k defined by

$$\begin{aligned} A''_k &= \begin{cases} 0 & k<n \\ (pP)^{k-2}(p-1)+A''_{k-2} & k\ge n \end{cases} \\ G_k &= \begin{cases} s_{k+1}^{p^{k+1}} & k<n \\ G_{k-2} & k\ge n \end{cases} \end{aligned}$$

Then, we have

LEMMA 3.7. *For $i\ge 0$,*

$$\begin{aligned} d(X_{n+i}) &\equiv (-1)^n u^{(pP)^iA}X_{n+i-2}^{p-1}d(X_{n+i-2}) \mod I(A_{n+i}+P^{n-\varepsilon(i)}) \\ &\equiv (-1)^n u^{A_{n+i}}w_0^{A''_{n+i}}G_{n+i} \mod I(A_{n+i}+1). \end{aligned}$$

Here $\varepsilon(i)=\max\{0,1-i\}$.

PROOF. We show the first congruence by induction. The congruence for $i=0$ follows from Lemma 3.3 and (3.4). Inductively suppose that the congruence holds for i. Then, we compute

$$\begin{aligned} d(X_{n+i}^{pP}) &\equiv (-1)^n u^{(pP)^{i+1}A}X_{n+i-2}^{(p-1)pP}d(X_{n+i-2}^{pP}) \mod I(pPA_{n+i}+pP^{n-\varepsilon(i)}) \\ d((-1)^{n+1}&u^{(pP)^{i+1}A}X_{n+i-2}^{(p-1)pP}Y_{i+1}) \\ &\equiv (-1)^{n+1}u^{(pP)^{i+1}A}X_{n+i-2}^{(p-1)pP}(d(X_{n+i-2}^{pP})-d(X_{n+i-1})) \end{aligned}$$

mod $I((pP)^{i+1}A+c_{i+1}+pPA_{n+i-2})$. Since $(pP)^{i+1}A+pPA_{n+i-2}=pPA_{n+i}$, $c_{i+1}>pP^{n-\varepsilon(i)}$ and $pPA_{n+i}+pP^{n-\varepsilon(i)}>A_{n+i+1}+P^{n-1}$, we obtain the congruence for $i+1$. Thus, the induction completes. □

Now we define integers a_k and a''_k, and elements x_k, g'_k and g_k as follows:

$$\begin{aligned} a_{in+j} &= p^jA_i \\ a''_{in+j} &= p^jA''_i \\ x_{in+j} &= X_i^{p^j} \\ g_{in+j} &= G_i^{p^j} \end{aligned}$$

for $i\ge 0$ and $0\le j<n$. We notice that the integer a_k is the same as the one in the introduction.

LEMMA 3.8. *These elements and integers satisfy the assumption of Proposition 1.3.*

PROOF. The first part of the assumption follows from the definition of elements and Lemmas 3.3 and 3.7.

We prove the second part by showing the following assertion on $\ell\ge 0$.

$(3.9)_{(\ell)}$ The elements $v_{m+n}^{(s-1)p^k+a''_k}g_k$ for non-negative integers s,k with $p\nmid s$ and $k<\ell$ are linearly independent over $E_{m-1}(n)_*$.

For $\ell < n^2$, it is trivial, since $g_k \neq g_{k'}$ if $k \neq k'$. Suppose that it holds for ℓ, and $v_{m+n}^{(s-1)p^\ell + a''_\ell} g_\ell = \sum_{k=0}^{\ell-1} \lambda_k v_{m+n}^{(s_k-1)p^k + a''_k} g_k$ for $\lambda_k \in E_{m-1}(n)_*$, where s_k is a nonnegative integer prime to p such that $(s-1)p^\ell + a''_\ell = (s_k-1)p^k + a''_k$. Then, since g_i's are generators, $\ell \equiv k \bmod n$, and $(s-1)p^\ell + a''_\ell = (s_k-1)p^k + a''_k$. Then, $a''_\ell \equiv (s_k-1)p^k + a''_k \bmod p^\ell$. If $k = in + j$, then $A''_{\ell'} \equiv (s_k-1)p^{in} + A''_n \bmod p^{\ell-j}$ for ℓ' such that $\ell = \ell' n + j$. The definition of integers A''_i implies that p divides s_k, and there is no integers s_k satisfying the congruence. Hence, $(3.7)_{(\ell+1)}$ holds, and the induction completes. □

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