# THE FIRST LINE OF THE BOCKSTEIN SPECTRAL SEQUENCE ON A MONOCHROMATIC SPECTRUM AT AN ODD PRIME

RYO KATO AND KATSUMI SHIMOMURA

Abstract. The chromatic spectral sequence is introduced in [8] to compute the $E_2$-term of the Adams-Novikov spectral sequence for computing the stable homotopy groups of spheres. The $E_1$-term $E_1^{s,t}(k)$ of the spectral sequence is an Ext group of $BP_*BP$-comodules. There are a sequence of Ext groups $E_1^{s,t}(n-s)$ for non-negative integers $n$ with $E_1^{s,t}(0) = E_1^{s,t}$, and Bockstein spectral sequences computing a module $E_1^{s,*}(n-s)$ from $E_1^{s-1,*}(n-s+1)$. So far, a small number of the $E_1$-terms are determined. Here, we determine the $E_1^{1,1}(n-1) = \mathrm{Ext}^1 M_{n-1}^1$ for $p > 2$ and $n > 3$ by computing the Bockstein spectral sequence with $E_1$-term $E_1^{0,s}(n)$ for $s = 1, 2$. As an application, we study the non-triviality of the action of $\alpha_1$ and $\beta_1$ in the homotopy groups of the second Smith-Toda spectrum $V(2)$.

## 1. Introduction

Let $p$ be a prime number, $\mathcal{S}_{(p)}$ the stable homotopy category of $p$-local spectra, and $S$ the sphere spectrum localized at $p$. Understanding homotopy groups $\pi_*(S)$ of $S$ is one of the principal problems in stable homotopy theory. The main vehicle for computing $\pi_*(S)$ is the Adams-Novikov spectral sequence based on the Brown-Peterson spectrum $BP$. $BP$ is the $p$-typical component of $MU$, the complex cobordism spectrum, and that it has homotopy groups $BP_* = \pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$ where $v_n$ is a canonical generator of degree $2p^n - 2$. In order to study the $E_2$-term of the Adams-Novikov spectral sequence, H. Miller, D. Ravenel and S. Wilson [8] introduced the chromatic spectral sequence. It was designed to compute the $E_2$-term, but has the following deeper connotation. Let $L_n\colon \mathcal{S}_{(p)} \to \mathcal{S}_{(p)}$ denote the Bousfield-Ravenel localization functor with respect to $v_n^{-1}BP$ (*cf.* [12]). It gives rise the chromatic filtration $\mathcal{S}_{(p)} \to \cdots \to L_n\mathcal{S}_{(p)} \to L_{n-1}\mathcal{S}_{(p)} \to \cdots \to L_0\mathcal{S}_{(p)}$ of the stable homotopy category of spectra, which is a powerful tool for understanding the category. The chromatic $n$th layer of the spectrum $S$ can be determined from the homotopy groups of $L_{K(n)}S$, the Bousfield localization of $S$ with respect to the $n$th Morava $K$-theory $K(n)$ that it has homotopy groups $K(n)_* = v_n^{-1}\mathbb{Z}/p[v_n]$ for $n > 0$ and $K(0)_* = \mathbb{Q}$. By the chromatic convergence theorem of Hopkins-Ravenel [13], $S$ is the inverse limit of the $L_nS$. Let $E(n)$ be the $n$th Johnson-Wilson spectrum $E(n)$ with $E(n)_* = v_n^{-1}\mathbb{Z}_{(p)}[v_1, \cdots, v_n]$ for $n > 0$ and $E(0) = K(0)$. It is Boufield equivalent to $v_n^{-1}BP$ and also to $K(0) \vee \cdots \vee K(n)$, i.e. $L_{E(n)} = L_n = L_{K(0)\vee\cdots\vee K(n)}$. We notice that $E(0) = H\mathbb{Q}$, the rational Eilenberg-MacLane spectrum, and $E(1)$ is the $p$-local Adams summand of periodic complex $K$-theory. Futhermore, $E(2)$ is closely related to elliptic cohomology. So far, we have no geometric interpretation of homology theories $K(n)$ or $E(n)$ when $n > 2$.

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From now on, we assume that the prime $p$ is odd. We explain the $E_1$-term of the chromatic spectral sequence. The Brown-Peterson spectrum $BP$ is a ring spectrum that induces the Hopf algebroid $(BP_*, BP_*(BP)) = (BP_*, BP_*[t_1, t_2, \dots])$ in the standard way [14], and we have an induced Hopf algebroid

$$(E(n)_*, E(n)_*(E(n))) = (E(n)_*, E(n)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E(n)_*)$$

where $E(n)_*$ is considered to be a $BP_*$-module by sending $v_k$ to zero for $k > n$. Then, the $E_1$-term is given by

$$E_1^{s,t}(n-s) = \mathrm{Ext}^t_{E(n)_*(E(n))}(E(n)_*, M^s_{n-s}).$$

Here, $M^s_{n-s}$ denotes the $E(n)_*(E(n))$-comodule $E(n)_*/(I_{n-s} + (v_{n-s}^\infty, v_{n-s+1}^\infty, \dots, v_{n-1}^\infty))$, in which $I_k$ denotes the ideal of $E(n)_*$ generated by $v_i$ for $0 \le i < k$ ($v_0 = p$), and $M/(w^\infty)$ for $w \in E(n)_*$ and an $E(n)_*$-module $M$ denotes the cokernel of the localization map $M \to w^{-1}M$. In order to study the stable homotopy groups $\pi_*(L_{K(n)}S)$, we study here the homotopy groups of the monochromatic component $M_nS$ of $S$ (see [12]). Then, the $E_2$-term $E_2^{s,t}(M_nS)$ of the Adams-Novikov spectral sequence for computing $\pi_*(M_nS)$ is the $E_1$-term $E_1^{n,s}(0)$ of the chromatic spectral sequence. In [8], the authors also introduced the $v_{n-s}$-Bockstein spectral sequence $E_1^{s-1,t+1}(n-s+1) \Rightarrow E_1^{s,t}(n-s)$ associated to a short exact sequence

$$0 \to M^{s-1}_{n-s+1} \xrightarrow{\varphi} M^s_{n-s} \xrightarrow{v_{n-s}} M^s_{n-s} \to 0$$

of $E(n)_*(E(n))$-comodules, where $\varphi(x) = x/v_{n-s}$. So far, the $E_1$-term $E_1^{s,t}(n-s)$ is determined in the following cases (*cf.* [14]):

$$\begin{aligned}
(s,t,n) &= (0,t,n) \quad \text{for (a) } n \le 2, \text{ (b) } n = 3,\ p > 3, \text{ (c) } t \le 2 \text{ by Ravenel [11],} \\
&\qquad\qquad \text{(Henn [2] for } n = 2 \text{ and } p = 3), \\
&= (1,0,n) \quad \text{for } n \ge 0 \text{ by Miller, Ravenel and Wilson [8],} \\
&= (s,t,n) \quad \text{for } n \le 2 \text{ by Shimomura and his colaborators: Arita [1],} \\
&\qquad\qquad \text{Tamura [20], Yabe [21] and Wang [22], ([15], [18], [19]),} \\
&= (1,1,3) \quad \text{by Shimomura [16], Hirata and Shimomura [3],} \\
&= (2,0,n) \quad \text{for } n > 3 \text{ by Shimomura [17], for } n = 3 \text{ by Nakai [9], [10].}
\end{aligned}$$

In this paper, we determine the structure of $E_1^{1,1}(n-1)$ for $n > 3$. The case $n = 3$, which is special, is treated in [16] and [3]. The result is the first step to understand $\pi_*(L_{K(n)}S)$ for $n > 3$ as explained above. We proceed to state the result.

In this paper, we consider only the cases $s = 0$ and $s = 1$, and, hereafter, put

$$v = v_n \quad \text{and} \quad u = v_{n-1}.$$

Furthermore, we put

$$F = \mathbb{Z}/p,$$

and consider the coefficient ring $K(n)_* = F[v_n^{\pm 1}] = F[v^{\pm 1}] = E(n)_*/I_n$,

$$A = E(n)_*/I_{n-1} \quad \text{and} \quad B = M^1_{n-1} = A/(u^\infty) = \text{Coker } (A \to u^{-1}A).$$

Since the ideal $I_{n-1}$ is invariant, $(A, \Gamma) = (A, E(n)_*(E(n))/I_{n-1})$ is a Hopf algebroid, and we use the abbreviation

$$\mathrm{Ext}^s M = \mathrm{Ext}^s_\Gamma(A, M)$$

for a $\Gamma$-comodule $M$. Then, the chromatic $E_1$-terms are

$$E_1^{0,t}(n) = \mathrm{Ext}^t K(n)_* \quad \text{and} \quad E_1^{1,t}(n-1) = \mathrm{Ext}^t B.$$

We have the $u$-Bockstein spectral sequence

$$E_1 = \mathrm{Ext}^* K(n)_* \Longrightarrow \mathrm{Ext}^* B \tag{1.1}$$

associated to the short exact sequence

$$0 \to K(n)_* \xrightarrow{\varphi} B \xrightarrow{u} B \to 0, \tag{1.2}$$

where $\varphi$ is a homomorphism defined by $\varphi(x) = x/u$.

Let $R$ be a ring, and let $R\langle g\rangle$ denote the $R$-module generated by $g$. The $E_1$-term of the $u$-Bockstein spectral sequence was determined by Ravenel [11] as follows:

**Theorem 1.3.** $\mathrm{Ext}^0 K(n)_* = K(n)_*$ *and*

$$\begin{array}{rcl} \mathrm{Ext}^1 K(n)_* & = & K(n)_* \langle h_i, \zeta_n : 0 \le i < n\rangle, \\ \mathrm{Ext}^2 K(n)_* & = & K(n)_* \langle \zeta_n h_i, b_i, g_i, k_i, h_j h_k : 0 \le i < n,\ 0 \le j < k-1 < n-1\rangle. \end{array}$$

In the theorem, the generators $h_i$ and $b_i$ are represented by $t_1^{p^i}$ and $\sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k} t_1^{kp^i} \otimes t_1^{(p-k)p^i}$ of the cobar complex $\Omega_\Gamma^* K(n)_*$, respectively, and $g_i$ and $k_i$ are given by the Massey products

$$g_i = \langle h_i, h_i, h_{i+1}\rangle \quad \text{and} \quad k_i = \langle h_i, h_{i+1}, h_{i+1}\rangle. \tag{1.4}$$

In order to determine the module $\mathrm{Ext}^0 B$, Miller, Ravenel and Wilson [8] introduced elements $x_i$ and integers $a_i$ in [8, (5.11) and (5.13)], where they denoted them by $x_{n,i}$ and $a_{n,i}$, such that $x_i \equiv v^{p^i} \bmod I_n$ with the action of the connecting homomorphism $\delta$ given in [8, (5.18)]:

$$\delta(v^s/u) = sv^{s-1}h_{n-1} \quad \text{and} \quad \delta(x_i^s/u^{a_i}) = sv^{(sp-1)p^{i-1}} h_{[i-1]} \text{ for } i \ge 1. \tag{1.5}$$

Hereafter, we let

$$[i] \in \{0, 1, \dots, n-2\}$$

be the principal representative of the integer $i$ module $n-1$. The elements $x_i$ and the integers $a_i$ are defined inductively by $x_0 = v$ and $a_0 = 1$, and for $i > 0$,

$$\begin{array}{rcl} x_i & = & \begin{cases} x_{i-1}^p & \text{for } i = 1 \text{ or } [i] \ne 1, \\ x_{i-1}^p - u^{b_{n,i}} v^{p^i - p^{i-1}+1} & \text{for } i > 1 \text{ and } [i] = 1, \text{ and} \end{cases} \\ a_i & = & \begin{cases} pa_{i-1} & \text{for } i = 1 \text{ or } [i] \ne 1, \\ pa_{i-1} + p - 1 & \text{for } i > 1 \text{ and } [i] = 1. \end{cases} \end{array} \tag{1.6}$$

Here, $b_{n,k(n-1)+1} = (p^n - 1)(p^{k(n-1)} - 1)/(p^{n-1} - 1)$. The result (1.5) determines the differentials of the Bockstein spectral sequence, which implies:

**Theorem 1.7.** ([8, Th. 5.10]) *As a $k_*$-module,*

$$\mathrm{Ext}^0 B = L_\infty \oplus \bigoplus_{p \nmid s, i \ge 0} L_{a_i} \langle x_i^s\rangle.$$

*Here, $k_* = k(n-1)_* = F[u]$, $L_i = k_*/(u^i)$ and $L_\infty = k_*/(u^\infty) = \varinjlim_i L_i$.*

This theorem together with (1.5) implies the following:

**Corollary 1.8.** *The cokernel of $\delta\colon \mathrm{Ext}^0 B \to \mathrm{Ext}^1 K(n)_*$ is the $F$-module generated by*

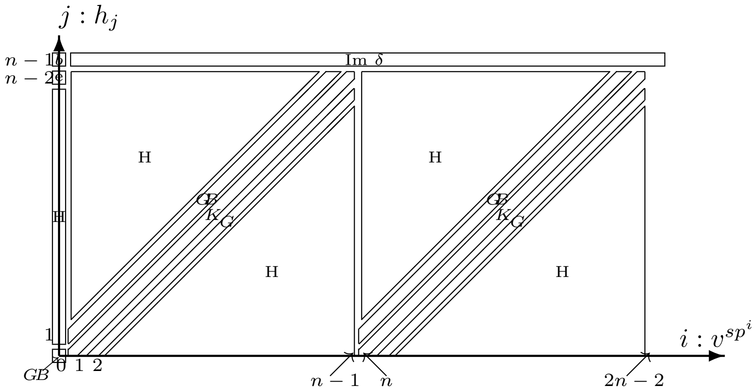
$$\begin{array}{ll} v^t \zeta_n, & v^{tp-1} h_{n-1}, \quad h_j \quad \textit{for } 0 \le j < n-1, \textit{ and} \\ v^{sp^k} h_j & \textit{for } 0 \le j < n-1, \textit{ where } [k] \ne [j],\ s \not\equiv -1\ (p), \textit{ or } s \equiv -1\ (p^2), \end{array}$$

*for integers $s$ and $t$ with $p \nmid s$.*

By Theorem 1.3, the module $\mathrm{Ext}^1 K(n)_*$ is the direct sum of $\zeta_n \mathrm{Ext}^0 K(n)_* = \zeta_n K(n)_*$, $F\langle h_j\rangle$ for $j \in \mathbb{Z}/(n-1)$ and the modules

$$V_{(i,j,s)} = F\left\langle v^{sp^i} h_j \right\rangle$$

for $(i,j,s) \in \mathbb{N} \times \mathbb{Z}/n \times \overline{\mathbb{Z}}$. Here, $\mathbb{N}$ denotes the set of non-negative integers, and $\overline{\mathbb{Z}} = \mathbb{Z} \setminus p\mathbb{Z}$. We partition $\mathbb{N} \times \mathbb{Z}/n$ as follows:




More precisely,

$$\begin{array}{rcl} H & = & \{(0,j) : 1 \le j < n-2\} \\ & & \cup\{(i,j) : i > 0,\ [i] \ne n-3, n-2,\ 2 + [i] \le j \le n-2\} \\ & & \cup\{(i,j) : i > 0,\ [i] \ne 0, 1,\ 0 \le j \le [i]-2\}, \\ GB & = & \{(i,[i]) : i \ge 0\}, \\ K & = & \{(i,[i]-1) : i > 0,\ [i] \ne 0\} \quad \text{and} \\ G & = & \{(i,[i]-2) : i > 1,\ [i] \ne 0, 1\}. \end{array}$$

We introduce notation

$$\begin{array}{rcl} V_{(0,n-2)} & = & \bigoplus_{s\in\overline{\mathbb{Z}}'} V_{(0,n-2,s)}, \\ V_{(0,n-1)} & = & \bigoplus_{t\in\mathbb{Z}} V_{(0,n-1,tp-1)} \;=\; F[v^{\pm p}]\left\langle v^{-1}h_{n-1}\right\rangle, \\ C_X & = & \bigoplus_{(i,j)\in X,\ s\in\overline{\mathbb{Z}}} V_{(i,j,s)} \quad \text{for a subset } X \subset \mathbb{N}\times\mathbb{Z}/n, \\ \overline{C}_{GB} & = & \bigoplus_{(i,j)\in GB}\left(\left(\bigoplus_{s\in\overline{\overline{\mathbb{Z}}}} V_{(i,j,s)}\right) \oplus \left(\bigoplus_{t\in\mathbb{Z}} V_{(i,j,tp^2-1)}\right)\right) \\ & = & \bigoplus_{(i,[i],s)\in\widetilde{GB}} V_{(i,j,s)} \oplus \bigoplus_{i\ge 0} F[v^{\pm p^{i+2}}]\left\langle v^{-p^i}h_{[i]}\right\rangle \quad \text{and} \\ C_O & = & F\left\langle \theta, h_j : j \in \mathbb{Z}/(n-1)\right\rangle. \end{array}$$

Here, for $e(i) = (p^i-1)/(p-1)$, $\theta = v^{e(n-2)}h_{n-2}$,

$$\begin{array}{rcl} \overline{\mathbb{Z}}' & = & \overline{\mathbb{Z}} \setminus \{e(n-2)\}, \quad \overline{\overline{\mathbb{Z}}} \;=\; \{n \in \overline{\mathbb{Z}} : p \nmid (s+1)\} \quad \text{and} \\ \widetilde{GB} & = & \{(i,[i],s) : s \in \overline{\overline{\mathbb{Z}}}\}. \end{array}$$

We also consider the subset $\boldsymbol{T}$ of $\mathbb{N}\times\mathbb{Z}/n\times\overline{\mathbb{Z}}$ defined by

$$\begin{array}{rcl} \boldsymbol{T} & = & \{(i,j,s) \in \mathbb{N}\times\mathbb{Z}/n\times\overline{\mathbb{Z}} :\ p\nmid(s+1) \text{ or } p^2 \mid (s+1) \text{ if } [i]=j, \\ & & \quad p \mid (s+1) \text{ if } (i,j)=(0,n-1), \text{ and } s \ne e(n-2) \text{ if } (i,j)=(0,n-2)\}. \end{array}$$

In this notation, the cokernel of $\delta$ in Corollary 1.8 is given by

$$\begin{aligned}\text{Coker }\delta &= \zeta_n K(n)_* \oplus C_O \oplus \textstyle\bigoplus_{(i,j,s)\in \boldsymbol{T}} V_{(i,j,s)} \\ &= \zeta_n K(n)_* \oplus C_O \oplus V_{(0,n-2)} \oplus V_{(0,n-1)} \oplus C_H \oplus C_K \oplus C_G \oplus \overline{C}_{GB}\end{aligned} \tag{1.9}$$

Finally, we consider the $k_*$-modules:

$$\begin{aligned}W_{(i,j,s)} &= L_{a(i,j,s)}\left\langle x_i^s h_j\right\rangle, \\ W_{(0,n-2)} &= \textstyle\bigoplus_{s\in\overline{\mathbb{Z}}'} W_{(0,n-2,s)}, \\ W_{(0,n-1)} &= \textstyle\bigoplus_{t\in\mathbb{Z}} W_{(0,n-1,tp-1)}, \\ B_X &= \textstyle\bigoplus_{(i,j)\in X,\ s\in\overline{\mathbb{Z}}} W_{(i,j,s)} \quad \text{for a subset } X\subset \mathbb{N}\times\mathbb{Z}/n, \\ \overline{B}_{GB} &= \textstyle\bigoplus_{(i,j)\in GB}\left(\left(\bigoplus_{s\in\overline{\overline{\mathbb{Z}}}} W_{(i,j,s)}\right)\oplus\left(\bigoplus_{t\in\mathbb{Z}} W_{(i,j,tp^2-1)}\right)\right) \quad \text{and} \\ C_\infty &= (K(n-1)_*/k_*)\left\langle \theta, h_j : j\in\mathbb{Z}/(n-1)\right\rangle.\end{aligned}$$

Here, $a(i,j,s)$ denotes an integer defined as follows: for $(i,j)=(0,n-2)$, $a(0,n-2,s)=2$ if $p\nmid s(s-1)$, and

$$a(0,n-2,s) = \begin{cases} a_l & p\nmid t,\ l>0,\ [l]\neq 0,n-2, \\ a_l+e(n-2)+p^{n-3} & p\nmid t,\ l>0,\ [l]=n-2, \\ a_l+1 & p\nmid t,\ l>0,\ [l]=0 \end{cases}$$

if $s=tp^l+e(n-2)$; for $(i,j)\in\{(0,n-1)\}\cup H\cup K\cup G\cup GB$,

$$a(i,j,s) = \begin{cases} p-1 & (i,j)=(0,n-1), \\ a_i & (i,j)\in H, \\ a_i+a_{i-1} & (i,j)\in K\cup G, \\ 2a_i & (i,j,s)\in \widetilde{GB}, \\ (p-1)a_{i+1} & (i,j)\in GB,\ p^2\mid (s+1). \end{cases}$$

**Theorem 1.10.** *The chromatic $E_1$-term $\mathrm{Ext}^1 B = \mathrm{Ext}^1 M^1_{n-1}$ is canonically isomorphic to the $k_*$-module*

$$\zeta_n \mathrm{Ext}^0 B \oplus C_\infty \oplus W_{(0,n-2)} \oplus W_{(0,n-1)} \oplus B_H \oplus B_K \oplus B_G \oplus \overline{B}_{GB}.$$

Let $V(n)$ be the $n$th Smith-Toda spectrum defined by $BP_*(V(n)) = BP_*/I_{n+1}$. As an application of the theorem, we study the action of $\alpha_1$ and $\beta_1$ on the elements $u^t$ $(t>0)$ in the Adams-Novikov $E_2$-term $E_2^*(V(n))$ in section 6. In particular, it leads us an geometric result for $n=4$. In [23], Toda constructed the self map $\gamma$ on $V(2)$ to show the existence of $V(3)$ for the prime $p>5$. We notice that $\gamma^t i\in\pi_*(V(2))$ for the inclusion $i\colon S\to V(2)$ to the bottom cell is detected by $u^t = v_3^t \in BP_*(V(2))$ in the Adams-Novikov spectral sequence.

**Theorem 1.11.** *Let $p>5$. Then $\gamma^t i\alpha_1$ and $\gamma^t i\beta_1$ are nontrivial in $\pi_*(V(2))$ for $t>0$.*

## 2. Bockstein spectral sequence

We compute the Bockstein spectral sequence by use of the following lemma.

**Lemma 2.1.** *Let $\delta\colon \mathrm{Ext}^s B \to \mathrm{Ext}^{s+1} K(n)_*$ be the connecting homomorphism associated to the short exact sequence (1.2). Suppose that* Coker $\delta = \bigoplus_k V_k \subset$

*$\mathrm{Ext}^1K(n)_*$ and $\bigoplus_k U_k \subset \mathrm{Ext}^2K(n)_*$ for $F$-modules $V_k$ and $U_k$, and there exist $u$-torsion $k_*$-modules $W_k$ fitting in a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V_k & \xrightarrow{\varphi'_*} & W_k & \xrightarrow{u} & W_k & \xrightarrow{\delta'} & U_k \\
 & & \downarrow & & {\scriptstyle f_k}\downarrow & & \downarrow{\scriptstyle f_k} & & \downarrow \\
0 & \longrightarrow & \mathrm{Coker}\,\delta & \xrightarrow{\varphi_*} & \mathrm{Ext}^1B & \xrightarrow{u} & \mathrm{Ext}^1B & \xrightarrow{\delta} & \mathrm{Ext}^2K(n)_*
\end{array}$$

*of exact sequences. Then,* $\mathrm{Ext}^1B = \bigoplus_k W_k$.

This follows immediately from [8, Remark 3.11].

Let $\widetilde{\theta}$ be an element of Corollary 5.8. Then, $\widetilde{\theta}/u^k$ and $h_j/u^k$ for $j \in \mathbb{Z}/(n-1)$ belong to $\mathrm{Ext}^1B$, and we define the map $f\colon C_\infty \to \mathrm{Ext}^1B$ by $f((u^{-k})\theta) = \widetilde{\theta}/u^k$ and $f((u^{-k})h_j) = h_j/u^k$ for $(u^{-k}) \in K(n-1)_*/k_*$, so that the short exact sequence

$$0 \to C_O \xrightarrow{1/u} C_\infty \xrightarrow{u} C_\infty \to 0 \tag{2.2}$$

yields a summand of Lemma 2.1.

Note that if a cocycle $z$ represents $\zeta_n$, then so does $z^p$. Therefore, we have $\zeta_n/u^j \in \mathrm{Ext}^1B$ represented by $z^{p^j}/u^j$. The exact sequence (1.2) induces the exact sequence $0 \to \mathrm{Ext}^0K(n)_* \xrightarrow{\varphi_*} \mathrm{Ext}^0B \xrightarrow{u} \mathrm{Ext}^0B \xrightarrow{\delta} \mathrm{Ext}^1K(n)_*$, and we have an exact sequence

$$0 \to \zeta_n\mathrm{Ext}^0K(n)_* \xrightarrow{\varphi_*} \zeta_n\mathrm{Ext}^0B \xrightarrow{u} \zeta_n\mathrm{Ext}^0B \xrightarrow{\delta} \zeta_n\mathrm{Ext}^1K(n)_*, \tag{2.3}$$

which is a summand of Lemma 2.1. Together with (2.2) and (2.3), Theorem 1.10 follows from Lemma 2.1 if the following sequence is exact for each $(i,j,s) \in \boldsymbol{T}$:

$$0 \to V_{(i,j,s)} \xrightarrow{\varphi'_*} W_{(i,j,s)} \xrightarrow{u} W_{(i,j,s)} \xrightarrow{\delta'} U_{(i,j,s)}, \tag{2.4}$$

where $U_{(i,j,s)}$ denotes an $F$-module generated by a single generator as follows: for $(i,j) = (0,n-2)$, $U_{(0,n-2,s)} = F\left\langle v^{s-2}k_{n-2}\right\rangle$ if $p \nmid s(s-1)$,

$$U_{(0,n-2,s)} = \begin{cases} F\left\langle v^{s-p^{l-1}}h_{[l-1]}h_{n-2}\right\rangle & p\nmid t,\ l>0,\ [l]\neq 0, n-2, \\ F\left\langle v^{s-p^{l-1}}b_{2n-5}\right\rangle & p\nmid t,\ l>0,\ [l]=n-2, \\ F\left\langle v^{s-p^{l-1}-1}g_{n-2}\right\rangle & p\nmid t,\ l>0,\ [l]=0; \end{cases}$$

if $s = tp^l + e(n-2)$; for $(i,j) \in \{(0,n-1)\} \cup H \cup K \cup G \cup GB$,

$$U_{(i,j,s)} = \begin{cases} F\left\langle v^{s-p+1}b_{n-1}\right\rangle & (i,j) = (0,n-1), \\ F\left\langle v^{(sp-1)p^{i-1}}h_{[i-1]}h_j\right\rangle & (i,j) \in H, \\ F\left\langle v^{(s-2)p}k_{n-1}\right\rangle & (i,j) = (1,0) \in K, \\ F\left\langle v^{(sp^2-p-1)p^{i-2}}k_{[i-2]}\right\rangle & (i,j) \in K,\ i>1, \\ F\left\langle v^{(sp^2-p-1)p^{i-2}}g_{[i-2]}\right\rangle & (i,j) \in G, \\ F\left\langle v^{s-p-1}g_{n-1}\right\rangle & (i,j,s) \in \widetilde{GB},\ i=0, \\ F\left\langle v^{(sp-2)p^{i-1}}g_{[i-1]}\right\rangle & (i,j,s) \in \widetilde{GB},\ i>0, \\ F\left\langle v^{(s+1-p)p^{i}}b_j\right\rangle & (i,j) \in GB,\ p^2 \mid (s+1). \end{cases}$$

Since the mapping $\boldsymbol{T} \to \{U_{(i,j,s)} : (i,j,s) \in \boldsymbol{T}\}$ assigning $(i,j,s)$ to $U_{(i,j,s)}$ is an injection, we see the following:

**Lemma 2.5.** *The direct sum of* $\zeta_n \mathrm{Ext}^1 K(n)_*$ *and* $U_{(i,j,s)}$ *for* $(i,j,s) \in \boldsymbol{T}$ *is a sub-$F$-module of* $\mathrm{Ext}^2 K(n)_*$.

The homomorphism $f_k$ in Lemma 2.1 on $W_{(i,j,s)}$ for $(i,j,s) \in \boldsymbol{T}$ is explicitly given by

$$f_{(i,j,s)}(x) = x/u^{a(i,j,s)}.$$

It follows that the homomorphism $\delta'$ on it is given by the composite $\delta(1/u^{a(i,j,s)})$. Hereafter we denote it by $\delta'_{(i,j,s)}$, that is, $\delta'_{(i,j,s)} = \delta(1/u^{a(i,j,s)})$, and consider a condition:

$(2.6)_{(i,j,s)}$ $\quad\delta'_{(i,j,s)}(x) = y$ for the generators $x \in W_{(i,j,s)}$ and $y \in U_{(i,j,s)}$.

Note that $\varphi'_*(\overline{x}) = u^{a(i,j,s)-1}x$ for the generators $\overline{x} \in V_{(i,j,s)}$ and $x \in W_{(i,j,s)}$, since $f_k\varphi'_*(\overline{x}) = \varphi_*(\overline{x}) = x/u$. Then,

**Lemma 2.7.** *For each* $(i,j,s) \in \boldsymbol{T}$, *if the condition* $(2.6)_{(i,j,s)}$ *holds, then* (2.4) *for* $(i,j,s)$ *is exact and yields a summand of Lemma 2.1.*

The relations in (1.5) show immediately

(2.8) *The condition* $(2.6)_{(i,j,s)}$ *holds for* $(i,j) \in H$.

*Proof of Theorem 1.10.* The theorem follows from Lemmas 2.1, 2.5 and 2.7 together with (2.2), (2.3), (2.8), Lemmas 3.7, 3.8, 4.1 and 5.9, in which the lemmas are proved below. Indeed, the direct sum of $\zeta_n \mathrm{Ext}^0 K(n)_*$, $C_O$ and $V_{(i,j,s)}$ for $(i,j,s) \in \boldsymbol{T}$ is the cokernel of $\delta$ by (1.9). $\square$

## 3. The summands on $V_{(0,n-1)}$ and $\overline{C}_{GB}$

We begin with stating some formulae on the Hopf algebroid $(A, \Gamma)$:

$$(3.1)\quad \begin{array}{rcl} 0 & = & vt_k^{p^n} + ut_{k+1}^{p^{n-1}} - u^{p^{k+1}}t_{k+1} - t_k\eta_R(v^{p^k}) \in \Gamma \quad \text{for } k < n, \\ \eta_R(u) & = & u, \quad \eta_R(v) \ = \ v + ut_1^{p^{n-1}} - u^p t_1, \\ \Delta(t_k) & = & \sum_{i=0}^k t_i \otimes t_{k-i}^{p^i} \quad \text{for } k < n, \text{ and} \\ \Delta(t_n) & = & \sum_{i=0}^n t_i \otimes t_{n-i}^{p^i} - ub_{n-2}. \end{array}$$

Then the connecting homomorphism $\delta\colon \mathrm{Ext}^1 B \to \mathrm{Ext}^2 K(n)_*$ is computed by the differential $d\colon \Omega^1_\Gamma A \to \Omega^2_\Gamma A$ of the cobar complex modulo an ideal, which is defined by

$$(3.2)\qquad d(x) = 1 \otimes x - \Delta(x) + x \otimes 1.$$

We also use the differential $d\colon \Omega^0_\Gamma A \to \Omega^1_\Gamma A$ defined by $d(w) = \eta_R(w) - \eta_L(w)$. For $w, w' \in \Omega^0_\Gamma A$ and $x \in \Omega^1_\Gamma A$, these differentials satisfy

$$(3.3)\quad \begin{array}{rcl} d(ww') & = & d(w)\eta_R(w') + wd(w'), \ d(wx) \ = \ d(w) \otimes x + wd(x), \text{ and} \\ d(x\eta_R(w)) & = & d(x)\eta_R(w) - x \otimes d(w). \end{array}$$

We also use the Steenrod operations $P^0$ and $\beta P^0$ on $\mathrm{Ext}^* C(j)$ for $j \geq 1$ and $\mathrm{Ext}^* B$ (*cf.* [6], [14]). Here, $C(j)$ denotes the comodule $A/(u^j)$, and we notice that $C(1) = K(n)_*$. Let $\widetilde{\Omega}^s M = \Omega^s_{E(n)_*(E(n))} M$ for an $E(n)_*(E(n))$-comodule $M$. Given a cocycle $x(j)$ of $\widetilde{\Omega}^s C(j)$, $\widetilde{x}(j)$ denotes a cochain of $\widetilde{\Omega}^s E(n)_*$ such that $\pi_j(\widetilde{x}(j)) = x(j)$ for the projection $\pi_j\colon \widetilde{\Omega}^s E(n)_* \to \widetilde{\Omega}^s C(j)$. Since $x(j)$ is a cocycle,

$d(\widetilde{x}(j)^p) = py_j + \sum_{i=1}^{n-2} v_i^p z_{j,i} + u^{jp} z_{j,n-1}$ for some elements $y_j$ and $z_{j,i} \in \widetilde{\Omega}^{s+1}E(n)_*$. Under this situation, the Steenrod operations are defined by

$$P^0([x(j)]) = [x(j)^p] \quad \text{and} \quad \beta P^0([x(j)]) = [y_j] \in \mathrm{Ext}^* C(jp), \quad \text{and}$$
$$P^0([x(j)/u^j]) = [x(j)^p/u^{jp}] \quad \text{and} \quad \beta P^0([x(j)/u^j]) = [y_j/u^{jp}] \in \mathrm{Ext}^* B.$$

Here, $[x]$ denotes the homology class represented by a cocycle $x$. In particular, the operation acts on our elements as follows:

$$\beta P^0(x_i/u^{a_i}) = \begin{cases} v^{p-1}h_{n-1}/u^{p-1} & i=0, \\ x_{i-1}^{p^2-1}h_{[i-1]}/u^{(p-1)a_i} & i>0, \end{cases} \quad \text{in } \mathrm{Ext}^1 B; \tag{3.4}$$

$$P^0(x_i^s h_k/u^j) = \begin{cases} x_{i+1}^s h_{k+1}/u^{jp} & k \neq n-2, \\ x_{i+1}^s h_0/u^{jp-p+1} & k = n-2; \end{cases} \quad \text{in } \mathrm{Ext}^1 B; \text{ and} \tag{3.5}$$
$$\beta P^0(x_i^s h_k) = x_{i+1}^s b_k \quad \text{in } \mathrm{Ext}^2 K(n)_*.$$

The following is a folklore (cf. [14, Corollary A1.5.5]):

$$P^0\delta = \delta P^0 \quad \text{and} \quad \beta P^0 \delta = -\delta\beta P^0 \quad \text{in } \mathrm{Ext}^* K(n)_*. \tag{3.6}$$

**Lemma 3.7.** *The condition* $(2.6)_{(i,j,s)}$ *holds for each* $(i,j,s) \in \{(0,n-1,tp-1),(i,j,tp^2-1) : t \in \mathbb{Z}, (i,j) \in GB\}$.

*Proof.* For $k \geq -1$, consider a generator $x(k,t) = x_k^{tp^2-1}h_{[k]}$ for $k \geq 0$ and $x(-1) = x_0^{tp-1}h_{n-1}$, and $\overline{(k,t)}$ denotes a triple $(k,[k],tp^2-1)$ if $k \geq 0$ and $(0,n-1,tp-1)$ if $k=-1$. Then, $(1/u^{a\overline{(k,t)}})(x(k,t)) = x_{k+2}^{t-1}\beta P^0(x_{k+1}/u^{a_{k+1}})$ for $k \geq -1$ by (3.4). Now, $\delta'_{\overline{(k,t)}}(x(k,t))$ equals

$$x_{k+2}^{t-1}\delta(\beta P^0(x_{k+1}/u^{a_{k+1}})) = -x_{k+2}^{t-1}(\beta P^0(x_k^{p-1}h_{\overline{[k]}})) = -x_{k+1}^{\nu(t)}b_{\overline{[k]}}$$

by (3.6), (1.5) and (3.5). Here, $(\nu(t),\overline{[k]}) = (tp-1,[k])$ if $k \geq 0$ and $= ((t-1)p, n-1)$ if $k=-1$. □

**Lemma 3.8.** *The condition* $(2.6)_{(i,[i],s)}$ *holds for* $(i,[i],s) \in \widetilde{GB}$.

*Proof.* We prove this by induction on $i$. By (3.1) and (3.2), we compute mod $(u^3)$

$$\begin{aligned} d(v^{s+1-p}t_1^{p^n}) &\equiv (s+1)uv^{s-p}t_1^{p^{n-1}} \otimes t_1^{p^n} + \tbinom{s+1}{2}u^2v^{s-p-1}t_1^{2p^{n-1}} \otimes t_1^{p^n} \\ d((s+1)uv^{s-p}t_2^{p^{n-1}}) &\equiv s(s+1)u^2v^{s-p-1}t_1^{p^{n-1}} \otimes t_2^{p^{n-1}} - (s+1)uv^{s-p}t_1^{p^{n-1}} \otimes t_1^{p^n} \end{aligned}$$

to obtain $\delta(v^s h_0/u^2) = s(s+1)v^{s-p-1}g_{n-1}$ and so

$$\delta'_{(0,0,s)}(v^s h_0) = s(s+1)v^{s-p-1}g_{n-1}.$$

Apply $P^0$ to it, and we obtain

$$\begin{aligned} \delta'_{(1,1,s)}(v^{sp}h_1) &= \delta(P^0(v^s h_0/u^2)) = P^0\delta(v^s h_0/u^2) = s(s+1)P^0(v^{s-p-1}g_{n-1}) \\ &= s(s+1)v^{sp-p^2-p}g_n = s(s+1)v^{sp-2}g_0. \end{aligned}$$

Here, we notice that $g_n = v^{p^2+p-2}g_0$ in $\mathrm{Ext}^2 K(n)_*$ by (3.1). Suppose inductively that $\delta'_{(i,1,s)}(x_i^s h_1) = s(s+1)v^{(sp-2)p^{i-1}}g_0$ for $[i]=1$, which is $(2.6)_{(i,1,s)}$. Note that $a_{i+j} = pa_{i+j-1}$ if $0 < j < n-2$, and we see that $P^0\delta'_{(i,j,s)} = \delta'_{(i+1,j+1,s)}P^0$ by (3.6). Therefore, $(P^0)^j$ for $j < n-2$ yields the equation for $\delta'_{a(i+j,j+1,s)}(x_{i+j}^s h_{j+1})$. At $i' = i+n-2$, for $t = (i',0,s)$, $\delta'_t(x_{i'}^s h_0) = \delta P^0(x_{i'-1}^s h_{n-2}/u^{a(i'-1,n-2,s)})$ (by (3.5)) $= s(s+1)v^{(sp-2)p^{i+n-3}}g_{n-2}$ by (3.6) and inductive hypothesis.

Note that $a_{i+n-1} = p^{n-1}a_i + p - 1$. Consider the connecting homomorphism $\delta_j\colon \mathrm{Ext}^1 M^1_{n-1} \to \mathrm{Ext}^2 C(j)$ associated to the short exact sequence $0 \to C(j) \xrightarrow{1/u^j} M^1_{n-1} \xrightarrow{u^j} M^1_{n-1} \to 0$. Then, $u^{j-1}\delta = \delta_j u^{j-1}$. Besides, $\delta_j(P^0)^k = (P^0)^k\delta$ if $p^k \geq j$. Now in $\mathrm{Ext}^2 C(p^2+p-1)$, $u^{p^2+p-2}\delta'_{(i+n-1,1,s)}(x^s_{i+n-1}h_1)$ equals

$$\begin{aligned} u^{p^2+p-2}\delta(x^s_{i+n-1}h_1/u^{p^{n-1}a+2(p-1)}) &= \delta_{p^2+p-1}(P^0)^{n-1}(x^s_i h_1/u^a) \\ = (P^0)^{n-1}(s(s+1)v^{(sp-2)p^{i-1}}g_0) &= s(s+1)v^{(sp-2)p^{i+n-2}}g_{n-1} \end{aligned}$$

for $a = a(i,[i],s)$, which equals $s(s+1)u^{p^2+p-2}v^{(sp-2)p^{i+n-2}}g_0$ by the relation $u^{p+2}g_{n-1} = u^{p^2+2p}g_0$. This relation follows from (1.4) and $uh_{n-1} = u^p h_0$ given by $d(v)$. □

## 4. The summands $C_G$ and $C_K$

We study the action of the connecting homomorphism $\delta$ by use of the Massey product. We notice that this is also shown by use of $P^0$-operation considered in the previous section, but we use the Massey product for the sake of simplicity.

**Lemma 4.1.** *The condition* $(2.6)_{(i,j,s)}$ *holds for* $(i,j) \in G \cup K$.

*Proof.* We consider the element $(1/u^{a(i,j,s)})(x^s_i h_j)$ the Massey product $\left\langle sx^{sp-1}_{i-1}/u^{a_{i-1}}, h_{[i-1]}, h_j \right\rangle$. Then, $\delta'_{(i,j,s)}(x^s_i h_j) = \delta\left\langle sx^{sp-1}_{i-1}/u^{a_{i-1}}, h_{[i-1]}, h_j \right\rangle = \left\langle s\delta(x^{sp-1}_{i-1}/u^{a_{i-1}}), h_{[i-1]}, h_j \right\rangle$, which equals $-\left\langle sv^{sp-2}h_{n-1}, h_0, h_0 \right\rangle = -sv^{(s-2)p}k_{n-1}$ if $i = 1$, and $-\left\langle sv^{(sp^2-p-1)p^{i-2}}h_{[i-2]}, h_{[i-1]}, h_j \right\rangle = \begin{cases} -sv^{(sp^2-p-1)p^{i-2}}k_{j-1} & j = [i-1], \\ -2sv^{(sp^2-p-1)p^{i-2}}g_j & j = [i-2] \end{cases}$ otherwise. Here, we note that $\langle h_i, h_{i+1}, h_i \rangle = 2g_i$. □

## 5. The summand $V_{(0,n-2)}$

Consider the elements $c_i = u^{p^i}h_{n-1+i}$ and $c'_i = u^{p^{i+1}}h_i$ of $\mathrm{Ext}^1 A$. The elements have internal degrees $|c_i| = |c'_i| = p^i e(n)q$ for $q = 2p-2$, and satisfy

$$c_i = c'_i, \quad c_i c_{i+1} = 0, \quad h_{n+i}c_i = 0 \quad \text{and} \quad h_{i+1}c_i = h_{i+1}c'_i = 0.$$

We consider the cochains $\overline{w}_k = u^{e(k-1)}ct_k^{p^{n-1}}$ of the cobar complex $\Omega^1_\Gamma A$. Then,

$$\overline{w}_k = -\overline{w}^p_{k-1}\eta_R(v) + u^{pe(k-2)}v^{p^{k-1}}ct_{k-1} + u^{p^k+pe(k-2)}ct_k \tag{5.1}$$

for $k > 1$ by (3.1). Let $w_k$ be a cochain of the cobar complex $\Omega^1_\Gamma A$ defined inductively by:

$$\begin{aligned} w_1 &= t_1^{p^{n-1}} - u^{p-1}t_1 = -\overline{w}_1 + u^{p-1}ct_1 \quad \text{and} \\ w_k &= w^p_{k-1}\eta_R(v) + (-1)^k u^{pe(k-2)}v^{p^{k-1}}ct_{k-1} \end{aligned} \tag{5.2}$$

and put

$$\begin{aligned} m'_k &= -\textstyle\sum_{i=1}^{k-1}(-1)^i u^{p^{i-1}}w^{p^i}_{k-i}\otimes\overline{w}_i \quad \text{and} \\ m_k &= u^{p^{k-1}}w_k + \textstyle\sum_{i=1}^{k-1}(-1)^i u^{p^{i-1}}v^{p^i e(k-i)}\overline{w}_i. \end{aligned} \tag{5.3}$$

**Lemma 5.4.** $d(v^{e(k)}) = m_k$. *Besides,* $d(w_k) = m'_k$ *if* $k \leq n$.

*Proof.* We prove the lemma inductively. Since $d(v) = uw_1 = m_1$, we see the case for $k = 1$. Indeed, $m'_1 = 0$.

Suppose that the equalities hold for $k-1$. Then, we compute by (3.3), (5.1) and (5.2),

$$\begin{aligned} d(v^{e(k)}) &= d(v^{pe(k-1)})\eta_R(v) + v^{pe(k-1)}d(v) \\ &= \left(u^{p^{k-1}}w_{k-1}^p + \textstyle\sum_{i=1}^{k-2}(-1)^i u^{p^i}v^{p^{i+1}e(k-1-i)}\overline{w}_i^p\right)\eta_R(v) - uv^{pe(k-1)}\left(\overline{w}_1 - u^{p-1}ct_1\right) \\ &= u^{p^{k-1}}\left(w_k - (-1)^k u^{pe(k-2)}v^{p^{k-1}}ct_{k-1}\right) - uv^{pe(k-1)}\left(\overline{w}_1 - u^{p-1}ct_1\right) \\ &\quad + \textstyle\sum_{i=1}^{k-2}(-1)^i u^{p^i}v^{p^{i+1}e(k-1-i)}\left(-\overline{w}_{i+1} + \left(u^{pe(i-1)}v^{p^i}ct_i + u^{p^{i+1}+pe(i-1)}ct_{i+1}\right)\right), \end{aligned}$$

which equals $m_k$, and similarly,

$$\begin{aligned} d(w_k) &= -\textstyle\sum_{i=1}^{k-2}(-1)^i u^{p^i}w_{k-1-i}^{p^{i+1}} \otimes \overline{w}_i^p\eta_R(v) + uw_{k-1}^p \otimes (\overline{w}_1 - u^{p-1}ct_1) \\ &\quad +(-1)^k u^{pe(k-2)}\left(u^{p^{k-1}}w_1^{p^{k-1}} \otimes ct_{k-1} + v^{p^{k-1}}d(ct_{k-1})\right) \\ &= -\textstyle\sum_{i=1}^{k-2}(-1)^i u^{p^i}w_{k-1-i}^{p^{i+1}} \otimes \left(-\overline{w}_{i+1} + \underline{u^{pe(i-1)}v^{p^i}ct_i + u^{p^{i+1}+pe(i-1)}ct_{i+1}}\right) \\ &\quad +uw_{k-1}^p \otimes (\overline{w}_1 - \underline{u^{p-1}ct_1}) \\ &\quad +\underline{(-1)^k u^{e(k-2)}\left(u^{p^{k-1}}w_1^{p^{k-1}} \otimes ct_{k-1} + v^{p^{k-1}}d(ct_{k-1})\right)} = m'_k \end{aligned}$$

Here, the underlined terms cancel each other if $k \le n$ by (5.2) and (3.1) with the relation $\Delta(cx) = T(c \otimes c)\Delta(x)$ for the switching map $T\colon \Gamma \otimes \Gamma \to \Gamma \otimes \Gamma$. $\square$

We also introduce an element

$$\overline{c}_k = h_{n+k-1} - u^{(p-1)p^k}h_k \in \mathrm{Ext}^1 A.$$

**Corollary 5.5.** *For each $0 < k < n$, the Massey products $\mu_k = \langle u^{p^k}, \overline{c}_k, c_{k-1}, c_{k-2}, \dots, c_1, c_0\rangle$ and $\mu'_k = \langle \overline{c}_k, c_{k-1}, c_{k-2}, \dots, c_1, c_0\rangle$ are defined. In fact, the cocycles $m_{k+1}$ and $m'_{k+1}$ represent elements of the Massey products $\mu_k$ and $\mu'_k$, respectively.*

In particular, we have

**Corollary 5.6.** *The Massey product $\left\langle u^{p^{n-3}}, \overline{c}_{n-3}, c_{n-4}, \dots, c_0\right\rangle \subset \mathrm{Ext}^1 A$ is defined and contains zero.*

**Lemma 5.7.** *The Massey product $\langle \overline{c}_{n-3}, c_{n-4}, \dots, c_0, h_{n-2}\rangle \subset \mathrm{Ext}^2 A$ contains zero.*

*Proof.* The Massey product $\langle \overline{c}_{n-3}, c_{n-4}, \dots, c_0, h_{n-2}\rangle$ contains

$$\langle h_{2n-4}, c_{n-4}, \dots, c_0, h_{n-2}\rangle - \left\langle u^{p^{n-2}-p^{n-3}}h_{n-3}, c_{n-4}, \dots, c_0, h_{n-2}\right\rangle.$$

It suffices show that the second term contains zero. Indeed, the first term does since a defining system cobounds $u^{e(n-3)}ct_{n-1}^{p^{n-2}}$. Since every Massey product $\langle h_j, h_{j-1}, \dots, h_{i+1}, h_i\rangle$ for $j - i \le n-2$ contains zero, all lower products contains zero, and we see that $\xi = \langle h_{n-3}, c_{n-4}, \dots, c_1, c_0, h_{n-2}\rangle$ is defined.

The statement of [4, Th.10] itself is applied to our case and says that there are elements $x_k \in \langle c_k, c_{k-1}, \dots, c_0, h_{n-2}, h_{n-3}, c_{n-4}, \dots, c_{k+1}\rangle$ for $0 \le k \le n-4$, $x_{n-3} \in \langle h_{n-3}, c_{n-4}, \dots, c_1, c_0, h_{n-2}\rangle$ and $x_{n-2} \in \langle h_{n-2}, h_{n-3}, c_{n-4}, \dots, c_1, c_0\rangle$ such that $\sum_{k=0}^{n-2} \pm x_k = 0$. Its proof tells us that we may take the elements $x_k$ arbitrary, and we take $x_k$ so that $x_k = 0$ for $0 \le k \le n-4$ and $x_{n-2} = 0$, whose relations follow from $d(ct_{n-1})$. Therefore, $x_{n-3} = 0$ and the lemma follows. $\square$

**Corollary 5.8.** *The Massey product* $\mu = \left\langle u^{p^{n-3}}, \bar{c}_{n-3}, c_{n-4}, \ldots, c_0, h_{n-2} \right\rangle$ *is defined and contain an element whose leading term is* $v^{e(n-2)}h_{n-2}$.

**Lemma 5.9.** *The condition* $(2.6)_{(i,j,s)}$ *holds for* $(i,j) = (0, n-2)$.

*Proof.* If $p \nmid s(s-1)$, it follows from the computation

$$\begin{array}{rcl} d(v^s t_1^{p^{n-2}}) & \equiv & suv^{s-1}t_1^{p^{n-1}} \otimes t_1^{p^{n-2}} + \binom{s}{2}u^2 t_1^{2p^{n-1}} \otimes t_1^{p^{n-2}} \mod (u^3) \\ d(suv^{s-1}ct_2^{p^{n-2}}) & \equiv & s(s-1)u^2 t_1^{p^{n-1}} \otimes ct_2^{p^{n-2}} - suv^{s-1}t_1^{p^{n-1}} \otimes t_1^{p^{n-2}} \mod (u^3). \end{array}$$

Suppose $s = tp^l + e(n-2)$ with $p \nmid t$ and $l > 0$. Let $\widetilde{\theta}$ denote an element of Corollary 5.8. We take a generator corresponding to $v^s h_{n-2}$ to be $v^{s-e(n-2)}\widetilde{\theta}$. We denote a representative of $\widetilde{\theta}$ by $m$, which is congruent to $v^{e(n-2)}t_1^{p^{n-2}} + uv^{pe(n-3)}ct_2^{p^{n-2}}$ modulo $(u^2)$. Then, $d(v^{s-e(n-2)}m) = tu^{a_l}v^{s-e(n-2)-p^{l-1}}t_1^{p^{[l-1]}} \otimes m \equiv tu^{a_l}v^{s-p^{l-1}}t_1^{p^{[l-1]}} \otimes t_1^{p^{n-2}}$. This shows the case for $[l] \neq 0, n-2$.

For $[l] = 0$, the similar computation shows that $d(v^{s-e(n-2)}m) \equiv tu^{a_l}v^{s-p^{l-1}}(t_1^{p^{n-2}} \otimes t_1^{p^{n-2}} + uv^{-1}t_1^{p^{n-1}+p^{n-2}} \otimes t_1^{p^{n-2}} + uv^{-1}t_1^{p^{n-2}} \otimes ct_2^{p^{n-2}})$, which yields $v^{s-1-p^{l-1}}g_{n-2}$. For $[l] = n-2$, $\widetilde{\theta}h_{n-3} \in u^{e(n-2)}\langle h_{2n-4}, h_{2n-5}, \ldots, h_{n-2}, h_{n-3}\rangle = \{u^{e(n-2)+p^{n-3}}b_{2n-5}\}$ in $C(p^{n-2})$. Indeed, $u^{e(n-3)}t_n^{p^{n-3}}$ yields the equality by (3.1). $\square$

## 6. On the action of $\alpha_1$ and $\beta_1$ on Greek letter elements

In this section, let $H^*M$ for a $BP_*(BP)$-comodule $M$ denote an Ext group $\mathrm{Ext}^*_{BP_*(BP)}(BP_*, M)$. Consider the comodule $N_{k-1}(j) = BP_*/(I_{k-1} + (v_{k-1}^j))$ ($v_0 = p$), and the connecting homomorphism $\partial_{k,j}$ associated to the short exact sequence $0 \to BP_*/I_{k-1} \xrightarrow{v_{k-1}^j} BP_*/I_{k-1} \to N_{k-1}(j) \to 0$. We abbreviate $\partial_{k,1}$ to $\partial_k$. Here we consider the Greek letter elements of $H^*BP_*/I_{n-1}$ defined by

$$\begin{array}{rcl} \overline{\alpha}_t^{(n-1)} & = & u^t \in H^0BP_*/I_{n-1} \quad \text{and} \\ \alpha_{(t/j)}^{(n)} & = & \partial_{n,j}(v^t) \in H^1BP_*/I_{n-1} \quad \text{for } v^t \in H^0N_{n-1}(j) \end{array}$$

for $t > 0$, and

$$\alpha_1 \ = \ \partial_1(v_1) = h_0 \in H^1BP_* \quad \text{and} \quad \beta_1 = \partial_1\partial_2(v_2) = b_0 \in H^2BP_*.$$

**Proposition 6.1.** *The elements* $\alpha_1$ *and* $\beta_1$ *act on the Greek letter elements as follows:*

$$\alpha_1\overline{\alpha}_t^{(n-1)} \neq 0 \in H^1BP_*/I_{n-1}, \quad \beta_1\overline{\alpha}_t^{(n-1)} \neq 0 \in H^2BP_*/I_{n-1};$$

*and if the Greek letter elements* $\alpha_{(sp^i/j)}^{(n)}$ *has an internal degree greater than* $2(p^n-1)(e(n-1)-1)$, *then*

$$\begin{array}{l} \alpha_1\alpha_{(sp^i/j)}^{(n)} \neq 0 \in H^2BP_*/I_{n-1} \text{ if } [i] \neq 0,\ p \nmid (s+1) \text{ or } p^2 \mid (s+1); \text{ and} \\ \beta_1\alpha_{(sp^i/j)}^{(n)} \neq 0 \in H^3BP_*/I_{n-1} \text{ if } n \neq 5,\ [i] \neq 1 \text{ or } p \nmid (s+1). \end{array}$$

In order to prove this, we make a chromatic argument: Let $N_k^0$ denote the $BP_*BP$-comodule $BP_*/I_k$, and put $M_k^0 = v_k^{-1}N_k^0$. We denote the cokernel of the inclusion $N_k^0 \to M_k^0$ by $N_k^1$, so that $0 \to N_k^0 \to M_k^0 \xrightarrow{\psi} N_k^1 \to 0$ is an exact sequence. Let $\widetilde{\partial}_{k+1} \colon H^sN_k^1 \to H^{s+1}N_k^0$ be the connecting homomorphism associated to the short exact sequence. We notice that $N_k^1 = \operatorname{colim}_j N_k(j)$ with inclusion $\varphi_j\colon N_k(j) \to N_k^1$ given by $\varphi_j(x) = x/u^j$, and that the connecting homomorphism $\partial_{n,j}\colon H^s N_{n-1}(j) \to H^{s+1} N_{n-1}^0$ factorizes to $\widetilde{\partial}_n \varphi_j$.

**Lemma 6.2.** *For an element $x_i^s/u^j \in H^0 N_{n-1}^1$ for $0 < j \le a_i$ ($j \le p^i$ if $s = 1$), $\alpha_1$ and $\beta_1$ act on it as follows:*

$$\begin{array}{ll} x_i^s \alpha_1/u^j \neq 0 \in H^1 N_{n-1}^1 & \text{if } [i] \neq 0,\ p \nmid (s+1) \text{ or } p^2 \mid (s+1)\text{; and} \\ x_i^s \beta_1/u^j \neq 0 \in H^2 N_{n-1}^1 & \text{if } n \neq 5,\ [i] \neq 1 \text{ or } p \nmid (s+1). \end{array}$$

*Proof.* A change of rings theorem of Miller and Ravenel [7] shows that the module $H^s M_{n-1}^1$ is isomorphic to $\operatorname{Ext}^s B$. By (1.5), we see that $x_i^s h_0/u \neq 0 \in \operatorname{Ext}^1 B$ unless $[i] = 0$, $p \mid (s+1)$ and $p^2 \nmid (s+1)$. This shows the first non-triviality. Similarly, since we have shown that (2.4) is exact, we see that $x_i^s \beta_1/u \neq 0 \in \operatorname{Ext}^2 B$ unless $n = 5$, $[i] = 1$ and $p \mid (s+1)$. □

**Lemma 6.3.** *Let $\xi_1$ denote $\alpha_1$ or $\beta_1$, and $x \in H^0 N_{n-1}^1$, and suppose that $x\xi_1$ has an internal degree greater than $2(p^{n-1} - 1)(e(n-1) - 1)$. If $x\xi_1 \in H^s N_{n-1}^1 \neq 0$, then $\widetilde{\partial}_n(x)\xi_1 \neq 0 \in H^{s+1} N_{n-1}^0$.*

*Proof.* It suffices to show that $x\xi_1$ is not in the image of $\psi_*\colon H^s M_{n-1}^0 \to H^s N_{n-1}^1$. Again the change of rings theorem shows that the module $H^s M_{n-1}^0$ is isomorphic to the module of Lemma 1.3 with substituting $n-1$ for $n$. Note that every generator of it except for $\zeta_{n-1}$ belongs to $H^s N_{n-1}^0$, and also is $u^{e(n-1)}\zeta_{n-1}$ (*cf.* [14]). It follows that every element of the image of $\psi_*$ has an internal degree no greater than $2(e(n-1) - 1)(p^{n-1} - 1)$. Thus the lemma follows. □

*Proof of Proposition 6.1.* The module $H^s M_{n-1}^0$ contains a submodule $k_* \langle h_0 \rangle$ if $s = 1$ and $k_* \langle b_0 \rangle$ if $s = 2$. Therefore, the first two relations hold. The other relations follow from Lemmas 6.2 and 6.3. □

*Proof of Theorem 1.11.* Note that $\overline{\alpha}_t^{(3)} = \overline{\gamma}_t = v_3^t$, and we obtain the theorem from Proposition 6.1 at $n = 4$. □

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Graduate shool of Mathematics, Nagoya University, Aichi, 464-8601, Japan
*E-mail address*: `ryo_kato_1128@yahoo.co.jp`

Department of Mathematics, Faculty of Science, Kochi University, Kochi, 780-8520, Japan
*E-mail address*: `katsumi@math.kochi-u.ac.jp`