

The modulo two homotopy groups of the L_2 -localization of the Ravenel spectrum

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Abstract. The Ravenel spectra $T(m)$ for non-negative integers m interpolate between the sphere spectrum and the Brown-Peterson spectrum. Let L_2 denote the Bousfield-Ravenel localization functor with respect to $v_2^{-1}BP$. In this paper, we determine the homotopy groups $\pi_*(L_2T(m) : \mathbb{Z}/2) = [M_2, L_2T(m)]_*$ for $m > 1$, where M_2 denotes the modulo two Moore spectrum.

1. Introduction

Let $\mathcal{S}_{(2)}$ denote the stable homotopy category of 2-local spectra, and $BP \in \mathcal{S}_{(2)}$ denote the Brown-Peterson ring spectrum. Then, $BP_* = \pi_*(BP) = \mathbb{Z}_{(2)}[v_1, v_2, \dots]$ and $BP_*(BP) = \pi_*(BP \wedge BP) = BP_*[t_1, t_2, \dots]$, which form a Hopf algebroid. The Adams-Novikov spectral sequence for computing the homotopy groups $\pi_*(X)$ of a spectrum X has the E_2 -term $E_2^*(X) = \text{Ext}_{BP_*(BP)}^*(BP_*, BP_*(X))$. Let $L_2 : \mathcal{S}_{(2)} \rightarrow \mathcal{S}_{(2)}$ be the Bousfield-Ravenel localization functor with respect to $v_2^{-1}BP$. Then, the E_2 -term $E_2^*(L_2S^0)$ for the sphere spectrum S^0 is determined in [12], but the homotopy groups $\pi_*(L_2S^0)$ stay undetermined. The Ravenel spectrum $T(m)$ for $m > 0$ is a ring spectrum characterized by $BP_*(T(m)) = BP_*[t_1, t_2, \dots] \subset BP_*(BP)$ as a $BP_*(BP)$ -comodule. The spectrum $T(m)$ interpolates between the sphere spectrum and the Brown-Peterson spectrum, and so the homotopy groups $\pi_*(L_2T(m))$ seem accessible if m is sufficiently large. Indeed, $\pi_*(L_2T(\infty)) = \pi_*(L_2BP)$ is determined by Ravenel [8]. Let M_k denote the mod k Moore spectrum defined by the cofiber sequence

$$(1.1) \quad S^0 \xrightarrow{2} S^0 \xrightarrow{i} M_k \xrightarrow{j} S^1.$$

For $m = 1$, $T(1) \wedge M_2$ is the Mahowald spectrum $X\langle 1 \rangle$ and the homotopy groups of $L_2X\langle 1 \rangle$ are determined in [11]. But even the homotopy groups of $L_2T(1) \wedge M_4$ are too complicated to be determined completely (cf. [2], [3]). Consider a spectrum $T(m)/(v_1^a)$ defined as a cofiber of the self-map $v_1^a : \Sigma^{2a}T(m) \rightarrow T(m)$ defined by the generator $v_1 \in \pi_2(T(m))$. We use the notation:

$$(1.2) \quad V_m(0) = T(m) \wedge M_2 \quad \text{and} \quad V_m(1)_a = T(m)/(v_1^a) \wedge M_2,$$

and abbreviate $V_m(1)_1$ to $V_m(1)$. In this paper, we consider the case where $m > 1$, and determine $\pi_*(L_2V_m(1))$ and $\pi_*(L_2V_m(0))$. The Adams-Novikov E_2 -term $E_2^*(L_2V_m(1))$ for $m > 1$ is determined by Ravenel [10] as follows:

$$(1.3) \quad E_2^*(L_2V_m(1)) = K_m(2)_* \otimes \wedge(h_{1,0}, h_{1,1}, h_{2,0}, h_{2,1})$$

for generators $h_{i,j} \in E_2^{1, 2^{m+i+j+1}-2^{j+1}}(L_2V_m(1))$ and $K_m(2)_* = v_2^{-1}\mathbb{Z}/2[v_2, v_3, \dots, v_{m+2}]$. We show that $V_m(1)$ is a $T(m)$ -module spectrum with M_2 -action, and then that all additive generators of the E_2 -term are permanent cycles and the extension problem of the spectral sequence is trivial.

Theorem 1.4. $\pi_*(L_2V_m(1)) = K_m(2)_* \otimes \wedge(h_{1,0}, h_{1,1}, h_{2,0}, h_{2,1})$ as a $\mathbb{Z}/2$ -module.

Let $\alpha : \Sigma^8 M_2 \rightarrow M_2$ denote the Adams map such that $BP_*(\alpha) = v_1^4$, and K_2^a denote a cofiber of α^a . Then, we show that $V_m(1)_{4a} = T(m) \wedge K_2^a$ in Lemma 2.4 and denote the telescope of $V_m(1)_4 \xrightarrow{\alpha} \dots \xrightarrow{\alpha} V_m(1)_{4a} \xrightarrow{\alpha} V_m(1)_{4a+4} \xrightarrow{\alpha} \dots$ by $V_m(1)_\infty$. By the v_1 -Bockstein spectral sequence, we determine the Adams-Novikov E_2 -term $E_2^*(L_2V_m(1)_\infty)$, whose structure is given in [4] without proof. Here we give a proof of it. Consider the integers e_n and a_n defined by

$$(1.5) \quad e_n = \frac{8^n - 1}{7} \quad \text{and} \quad a_n = \begin{cases} 1 & n = 0, \\ 3e_{k+1} - 1 & n = 3k + 1, \\ 6e_{k+1} & n = 3k + 2, \\ 12e_{k+1} & n = 3k + 3. \end{cases}$$

We introduce modules

$$\begin{aligned} E_m(2)_* &= v_2^{-1}\mathbb{Z}_{(2)}[v_1, v_2, \dots, v_{m+2}], \\ Q(k) &= E_{m-1}(2)_*/(2, v_1^{a_k})[x_{k+1}]\langle x_k/v_1^{a_k} \rangle, \end{aligned}$$

where $x_n \in E_m(2)_*$ is an element defined in (4.1) such that $x_n \equiv v_{m+2}^{2^n}$ modulo $(2, v_1)$, and $x_n/v_1^{a_n} \in E_2^0(L_2V_m(1)_\infty)$ by Proposition 4.3. We also introduce homology classes ζ and ζ_n of $E_2^1(V_m(0))$, which correspond to elements $v_{m+2}h_{1,1}$ and $v_{m+2}^{2^l} \zeta_l \in E_2^1(L_2V_m(1))$ for $n = 3k + l$ with $l \in \{1, 2, 3\}$, respectively, where ζ_l corresponds to $h_{1,0}$ if $l = 1$, and $h_{2,l-2}$ if $l = 2, 3$.

Proposition 1.6. (cf. [4]) *The E_2 -term of Adams-Novikov spectral sequence for computing $\pi_*(L_2V_m(1)_\infty)$ is isomorphic to the direct sum of $Q(0) \otimes \wedge(h_{1,0}, h_{2,0}, h_{2,1})$ and the tensor product of $\wedge(\zeta)$ and*

$$E_{m-1}(2)_*/(2, v_1^\infty) \oplus \bigoplus_{k>0} Q(k) \otimes \wedge(\zeta_{k+1}, \zeta_{k+2})$$

as a $\mathbb{Z}/2[v_1]$ -module.

By noticing that $x_n \in E_2^0(L_2V_m(1)_{a_n})$ survives to $\pi_*(L_2V_m(1)_{a_n})$ in Lemma 5.1, we see that all additive generators of Proposition 1.6 are permanent cycles.

Theorem 1.7. *The homotopy groups $\pi_*(L_2V_m(1)_\infty)$ are isomorphic to the Adams-Novikov E_2 -term given in Proposition 1.6.*

Consider the cofiber sequence

$$(1.8) \quad V_m(0) \xrightarrow{\eta} v_1^{-1}V_m(0) \xrightarrow{p} V_m(1)_\infty \rightarrow \Sigma V_m(0)$$

for the localization map η . Here, we introduce algebras

$$k_m(1)_* = \mathbb{Z}/2[v_1, v_2, \dots, v_{m+1}] \quad \text{and} \quad K_m(1)_* = v_1^{-1}k_m(1)_*.$$

Ravenel showed the following

Proposition 1.9. (cf. [10]) *The homotopy groups $\pi_*(v_1^{-1}V_m(0))$ are isomorphic to $K_m(1)_* \otimes \wedge(h_{1,0})$.*

There is a relation between $h_{1,0}$ and ζ , which is shown in section four:

Lemma 1.10. *The induced homomorphism p_* from p in (1.8) assigns $h_{1,0}/v_1^j \in E_2^1(v_1^{-1}V_m(0))$ to $\zeta/v_1^{j-2} \in E_2^1(L_2V_m(1)_\infty)$.*

Observing the correspondence in the Adams-Novikov E_2 -terms, we obtain

Corollary 1.11. *The homotopy groups $\pi_*(L_2V_m(0))$ are isomorphic to the direct sum of $\Sigma^{-1}Q(0) \otimes \wedge(h_{1,0}, h_{2,0}, h_{2,1})$ and the tensor product of $\wedge(\zeta)$ and*

$$k_m(1)_* \oplus \Sigma^{-1}k_m(1)_*/(2, v_1^\infty, v_2^\infty) \oplus \bigoplus_{k>0} \Sigma^{-1}Q(k) \otimes \wedge(\zeta_{k+1}, \zeta_{k+2})$$

as a $\mathbb{Z}/2[v_1]$ -module.

In the next section, we observe about an action of the Moore spectrum M_2 on $V_m(1)_t$ and a ring structure of $V_m(1)_{4t}$, in order to study the Adams-Novikov differential and the extension problem of the spectral sequence in the following sections. We prove Theorem 1.4 in section three. Section four is devoted to show Proposition 1.6. We end by proving Theorem 1.7 in the last section.

2. The spectrum $T(m) \wedge K_k^t$

We work in the stable homotopy category of spectra localized at the prime two. Let BP denote the Brown-Peterson spectrum. Then, we have the Adams-Novikov spectral sequence

$$E_2^{s,t}(X) = \text{Ext}_\Gamma^{s,t}(A, BP_*(X)) \implies \pi_*(X).$$

Here (A, Γ) is the associated Hopf algebroid such that

$$(A, \Gamma) = (BP_*, BP_*(BP)) = (\mathbb{Z}_{(2)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots])$$

for the Hazewinkel generators $v_k \in BP_{2^{k+1}-2}$ and the generators $t_k \in BP_{2^{k+1}-2}(BP)$.

Let M_k and K_k^t for $k = 2, 4$ and $t > 0$ denote spectra defined by the cofiber sequences

$$S^0 \xrightarrow{2} S^0 \xrightarrow{i} M_k \xrightarrow{j} S^1 \quad \text{and} \quad \Sigma^{8t} M_k \xrightarrow{\alpha^t} M_k \xrightarrow{i_k^t} K_k^t \xrightarrow{j_k^t} \Sigma^{8t+1} M_k.$$

Here α denotes the Adams map such that $BP_*(\alpha) = v_1^4$. Note that M_4 and K_4^t are ring spectra (cf. [5]). The Ravenel spectrum $T(m)$ is characterized by $BP_*(T(m)) = A[t_1, \dots, t_m] \subset \Gamma$ as Γ -comodules, and is a ring spectrum, whose multiplication and unit map we denote by μ and ι , respectively. Throughout the paper, we fix a positive integer m . Let $(A, \Gamma_m) = (A, \Gamma/(t_1, t_2, \dots, t_m))$ be the Hopf algebroid associated with (A, Γ) ,

and consider a spectrum X such that $BP_*(X) = M \otimes_A A[t_1, \dots, t_m]$ for a Γ -comodule M . Then, we have an isomorphism

$$(2.1) \quad E_2^*(X) = \text{Ext}_{\Gamma_m}^*(A, M)$$

by the change of rings theorem (*cf.* [10]). By observing the reduced cobar complex for the Ext group, we have

$$(2.2) \quad \text{The } E_2\text{-term has the vanishing line of the slope } 1/(q_m - 1) \text{ if } M \text{ is } (-1)\text{-connected.}$$

Hereafter, we put

$$(2.3) \quad q_m = 2^{m+2} - 2$$

which is the degree of $u_1 = v_{m+1}$ and $s_1 = t_{m+1}$. This shows $\pi_2(T(m)) = BP_2 = \mathbb{Z}_{(2)}\{v_1\}$ if $m > 0$. Let $T(m)/(v_1^a)$ for an integer $a > 0$ denote the cofiber of $\tilde{v}_1^a : \Sigma^{8a}T(m) \rightarrow T(m)$, where $\tilde{v}_1 : \Sigma^8T(m) \rightarrow T(m)$ is the composite

$$\tilde{v}_1 : \Sigma^8T(m) = S^8 \wedge T(m) \xrightarrow{v_1 \wedge T(m)} T(m) \wedge T(m) \xrightarrow{\mu} T(m).$$

Lemma 2.4. *For $k = 2, 4$ and $a > 0$, $T(m)/(v_1^{4a}) \wedge M_k = T(m) \wedge K_k^a$. In particular, $T(m) \wedge K_2^a \wedge M_4 = T(m)/(v_1^{4a}) \wedge M_2 \wedge M_4 = T(m) \wedge M_2 \wedge K_4^a$.*

Proof. Since $\pi_8(T(m) \wedge M_k) = BP_8/(k) = \mathbb{Z}/k\{v_1^4, v_1v_2\}$ by (2.2), we see that $v_1^4 \wedge M_k = \iota \wedge \alpha i \in \pi_8(T(m) \wedge M_k)$. Indeed, both of these elements are assigned to $v_1^4 \in BP_8(T(m) \wedge M_i)$ under the homomorphism induced from the unit map of BP . It extends to $v_1^4 \wedge M_k = \iota \wedge \alpha : M_k \rightarrow T(m) \wedge M_k$, since $[M_k, T(m) \wedge M_k]_8 = \pi_8(T(m) \wedge M_k)$. Indeed, $\pi_9(T(m) \wedge M_k) = BP_9/(k) = 0$. We further extend it to a self-map $A = \tilde{v}_1^4 \wedge M_k = T(m) \wedge \alpha : T(m) \wedge M_k \rightarrow T(m) \wedge M_k$ by the ring structure of $T(m)$. Now the cofiber of A^a is $T(m)/(v_1^{4a}) \wedge M_k = T(m) \wedge K_k^a$. \square

This lemma implies

$$(2.5) \quad V_m(1)_{4a} = T(m) \wedge K_2^a$$

for the spectrum $V_m(1)_{4a}$ in (1.2).

Lemma 2.6. *Let F denote one of the spectra M_k and K_k^a for $k = 2, 4$ and $a > 0$. Then, there is a pairing $\nu_F : F \wedge F \rightarrow T(m) \wedge F$ such that $\nu_F \circ (F \wedge i_F) = \iota \wedge F : F \rightarrow T(m) \wedge F$ for $m > 0$. Here $i_F : S^0 \rightarrow F$ denotes the inclusion to the bottom cell.*

Proof. The pairing for $F = M_4$ or K_4^a is the composite $(\iota \wedge F \wedge F)(T(m) \wedge \mu_F)$ for the multiplication μ_F of the ring spectrum of F (see [5]).

For $F = M_2$, we see that $\pi_0(T(m) \wedge M_2) = BP_0/(2) = \mathbb{Z}/2$ and $\pi_1(T(m) \wedge M_2) = BP_1/(2) = 0$ by (2.2), and so $[M_2, T(m) \wedge M_2]_0 = \mathbb{Z}/2$.

Note that $M_2 \wedge M_4 = M_2 \vee \Sigma M_2$. Then, by Lemma 2.4,

$$\begin{aligned} T(m) \wedge M_2 \wedge K_4^a &= T(m)/(v_1^{4a}) \wedge M_2 \wedge M_4 = T(m)/(v_1^{4a}) \wedge (M_2 \vee \Sigma M_2) \\ &= T(m)/(v_1^{4a}) \wedge M_2 \vee \Sigma T(m)/(v_1^{4a}) \wedge M_2 = T(m) \wedge K_2^a \vee \Sigma T(m) \wedge K_2^a. \end{aligned}$$

We also see that $T(m) \wedge K_2^a \wedge K_4^a = T(m)/(v_1^{4a}) \wedge K_2^a \wedge M_4 = T(m)/(v_1^{4a}) \wedge (K_2^a \vee \Sigma K_2^a)$, and so $T(m) \wedge K_2^a \wedge K_4^a \wedge M_2 = T(m) \wedge K_2^a \wedge K_2^a \vee \Sigma T(m) \wedge K_2^a \wedge K_2^a$. Then,

$$\begin{aligned} T(m) \wedge M_2 \wedge K_4^a \wedge K_4^a \wedge M_2 &= T(m) \wedge K_2^a \wedge K_4^a \wedge M_2 \vee \Sigma T(m) \wedge K_2^a \wedge K_4^a \wedge M_2 \\ &= T(m) \wedge K_2^a \wedge K_2^a \vee \Sigma T(m) \wedge K_2^a \wedge K_2^a \vee \Sigma T(m) \wedge K_2^a \wedge K_2^a \wedge M_2. \end{aligned}$$

Let $\mu_K : K_4^a \wedge K_4^a \rightarrow K_4^a$ denote the multiplication of the ring spectrum K_4^a , and $\tilde{\nu}$ be the composite $T(m) \wedge M_2 \wedge M_2 \xrightarrow{T(m) \wedge \nu_{M_2}} T(m) \wedge T(m) \wedge M_2 \xrightarrow{\mu \wedge M_2} T(m) \wedge M_2$. Then the desired pairing is a composite

$$\begin{aligned} K_2^a \wedge K_2^a &\xrightarrow{\iota \wedge K \wedge K} T(m) \wedge K_2^a \wedge K_2^a \xrightarrow{inc \wedge K_2^a} T(m) \wedge M_2 \wedge K_4^a \wedge K_4^a \wedge M_2 \xrightarrow{switch} \\ &T(m) \wedge M_2 \wedge M_2 \wedge K_4^a \wedge K_4^a \xrightarrow{\tilde{\nu}} T(m) \wedge M_2 \wedge K_4^a \wedge K_4^a \xrightarrow{T(m) \wedge M_2 \wedge \mu_K} T(m) \wedge M_2 \wedge K_4^a \xrightarrow{prj} T(m) \wedge K_2^a. \end{aligned}$$

\square

Corollary 2.7. *The spectra $V_m(0)$ and $V_m(1)_{4a}$ for $a > 0$ are ring spectra.*

We say that a spectrum X has M_2 -action, if there is a pairing $\varphi_X : X \wedge M_2 \rightarrow X$ such that $\varphi_X(X \wedge i) = id_X$. Here $i : S^0 \rightarrow M_2$ is the inclusion of (1.1) and $id_X : X \rightarrow X$ denotes the identity map.

Lemma 2.8. $V_m(1)_t$ has M_2 -action.

Proof. Since $T(m)$ is an associative ring spectrum, $T(m)/(v_1^t)$ is a $T(m)$ -module spectrum. The action $\varphi_{V_m(1)_t}$ is defined by the composite $V_m(1)_t \wedge M_2 = T(m)/(v_1^t) \wedge M_2 \wedge M_2 \xrightarrow{T(m)/(v_1^t) \wedge \nu_{M_2}} T(m)/(v_1^t) \wedge T(m) \wedge M_2 \rightarrow T(m)/(v_1^t) \wedge M_2 = V_m(1)_t$. \square

Since $V_m(1)_t$ is a $T(m)$ -module spectrum, it implies the following

Corollary 2.9. $V_m(1)_t$ is a $V_m(0)$ -module spectrum.

3. The homotopy groups of $L_2V_m(1)$

Note that if $BP_*(X)$ is $(2, v_1)$ -nil, then $BP_*(L_2X) = v_2^{-1}BP_*(X)$, since L_2 is smashing (cf. [8], [9]). Therefore, the Adams-Novikov E_2 -term $E_2^*(L_2V_m(1)_t)$ is $\text{Ext}_{\Gamma}^*(A, v_2^{-1}BP_*/(2, v_1^t)[t_1, \dots, t_m])$, which is isomorphic to

$$E_2^*(L_2V_m(1)_t) = \text{Ext}_{\Gamma_m}^*(A, v_2^{-1}BP_*/(2, v_1^t))$$

by (2.1). Consider a spectrum

$$E_m(2) = v_2^{-1}BP\langle m+2 \rangle$$

for the Johnson-Wilson spectrum $BP\langle m+2 \rangle$. Then we obtain a Hopf algebroid

$$(E_m(2)_*, \Sigma_m(2)) = (v_2^{-1}\mathbb{Z}_{(2)}[v_1, v_2, \dots, v_{m+2}], E_m(2)_* \otimes_A \Gamma_m \otimes_A E_m(2)_*).$$

Since

$$v_2^{-1}BP_*/J \xrightarrow{1 \otimes \eta_R} E_m(2)_*/J \otimes_A \Gamma_m$$

for an invariant regular ideal $J = (2^b, v_1^a)$ is a faithfully flat extension, we have an isomorphism

$$\text{Ext}_{\Gamma_m}^*(A, BP_*/J) \cong \text{Ext}_{\Sigma_m(2)}^*(E_m(2)_*, E_m(2)_*/J)$$

by a theorem of Hopkins' (cf. [1, Th. 3.3]). Note that $m+2$ is the smallest number n , for which $v_2^{-1}BP_*/J \xrightarrow{1 \otimes \eta_R} v_2^{-1}BP\langle n \rangle_*/J \otimes_A \Gamma_m$ is a faithfully flat extension. We use the abbreviation

$$(3.1) \quad H^*M = \text{Ext}_{\Sigma_m(2)}^*(E_m(2)_*, M)$$

for a $\Sigma_m(2)$ -comodule M . We compute the Ext group H^*M by the reduced cobar complex $\tilde{\Omega}_{\Sigma_m(2)}^*M$ (cf. [10]). Since the differentials of the cobar complex are defined by the right unit $\eta_R : E_m(2)_* \rightarrow \Sigma_m(2)$ and the diagonal $\Delta : \Sigma_m(2) \rightarrow \Sigma_m(2) \otimes_{E_m(2)_*} \Sigma_m(2)$, we write down here some formulas on them based on the Hazewinkel and the Quillen formulas:

$$(3.2) \quad \begin{aligned} v_n &= 2\ell_n - \sum_{k=1}^{n-1} \ell_k v_{n-k}^{2^k} \in \mathbb{Q} \otimes A = \mathbb{Q}[\ell_1, \ell_2, \dots], \\ \eta_R(\ell_n) &= \sum_{k=0}^n \ell_k t_{n-k}^{2^k} \in \mathbb{Q} \otimes \Gamma = \mathbb{Q} \otimes A[t_1, t_2, \dots] \quad \text{and} \\ \sum_{i+j=n} \ell_i \Delta(t_j^{2^i}) &= \sum_{i+j+k=n} \ell_i t_j^{2^i} \otimes t_m^{2^{i+j}} \in \mathbb{Q} \otimes \Gamma \otimes_A \Gamma. \end{aligned}$$

Hereafter, we put $v_2 = 1$ and use the following notation:

$$u_i = v_{m+i} \quad \text{and} \quad s_i = t_{m+i}.$$

Since the structure maps preserve degrees, we may recover v_2 's from its degrees. Then, we obtain the following two lemmas immediately from (3.2) by a routine computation:

Lemma 3.3. The right unit $\eta_R : A \rightarrow \Gamma_m/(2)$ acts as follows:

$$\begin{aligned} \eta_R(v_n) &= v_n \quad \text{for } n \leq m+1, \\ \eta_R(u_2) &= u_2 + v_1 s_1^2 + v_1^{2^{m+1}} s_1, \\ \eta_R(u_3) &\equiv u_3 + s_1^4 + s_1 + v_1 r_1 \pmod{(2, v_1^{2^{m+2}})}, \\ \eta_R(u_4) &\equiv u_4 + s_2^4 + s_2 + v_3 s_1^8 + v_3^{2^{m+1}} s_1 \pmod{(2, v_1)} \end{aligned}$$

for $r_1 = s_2^2 + v_1 u_2 s_1^2$.

This yields the relations in $\Sigma_m(2)$:

$$(3.4) \quad s_1^4 + s_1 \equiv v_1 r_1 \pmod{(2, v_1^{2^{m+2}})} \quad \text{and} \quad s_2^4 + s_2 + v_3 s_1^8 + v_3^{2^{m+1}} s_1 \equiv 0 \pmod{(2, v_1)}.$$

Lemma 3.5. The diagonal Δ behaves on the generators s_i as follows:

$$\begin{aligned} \Delta(s_1) &= s_1 \otimes 1 + 1 \otimes s_1, \\ \Delta(s_2) &= s_2 \otimes 1 + 1 \otimes s_2 + v_1 s_1 \otimes s_1, \\ \Delta(s_3) &\equiv s_3 \otimes 1 + 1 \otimes s_3 + v_2 s_1^2 \otimes s_1^2 \pmod{(2, v_1)}. \end{aligned}$$

Lemma 3.6. *Let z denote an element defined by $r_1^4 + r_1 + v_3^2 s_1^4 + v_3^{2m+2} s_1^2 = v_1 z$. Then the cochains $r_1, z \in \widetilde{\Omega}_{\Sigma_m(2)}^1 E_m(2)_*/(2)$ are cocycles. Besides, $z \equiv u_2 s_1^2$ modulo (v_1^2) .*

Proof. Since $v_1 \in \widetilde{\Omega}_{\Sigma_m(2)}^0 E_m(2)_*/(2)$ and $s_1 \in \widetilde{\Omega}_{\Sigma_m(2)}^1 E_m(2)_*/(2)$ are both cocycles, so is r_1 by the relation $v_1 r_1 = s_1^4 + s_1 \in \Sigma_m(2)$ in (3.4). Furthermore, $v_3 \in \widetilde{\Omega}_{\Sigma_m(2)}^0 E_m(2)_*/(2)$ is a cocycle. It follows similarly from its definition that z is a cocycle. By the definition of r_1 , $r_1^4 + r_1 \equiv s_2^8 + s_2^2 + v_1 u_2 s_1^2 \equiv v_1 u_2 s_1^2 + v_3^2 s_1^{16} + v_3^{2m+2} s_1^2$ modulo $(2, v_1^2)$ by (3.4). \square

We now work as [6].

Lemma 3.7. *$u_2^t \in E_2^0(V_m(1))$ and $u_2^t h_{2,0} \in E_2^1(V_m(1))$ for each $t > 0$ are permanent cycles.*

Proof. For $t = 1$, the lemma is seen by (2.2). Consider the cofiber sequence $\Sigma^2 V_m(0) \xrightarrow{v_1} V_m(0) \xrightarrow{i_1} V_m(1) \xrightarrow{j_1} \Sigma^3 V_m(0)$. Put $d(u_2^t) = v_1 k_t' \in \widetilde{\Omega}_{\Sigma_m(2)}^1 E_m(2)_*/(2)$ by virtue of Lemma 3.3, and let $k_t \in E_2^1(V_m(0))$ be the homology class of the cocycle k_t' . Then, $k_1 = h_{1,1}$, $v_1 k_t = 0$ and $k_{t+1} = \langle k_1, v_1, k_t \rangle$. Indeed, $\langle k_1, v_1, k_t \rangle$ is the class of $k_1' \eta_R(u_2^t) + u_2 k_t' = d(u_2^{t+1})/v_1 = k_{t+1}'$. Besides, $\delta(u_2^t) = k_t$ for the connecting homomorphism associated to the cofiber sequence. Let $\xi_1 \in \pi_{q_m-1}(V_m(0))$ denote the homotopy element detected by k_1 . Then, $v_1 \xi_1 = \xi_1 v_1 = 0$.

Suppose now that $u_2^t \in E_2^0(V_m(1))$ is a permanent cycle. Then, k_t is a permanent cycle that detects the element $\xi_t = j_1 u_2^t$ by the Geometric Boundary Theorem. Since $v_1 \xi_t = 0$, the Toda bracket $\{\xi_1, v_1, \xi_t\}$ is defined, which is detected by the Massey product $\langle k_1, v_1, k_t \rangle$. Note here that the Toda bracket is defined since $V_m(0)$ is a ring spectrum. It follows that k_{t+1} is a permanent cycle and detects a homotopy element, which we denote by ξ_{t+1} . Since the Massey product $\langle v_1, k_1, v_1 \rangle$ is zero in the E_2 -term $E_2^{0, q_m+4}(V_m(0))$, we see that $\{v_1, \xi_1, v_1\} = 0$ by (2.2). Now we compute $v_1 \{\xi_1, v_1, \xi_t\} = \{v_1, \xi_1, v_1\} \xi_t = 0$, and ξ_{t+1} is pulled back to u_2^{t+1} under the map j_1 .

Turn to $u_2^t h_{2,0}$. In this case a similar argument works. For the connecting homomorphism δ , $\delta(u_2^t h_{2,0}) = \langle h_{1,0}^2, v_1, k_t \rangle$, which detects a homotopy element $\{\eta_0^2, v_1, \xi_t\}$, where η_0 denotes an element detected by $h_{1,0}$. Applying v_1 shows $\{v_1, \eta_0^2, v_1\} \xi_t = 0$. Indeed, $\{v_1, \eta_0^2, v_1\}$ is detected by $E_2^{s, 2q_m+4+s}(V_m(0))$ for $s > 2$. \square

Lemma 3.8. *The elements $h_{1,0}, h_{1,1} \in E_2^1(V_m(0))$ and $h_{2,1} \in E_2^1(L_2 V_m(0))$ are permanent cycles.*

Proof. $h_{1,0}, h_{1,1}$ are seen immediately by (2.2).

The cobar module $\widetilde{\Omega}_{\Gamma_m}^{4, 4q_m+6} BP_*/(2)$ is generated by $v_1^3 s_1^{\otimes 4}$ and $v_2 s_1^{\otimes 4}$ by degree reason. The first generator cobounds $v_2^2 s_2 \otimes s_1 \otimes s_1$, and we obtain $E_2^{4, 4q_m+6}(V_m(0)) = \mathbb{Z}/2\{v_2 h_{1,0}^4\}$. Put $d_3(h_{2,1}) = av_2 h_{1,0}^4 \in E_2^{4, 4q_m+6}(V_m(0))$ for $a \in \mathbb{Z}/2$. Let w be an element fit in $d(s_3) = v_2 s_1^2 \otimes s_1^2 + v_1 w$ by virtue of Lemma 3.5. Then, $d(w) = 0$ in the cobar complex $\widetilde{\Omega}_{\Sigma_m(2)}^3 E_m(2)_*/(2)$, and we see that $s_1^{\otimes 4}$ cobounds $s_3^2 \otimes s_1 \otimes s_1 + v_1 w^2 \otimes s_2 + (r_1 \otimes s_1 + s_1 \otimes r_1 + v_1 r_1 \otimes r_1) \otimes s_2$ (in which we set $v_2 = 1$). It follows that $d_3(h_{2,1}) = av_2 h_{1,0}^4 = 0 \in E_2^4(L_2 V_m(0))$ as desired. Indeed, $v_2 h_{1,0}^4 = v_1 g h_{1,0}^2 = 0$, since $v_2 h_{1,0}^2 = v_1 g$ for an element g and $v_1 h_{1,0}^2 = 0$ by $d(s_2) = v_1 s_1 \otimes s_1$. \square

Proof of Theorem 1.4. Every element $x \in E_2^s(L_2 V_m(1))$ is decomposed as $x = x' x''$ for $x' \in \mathbb{Z}/2[u_2] \otimes \wedge(h_{2,0})$ and $x'' \in K_{m-1}(2)_* \otimes \wedge(h_{1,0}, h_{1,1}, h_{2,1})$. Note that $K_{m-1}(2)_* \otimes \wedge(h_{1,0}, h_{1,1}, h_{2,1}) \subset E_2^*(L_2 V_m(0))$. Since x' (resp. x'') is a permanent cycle of the Adams-Novikov spectral sequence for computing $\pi_*(L_2 V_m(1))$ (resp. $\pi_*(L_2 V_m(0))$) by Lemma 3.7 (resp. 3.8), we obtain that the element x is a permanent cycle from Corollary 2.9. We see that the extension problem is trivial by Lemma 2.8. Indeed, $\mathbb{Z}/2 = \pi_0(M_2)$ acts on $\pi_*(L_2 V_m(1))$. \square

4. The elements x_n

We introduce the integer b_n for $n \geq 0$ by

$$b_n = \begin{cases} a_n - 8 & n \equiv 1 \pmod{3} \\ a_n - 3 & n \equiv 2 \pmod{3} \\ 0 & n \equiv 0 \pmod{3}, \end{cases}$$

and the elements $x_n \in E_m(2)_*$ defined by

$$(4.1) \quad x_n = x_{n-1}^2 + v_1^{b_n} y_{n-1}, \quad \text{where} \quad y_n = \begin{cases} 0 & n \leq 0 \text{ or } n \equiv 2 \pmod{3} \\ x_0 & n = 1 \\ x_2 + v_1^2 v_3^4 x_1^2 + v_1^4 v_3^{2^{m+3}} x_1 & n = 3 \\ x_{n-2} y_{n-3} & n \equiv 0, 1 \pmod{3} \text{ and } n \geq 4. \end{cases}$$

We also consider cocycles $z_n \in \Sigma_m(2)$:

$$(4.2) \quad z_n = \begin{cases} s_1^{2^{n+1}} & n = 0, 1 \\ r_1^{2^{n-1}} & n = 2, 3 \\ x_{n-3} z_{n-3} & n > 3. \end{cases}$$

Proposition 4.3. *For the differential $d : \Omega_{\Sigma_m(2)}^0 E_m(2)_*/(2) \rightarrow \Omega_{\Sigma_m(2)}^1 E_m(2)_*/(2)$ of the cobar complex,*

$$d(x_n) = v_1^{a_n} z_n.$$

Proof. For $n = 0$ and 1 , it is immediate from Lemma 3.3, and the cases for $n = 2$ and 3 follow from the computation $d(x_2) = d(u_2^4 + v_1^3 u_2) = v_1^4 s_1^8 + v_1^4 s_1^2 = v_1^6 r_1^2$ by (3.4). For $n = 4$,

$$\begin{aligned} d(x_4) &\equiv d(x_2^4 + v_1^{18} x_2 + v_1^{20} v_3^4 x_1^2 + v_1^{22} v_3^{2^{m+3}} x_1) \\ &\equiv v_1^{24} r_1^8 + v_1^{24} r_1^2 + v_1^{24} v_3^4 s_1^8 + v_1^{24} v_3^{2^{m+3}} s_1^4 \equiv v_1^{26} z^2 \equiv v_1^{26} x_1 z_1 \pmod{(2, v_1^{28})} \end{aligned}$$

by the definition of z .

Suppose inductively that $d(x_{3k+1}) = v_1^{a_{3k+1}} x_{3k-2} z_{3k-2} \pmod{(2, v_1^{a_{3k+1}+2})}$ for $k > 0$.

$$\begin{aligned} d(x_{3k+1}^2) &\equiv v_1^{2a_{3k+1}} x_{3k-2}^2 z_{3k-2}^2 \pmod{(2, v_1^{2a_{3k+1}+4})} \\ d(v_1^{a_{3k+2}-3} y_{3k+1}) &\equiv d(v_1^{a_{3k+2}-3} x_{3k-1} y_{3k-2}) \\ &\equiv v_1^{a_{3k+2}-3} x_{3k-1} (v_1 z_{3k-2}^2 + v_1^3 z_{3k-1}) \pmod{(2, v_1^{a_{3k+2}-3+a_{3k-1}})} \end{aligned}$$

and the sum shows $d(x_{3k+2}) \equiv v_1^{a_{3k+2}} x_{3k-1} z_{3k-1} \pmod{(2, v_1^{a_{3k+2}+2})}$. Similarly,

$$\begin{aligned} d(x_{3k+2}^4) &\equiv v_1^{4a_{3k+2}} x_{3k-1}^4 z_{3k-1}^4 \pmod{(2, v_1^{4a_{3k+2}+8})} \\ d(v_1^{a_{3k+4}-8} y_{3k+3}) &\equiv d(v_1^{a_{3k+4}-8} x_{3k+1} y_{3k}) \\ &\equiv v_1^{a_{3k+4}-8} x_{3k+1} (v_1^6 z_{3k}^2 + v_1^8 z_{3k+1}) \pmod{(2, v_1^{a_{3k+4}-8+a_{3k+1}})} \end{aligned}$$

and we have $d(x_{3k+4}) = v_1^{a_{3k+4}} x_{3k+1} z_{3k+1} \pmod{(2, v_1^{a_{3k+4}+2})}$, which completes the induction. \square

Proof of Lemma 1.10. It suffices to show that $h_{1,0}/v_1^j \in E_2^1(L_2 V_m(1)_\infty)$ equals ζ/v_1^{j-2} . The element $h_{1,0}/v_1^j$ is represented by s_1/v_1^j . We make a computation in the cobar complex

$$\begin{aligned} d(u_2^2/v_1^{j+2}) &= s_1^4/v_1^j = s_1/v_1^j + r_1/v_1^{j-1} \\ d(v_3^2 u_2^2/v_1^{j+1}) &= v_3^2 s_1^4/v_1^{j-1} \\ d(v_3^{2^{m+2}} u_2/v_1^j) &= v_3^{2^{m+2}} s_1^2/v_1^{j-1} \\ d(x_2^2/v_1^{j+1}) &= r_1^4/v_1^{j-1}. \end{aligned}$$

by Lemma 3.3 and Proposition 4.3. The sum yields the homologous relation $s_1/v_1^j \sim z/v_1^{j-2}$ by Lemma 3.6, and so $h_{1,0}/v_1^j = \zeta/v_1^{j-2}$ in $E_2^1(L_2 V_m(1)_\infty)$. \square

Proof of Proposition 1.6. We consider the v_1 -Bockstein spectral sequence given by the short exact sequence $0 \rightarrow E_m(2)_*(V_m(1)) \xrightarrow{\varphi} E_m(2)_*(V_m(1)_\infty) \xrightarrow{v_1} E_m(2)_*(V_m(1)_\infty) \rightarrow 0$ for φ given by $\varphi(x) = x/v_1$. Let B^* denote the $\mathbb{Z}/2[v_1]$ -module of the proposition. Then, it is easy to see that B^s contains the image of $\varphi_* : E_2^s(L_2 V_m(1)) \rightarrow E_2^s(L_2 V_m(1)_\infty)$ and that Proposition 4.3 defines a homomorphism $f : B^s \rightarrow E_2^s(L_2 V_m(1)_\infty)$. We also consider the composite $\partial = \delta \circ f : B^s \rightarrow E_2^{s+1}(L_2 V_m(1))$, where $\delta : E_2^s(L_2 V_m(1)_\infty) \rightarrow E_2^{s+1}(L_2 V_m(1))$ denotes the connecting homomorphism associated to the short exact sequence. By [7, Remark 3.11], it suffices to show the sequence

$$(4.4) \quad 0 \rightarrow \text{Coker } \partial \xrightarrow{\varphi_*} B^* \xrightarrow{v_1} B^* \xrightarrow{\partial} \text{Im } \partial \rightarrow 0$$

is exact.

We decompose $E_2^*(L_2V_m(1))$ into the direct sum of $M_C = K_{m-1}(2)_*[u_2^2]\{u_2\} \otimes \wedge(h_{10}, h_{20}, h_{21})$, $M_I = K_{m-1}(2)_*[u_2^2]\{h_{11}\} \otimes \wedge(h_{10}, h_{20}, h_{21})$ and $N \otimes \wedge(\zeta) = K_{m-1}(2)_*[u_2^2] \otimes \wedge(h_{10}, h_{20}, h_{21}, \zeta)$. We notice that for non-negative integers n and r with $r < 8$, there exist uniquely non-negative integers t and q such that $n = 8^qt + re_q$. By this fact, we decompose summands of N as follows:

$$\begin{aligned}
& K_{m-1}(2)_*[u_2^2] \\
&= K_{m-1}(2)_* \oplus \bigoplus_{k \geq 1} \underline{x_k K_{m-1}(2)_*[x_{k+1}]_A}, \\
& K_{m-1}(2)_*[u_2^2]h_{10} \\
&= \bigoplus_{q \geq 0} \left(\left(\underline{x_{3q+2} K_{m-1}(2)_*[x_{3q+3}]_a} \oplus \underline{x_{3q+3} K_{m-1}(2)_*[x_{3q+4}]_b} \right) \zeta_{3q+4} \oplus \underline{\underline{K_{m-1}(2)_*[x_{3q+2}] \zeta_{3q+1}}_A} \right), \\
& K_{m-1}(2)_*[u_2^2]h_{20} \\
&= \bigoplus_{q \geq 0} \left(\underline{x_{3q+3} K_{m-1}(2)_*[x_{3q+4}] \zeta_{3q+5}}_c \oplus \left(\underline{x_{3q+1} K_{m-1}(2)_*[x_{3q+2}]_d} \oplus \underline{\underline{K_{m-1}(2)_*[x_{3q+3}]_A}} \right) \zeta_{3q+2} \right), \\
& K_{m-1}(2)_*[u_2^2]h_{21} \\
&= \bigoplus_{q \geq 0} \left(\underline{x_{3q+1} K_{m-1}(2)_*[x_{3q+2}]_e} \oplus \underline{x_{3q+2} K_{m-1}(2)_*[x_{3q+3}]_f} \oplus \underline{\underline{K_{m-1}(2)_*[x_{3q+4}]_A}} \right) \zeta_{3q+3}, \\
& K_{m-1}(2)_*[u_2^2]h_{10}h_{20} \\
&= \bigoplus_{q \geq 0} \left(\underline{\underline{K_{m-1}(2)_*[x_{3q+3}] \zeta_{3q+4} \zeta_{3q+2}}_a} \oplus \underline{x_{3q+3} K_{m-1}(2)_*[x_{3q+4}] \zeta_{3q+4} \zeta_{3q+5}}_B \oplus \underline{\underline{K_{m-1}(2)_*[x_{3q+2}] \zeta_{3q+1} \zeta_{3q+2}}_d} \right), \\
& K_{m-1}(2)_*[u_2^2]h_{20}h_{21} \\
&= \bigoplus_{q \geq 0} \left(\underline{\underline{K_{m-1}(2)_*[x_{3q+4}] \zeta_{3q+3} \zeta_{3q+5}}_c} \oplus \left(\underline{x_{3q+1} K_{m-1}(2)_*[x_{3q+2}]_B} \oplus \underline{\underline{K_{m-1}(2)_*[x_{3q+3}]_f}} \right) \zeta_{3q+2} \zeta_{3q+3} \right), \\
& K_{m-1}(2)_*[u_2^2]h_{10}h_{21} \\
&= \bigoplus_{q \geq 0} \left(\left(\underline{K_{m-1}(2)_*[x_{3q+3}] x_{3q+2}}_B \oplus \underline{\underline{K_{m-1}(2)_*[x_{3q+4}]_b}} \right) \zeta_{3q+4} \zeta_{3q+3} \oplus \underline{\underline{K_{m-1}(2)_*[x_{3q+2}] \zeta_{3q+1} \zeta_{3q+3}}_e} \right), \\
& K_{m-1}(2)_*[u_2^2]h_{10}h_{20}h_{21} \\
&= \bigoplus_{k \geq 1} \underline{\underline{K_{m-1}(2)_*[x_{k+1}] \zeta_k \zeta_{k+1} \zeta_{k+2}}_B}.
\end{aligned}$$

Here, \underline{M}_X and $\underline{\underline{M}}'_X$ for modules M and M' mean that M and M' are isomorphic under a Bockstein differential d_r for some r so that $d_r(M) = M'$, which is seen by Proposition 4.3. Let N_C (resp. N_I) be the direct sum of single (resp. double) underlined submodules of N , and put $\tilde{M} = Q(0) \otimes \wedge(h_{1,0}, h_{2,0}, h_{2,1})$, $\tilde{N} = \bigoplus_{k > 0} Q(k) \otimes \wedge(\zeta_{k+1}, \zeta_{k+2})$. Then we have the three exact sequences

$$\begin{aligned}
0 \rightarrow M_C \xrightarrow{\varphi_*} \tilde{M} \xrightarrow{v_1} \tilde{M} \rightarrow M_I \rightarrow 0, \quad 0 \rightarrow N_C \xrightarrow{\varphi_*} \tilde{N} \xrightarrow{v_1} \tilde{N} \rightarrow N_I \rightarrow 0 \quad \text{and} \\
0 \rightarrow K_{m-1}(2)_* \rightarrow E_{m-1}(2)_*/(2, v_1^\infty) \rightarrow E_{m-1}(2)_*/(2, v_1^\infty) \rightarrow 0,
\end{aligned}$$

the direct sum of which yields the sequence (4.4). \square

5. The Adams-Novikov E_∞ -term for $\pi_*(L_2T(m) \wedge M_2)$

We first show that all elements of the Adams-Novikov E_2 -term for $\pi_*(L_2V_m(1)_\infty)$ are permanent cycles. Take an element $x/v_1^t \in E_2^0(L_2V_m(1)_\infty)$. Then $x \in E_2^0(L_2V_m(1)_t)$. Thus, if $x = y^2/v_1^t$ for some $y \in E_2^0(L_2V_m(1)_{4t})$, then x is a permanent cycle. So it is sufficient to show that $d_3(x_n) = 0 \in E_2^3(L_2V_m(1)_{a_n})$ for each $n \geq 0$. We consider the integer

$$\varepsilon_n = \begin{cases} 2 & n \not\equiv 0 \pmod{3} \\ 0 & n \equiv 0 \pmod{3} \end{cases}$$

so that $V_m(1)_{a_n + \varepsilon_n}$ is a ring spectrum by Corollary 2.7.

Lemma 5.1. $d_3(x_n) = 0 \in E_2^3(L_2V_m(1)_{a_n})$ for $n \geq 0$.

Proof. For $n = 0$, it is shown in Lemma 3.7.

Suppose that $d_3(x_n) = \xi \in E_2^3(L_2V_m(1)_{a_n})$ for $n > 0$. Send this to $E_2^3(L_2V_m(1)_{a_{n-1}})$, and we see that $\xi = d_3(x_n) = d_3(x_{n-1}^2) \in E_2^3(L_2V_m(1)_{a_{n-1}})$. Then, the map $v_1^{\varepsilon_{n-1}} : E_2^3(L_2V_m(1)_{a_{n-1}}) \rightarrow E_2^3(L_2V_m(1)_{a_{n-1} + \varepsilon_{n-1}})$ assigns $v_1^{\varepsilon_{n-1}} \xi$ to $v_1^{2\varepsilon_{n-1}} \xi = d_3((v_1^{\varepsilon_{n-1}} x_{n-1})^2)$, which is zero, since $v_1^{\varepsilon_{n-1}} x_{n-1} \in E_2^0(L_2V_m(1)_{a_{n-1} + \varepsilon_{n-1}})$ and $V_m(1)_{a_{n-1} + \varepsilon_{n-1}}$ is a ring spectrum. It follows that $\xi = v_1^{a_{n-1} - \varepsilon_{n-1}} \xi'$ for some $\xi' \in E_2^3(L_2V_m(1)_{a_n - a_{n-1} + \varepsilon_{n-1}})$.

Note that this works even if $n = 1$, though $V_m(1)$ is not a ring spectrum. Consider the commutative diagram

$$\begin{array}{ccccccc}
V_m(1) & \xlongequal{\quad} & V_m(1) & \xrightarrow{\quad} & * & \xrightarrow{\quad} & V_m(1) \\
\downarrow v_1^{a_n - a_{n-1} + \varepsilon_{n-1}} & & \downarrow v_1^{a_n} & & \downarrow & & \downarrow v_1^{a_n - a_{n-1} + \varepsilon_{n-1}} \\
V_m(1)_{a_n - a_{n-1} + \varepsilon_{n-1} + 1} & \xrightarrow{v_1^{a_{n-1} - \varepsilon_{n-1}}} & V_m(1)_{a_n + 1} & \xrightarrow{i_v} & V_m(1)_{a_{n-1} - \varepsilon_{n-1}} & \xrightarrow{j_v} & V_m(1)_{a_n - a_{n-1} + \varepsilon_{n-1} + 1} \\
\downarrow & & \downarrow & & \parallel & & \downarrow p \\
V_m(1)_{a_n - a_{n-1} + \varepsilon_{n-1}} & \xrightarrow{v_1^{a_{n-1} - \varepsilon_{n-1}}} & V_m(1)_{a_n} & \xrightarrow{i'_v} & V_m(1)_{a_{n-1} - \varepsilon_{n-1}} & \xrightarrow{j'_v} & V_m(1)_{a_n - a_{n-1} + \varepsilon_{n-1}},
\end{array}$$

in which rows are cofiber sequences. Let $\langle x \rangle \in \pi_*(X)$ denote a homotopy element detected by $x \in E_2^*(X)$. Noticing that $x_n \in E_2^0(L_2V_m(1)_{a_n - 1 - \varepsilon_{n-1}})$ is a permanent cycle, we see that $j_{v*}(\langle x_n \rangle) = \langle v_1^{a_n - a_{n-1} + \varepsilon_{n-1}} \zeta_n \rangle$ and $j'_{v*}(\langle x_n \rangle) = \langle \xi' \rangle$, and so $p_*(\langle v_1^{a_n - a_{n-1} + \varepsilon_{n-1}} \zeta_n \rangle) = \langle \xi' \rangle$. Since $\langle \zeta_n \rangle \in \pi_*(L_2V_m(1))$ by Theorem 1.4, we obtain $\langle \xi' \rangle = 0$, and $\langle x_n \rangle$ is in the image under the map i'_{v*} . It follows that there is a permanent cycle $x'_n \in E_2^0(L_2V_m(1)_{a_n})$, whose leading term is x_n , such that $i_{v*}(\langle x'_n \rangle) = \langle x_n \rangle \in \pi_*(L_2V_m(1)_{a_n - 1 - \varepsilon_{n-1}})$. The lemma now follows by replacing x_n by x'_n . \square

References

- [1] M. Hovey and H. Sadofsky, Invertible spectra in the $E(n)$ -local stable homotopy category, *J. London Math. Soc.* **60** (1999) 284–302.
- [2] I. Ichigi and K. Shimomura, Subgroups of $\pi_*(L_2T(1))$ at the prime two, *Hiroshima Math. J.* **33** (2003), 359–369.
- [3] I. Ichigi, K. Shimomura and X. Wang, On subgroups of $\pi_*(L_2T(1) \wedge M(2))$ at the prime two, to appear in *Bulletin of the Mexican Mathematical Society*. **13**, (2007).
- [4] Y. Kamiya and K. Shimomura, The homotopy groups $\pi_*(L_2V(0) \wedge T(k))$, *Hiroshima Math. J.* **31** (2001), 391–408.
- [5] M. Mahowald, The construction of small ring spectra, *Geom. Appl. Homotopy Theory II*, Proc. Conf., Evanston 1977, Lect. Notes Math. **658** (1978), 234–239.
- [6] M. Mahowald and K. Shimomura, The Adams-Novikov spectral sequence for the L_2 localization of a v_2 spectrum, the Proceedings of the International Congress in Algebraic Topology, Edited by M. Tangora, 1991, *Contemporary Math.* **146** (1993), 237–250.
- [7] H. R. Miller, D. C. Ravenel, and W. S. Wilson, Periodic phenomena in the Adams-Novikov spectral sequence, *Ann. of Math.* **106** (1977), 469–516.
- [8] D. C. Ravenel, Localization with respect to certain periodic homology theories, *Amer. J. Math.* , **106** (1984), 351–414.
- [9] D. C. Ravenel, *Nilpotence and Periodicity in Stable Homotopy Theory*, Annals of Mathematics Studies, Number 128, Princeton, 1992.
- [10] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, AMS Chelsea Publishing, Providence, 2004.
- [11] K. Shimomura, The homotopy groups of the L_2 -localized Mahowald spectrum $X(1)$, *Forum Math.* **7** (1995), 685–707.
- [12] K. Shimomura and X. Wang, The Adams-Novikov E_2 -term for $\pi_*(L_2S^0)$ at the prime 2 , *Math. Z.* **241** (2002), 271–311.

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