THE HOMOTOPY GROUPS OF L_2-LOCALIZATION OF THE RAVENEL SPECTRA $T(m)/v_1$ AT THE PRIME TWO

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ABSTRACT. The Ravenel spectra $T(m)$ for non-negative integers m interpolate between the sphere spectrum and the Brown-Peterson spectrum. It admits an essential self-map $\alpha\colon \Sigma^{2p-2}T(m) \to T(m)$, whose cofiber we denote by $T(m)/v_1$. In this note, we work in the two-local stable homotopy category and study the homotopy groups of the Bousfield localization of $T(m)/v_1$ with respect to the v_2-inverted Brown-Peterson spectrum.

1. INTRODUCTION

In the stable homotopy category of spectra localized at an odd prime number p, the second author, A. Yabe and X. Wang ([11], [9]) determined the structure of the homotopy groups of the sphere spectrum L_2S^0 localized with respect to the v_2-localized Brown-Peterson spectrum $v_2^{-1}BP$ by use of the Adams-Novikov spectral sequence

$$E_2^*(X) = \mathrm{Ext}^*_{BP_*(BP)}(BP_*, BP_*(X)) \Longrightarrow \pi_*(X).$$

Here, the E_2-term is the Ext group of the category of $BP_*(BP)$-comodules. At the prime two, the second author and X. Wang ([10]) determined only the E_2-term of the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(L_2S^0)$, and we are interested in the stable homotopy category of spectra localized at the prime two. In his book [8], Ravenel constructed the spectrum $T(m)$ for each $m \geq 0$ characterized by

$$(1.1) \qquad BP_*(T(m)) = BP_*[t_1, \dots, t_m] \subset BP_*(BP) = BP_*[t_1, t_2, \dots]$$

as a $BP_*(BP)$-comodule. These spectra admit maps $T(m) \to T(m+1)$ inducing the inclusion on BP_*-homology, and $T(0)$ and $T(\infty)$ are the sphere and the Brown-Peterson spectra, respectively. The homotopy groups of $L_2T(\infty)$ are determined by Ravenel as $BP_* \oplus BP_*/(2^\infty, v_1^\infty, v_2^\infty)$ in [7]. We have partial results [2] and [4] on subgroups of the homotopy groups $\pi_*(L_2T(1))$. We use the 2- and the v_1-Bockstein spectral sequences to determine it for $m \geq 1$ in two different order :

1) the v_1-Bockstein spectral sequence first and then the 2-Bockstein spectral sequence;
2) the 2-Bockstein spectral sequence first and then the v_1-Bockstein spectral sequence.

As the first step in the order 1), the v_1-Bockstein spectral sequence is computed in [3], and we obtain the homotopy groups of $L_2T(m) \wedge M$ for the modulo two Moore spectrum M. In this paper, we consider the first step of the order 2).

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Let $T(m)/v_1$ denote the cofiber of $\alpha\colon \Sigma^2 T(m) \to T(m)$ for $m > 0$ such that $BP_*(\alpha) = v_1 - 2t_1$, whose existence is shown in section two. We then define a spectrum C by the cofiber sequence

$$T(m)/v_1 \xrightarrow{\eta} 2^{-1}T(m)/v_1 \longrightarrow C \longrightarrow \Sigma T(m)/v_1 \tag{1.2}$$

for the localization map $\eta\colon T(m)/v_1 \to 2^{-1}T(m)/v_1$. We first determine the Adams-Novikov E_2-term of L_2C in Proposition 3.8 by use of the 2-Bockstein spectral sequence associated to the cofiber sequence

$$D \xrightarrow{\iota} C \xrightarrow{2} C \xrightarrow{\kappa} \Sigma D, \tag{1.3}$$

where D denotes the spectrum $T(m)/v_1 \wedge M$ for the mod 2 Moore spectrum M. The E_2-term of the Adams-Novikov spectral sequence for $\pi_*(L_2D)$ is determined by Ravenel (*cf.* [8]) as follows:

$$E_2^*(L_2D) = K_m(2)_* \otimes \wedge(g_{10}, g_{11}, g_{20}, g_{21}), \tag{1.4}$$

where

$$K_m(2)_* = v_2^{-1}\mathbb{Z}/2[v_2, \dots, v_{m+2}], \tag{1.5}$$

and g_{ij} denotes the element of bidegree $(1, 2^{j+1}(2^{m+i}-1))$, which is denoted by $h_{m+i,j}$ in [8]. Next, we show that every element of the Adams-Novikov E_2-term $E_2^*(L_2C)$ is a permanent cycle in Lemma 3.12, and the extension problem of the spectral sequence is trivial in Lemma 3.13. These show the homotopy groups of L_2C are isomorphic to the E_2-term.

In order to state our result, we introduce notation: the algebra

$$E_m(2)_* = v_2^{-1}\mathbb{Z}_{(2)}[v_1, v_2, \dots, v_{m+2}]$$

such that $K_m(2)_* = E_m(2)_*/(2, v_1)$, the elements

$$u_i = v_{m+i} \in BP_* \quad \text{for } i \geq 1,$$

the algebras

$$\begin{array}{rcl} R & = & E_{m-2}(2)_*/(v_1) \;=\; v_2^{-1}\mathbb{Z}_{(2)}[v_2, \dots, v_m], \\ R^{(n)} & = & R\left[u_1^{2^n}, u_2^{2^n}\right] \quad \text{and} \\ R_j^{(n)} & = & R\left[u_j^{2^n}\right], \end{array}$$

and the submodules of $E_m(2)_*/(2^\infty, v_1) = R[u_1, u_2] \otimes \mathbb{Q}/\mathbb{Z}_{(2)}$:

$$\begin{array}{rcl} \overline{M}(i) & = & \bigoplus_{j=1}^2 R_j^{(i+1)}/(2^{i+1})\left\{u_j^{2^i}/2^{i+1}\right\}, \\ M^0(i) & = & R^{(i+1)}/(2^{i+1})\left\{u_1^{2^i}u_2^{2^{i+1}}/2^{i+1}, u_2^{2^i}u_1^{2^{i+1}}/2^{i+1}, u_1^{2^i}u_2^{2^i}/2^{i+1}\right\} \quad \text{and} \\ M^1(i) & = & R^{(i+1)}/(2^{i+1})\Big\{u_2^{2^i}u_1^{2^{i+1}}\overline{g}_{10}/2^{i+1}, u_1^{2^i}u_2^{2^{i+1}}\overline{g}_{20}/2^{i+1}, \\ & & \qquad u_1^{2^i}u_2^{2^i}\overline{g}_{10}/2^{i+1} = u_1^{2^i}u_2^{2^i}\overline{g}_{20}/2^{i+1}\Big\}. \end{array}$$

Here, $\overline{g}_{j0}$ is an element such that $\overline{g}_{j0}/2 = u_j^{-1}g_{j0}/2$, whose existence is shown in Lemma 3.2.

Theorem (1.6) *The homotopy groups $\pi_*(L_2C)$ for $m > 1$ are isomorphic to the tensor product of $\wedge(g_{11}, g_{21})$ and the direct sum of the modules $R/(2^\infty)$, $\overline{M}(i)$, $M^0(i)$ and $M^1(i)$ for $i \geq 0$.*

Since the E_2-term $E_2^*(2^{-1}T(m)/v_1)$ is isomorphic to $H\mathbb{Q}_*(L_2T(m)/v_1)=\mathbb{Q}[v_2,v_3,\dots,v_m]$ by [8, Cor 6,5,7], the homotopy groups of $L_2T(m)/v_1$ are obtained by observing the homotopy exact sequence associated to the cofiber sequence (1.2).

Corollary (1.7) *The homotopy groups $\pi_*(L_2T(m)/v_1)$ for $m>1$ are isomorphic to the direct sum of the modules $\mathbb{Z}_{(2)}[v_2,v_3,\dots,v_m]$, $\Sigma^{-1}R/(2^\infty)\{g_{11},g_{21},g_{11}g_{21}\}$ and $\bigoplus_{i\geq 0}\Sigma^{-1}(\overline{M}(i)\oplus M^0(i)\oplus M^1(i))\otimes\wedge(g_{11},g_{21})$. Here, Σ denotes a shift of dimension.*

We note that the homotopy groups of $L_2T(1)/v_1$ are given in [6]. The structure of $\pi_*(L_2T(m)/v_1)$ for $m>1$ in Corollary 1.7 is less complicated than that of the case for $m=1$. So it seems that it is useful to determine the homotopy groups $\pi_*(L_2T(m))$ for $m>1$ completely. For $m=1$, we know the structure of subgroups of $\pi_*(L_2T(1))$ (*cf.* [2], [4]).

2. A change of rings theorem and structure maps

We work in the stable homotopy category of spectra localized at the prime two. Let BP denote the Brown-Peterson ring spectrum, and consider the Hopf algebroid (A,Γ) associated with it, where

$$\begin{array}{rcl} A & = & \pi_*(BP) \;=\; BP_* \;=\; \mathbb{Z}_{(2)}[v_1,v_2,\dots], \\ \Gamma & = & BP_*(BP) \;=\; BP_*[t_1,t_2,\dots]. \end{array}$$

The Hopf algebroid Γ gives rise to another one

$$(A,\Gamma_m)=(A,\Gamma/(t_1,\dots,t_m))=(A,BP_*[t_{m+1},t_{m+2},\dots]).$$

Recall the Ravenel spectrum $T(m)$ in (1.1) for $m\geq 0$, which is a ring spectrum with multiplication $\mu\colon T(m)\wedge T(m)\to T(m)$. Then, Ravenel showed in [8] the change of rings theorem

$$E_2^*(T(m)\wedge X)=\mathrm{Ext}^*_{\Gamma_m}(A,BP_*(X))$$

for a spectrum X. If X is the sphere spectrum S^0, then we have an element $v_1\in\mathrm{Ext}^{0,2}_{\Gamma_m}(A,A)$ for $m>0$. This element is represented by v_1-2t_1 in the cobar complex $\Omega^0_\Gamma BP_*(T(m))$ for computing $E_2^*(T(m))$. Since $E_2^{s,1+s}(T(m))=0$ by observing the reduced cobar complex, the element v_1 survives to a homotopy element $\alpha'\in\pi_2(T(m))$. We now let $T(m)/v_1$ denote the cofiber of the composite $\alpha\colon\Sigma^2T(m)=T(m)\wedge S^2\xrightarrow{1\wedge\alpha'}T(m)\wedge T(m)\xrightarrow{\mu}T(m)$.

Let M and M_∞ be the modulo two Moore spectrum and the cofiber of the localization map $S^0\to S\mathbb{Q}$, respectively. In this paper we consider the spectra

$$D=T(m)/v_1\wedge M\quad\text{and}\quad C=T(m)/v_1\wedge M_\infty.$$

These fit in the cofiber sequence (1.3). The BP_*-homologies of the L_2-localizations of these spectra are

$$\begin{array}{rcl} BP_*(L_2D) & = & v_2^{-1}BP_*/(2,v_1)[t_1,\dots,t_m]\quad\text{and} \\ BP_*(L_2C) & = & v_2^{-1}BP_*/(2^\infty,v_1)[t_1,\dots,t_m]. \end{array}$$

Consider a spectrum

$$E_m(2)=v_2^{-1}BP\langle m+2\rangle$$

for the Johnson-Wilson spectrum $BP\langle m{+}2\rangle$ such that $\pi_*(BP\langle m{+}2\rangle)=\mathbb{Z}_{(2)}[v_1,v_2,\dots,v_{m+2}]$. Since

$$v_2^{-1}BP_*/J\xrightarrow{1\otimes\eta_R}E_m(2)_*/J\otimes_A\Gamma_m$$

for an invariant regular ideal J of length two is a faithfully flat extension, we have an isomorphism

$$\mathrm{Ext}^*_{\Gamma_m}(A, v_2^{-1}BP_*/J) \cong \mathrm{Ext}^*_{\Sigma_m(2)}(E_m(2)_*, E_m(2)_*/J) \tag{2.1}$$

shown by the same way as the proofs of the change of rings theorem in [1]. Here,

$$\Sigma_m(2) = E_m(2)_* \otimes_A \Gamma_m \otimes_A E_m(2)_*$$

is the induced Hopf algebroid, and

$$\Sigma_m(2) = E_m(2)_*[t_1, t_2, \dots]/(\eta_R(v_{m+k}) : k > 2). \tag{2.2}$$

Note that $m+2$ is the smallest number n such that

$$v_2^{-1}BP_*/J \xrightarrow{1\otimes\eta_R} v_2^{-1}BP\langle n\rangle_*/J \otimes_A \Gamma_m$$

is a faithfully flat extension.

Proposition (2.3) *The Adams-Novikov E_2-terms for computing $\pi_*(L_2C)$ and $\pi_*(L_2D)$ are isomorphic to*

$$\begin{array}{rcl} E_2^*(L_2C) & = & \mathrm{Ext}^*_{\Sigma_m(2)}(E_m(2)_*, E_m(2)_*/(2^\infty, v_1)) \quad \textit{and} \\ E_2^*(L_2D) & = & \mathrm{Ext}^*_{\Sigma_m(2)}(E_m(2)_*, E_m(2)_*/(2, v_1)). \end{array}$$

Proof. The isomorphism on $E_2^*(L_2D)$ follows from (2.1). Since $L_2C = \mathrm{hocolim}_k\, L_2T(m) \wedge M_k$ for the mod 2^k Moore spectrum M_k, the change of rings theorem (2.1) also shows the isomorphism on $E_2^*(L_2C)$. □

Consider the Hopf algebroid $(E_m(2)_*, \Sigma_m(2))$ (see (2.2)). We read off the behavior of the right unit $\eta_R\colon E_m(2)_* \to \Sigma_m(2)$ and the diagonal $\Delta\colon \Sigma_m(2) \to \Sigma_m(2) \otimes_{E_m(2)_*} \Sigma_m(2)$ from that of Γ_m. Hereafter we set $v_2 = 1$ and use the notation

$$u_i = v_{m+i} \quad \text{and} \quad s_i = t_{m+i}$$

for $i = 1, 2$. Recall the Hazewinkel and the Quillen formulas:

$$\begin{array}{l} v_n = 2\ell_n - \sum_{k=1}^{n-1} \ell_k v_{n-k}^{2^k} \in \mathbb{Q} \otimes A = \mathbb{Q}[\ell_1, \ell_2, \dots], \\ \eta_R(\ell_n) = \sum_{k=0}^{n} \ell_k t_{n-k}^{2^k} \in \mathbb{Q} \otimes \Gamma = \mathbb{Q} \otimes A[t_1, t_2, \dots] \quad \text{and} \\ \sum_{i+j=n} \ell_i \Delta(t_j^{2^i}) = \sum_{i+j+k=n} \ell_i t_j^{2^i} \otimes t_k^{2^{i+j}} \in \mathbb{Q} \otimes \Gamma \otimes_A \Gamma. \end{array}$$

Then a routine computation shows

Lemma (2.4) *The right unit $\eta_R\colon A \to \Gamma_m$ and the diagonal $\Delta\colon \Gamma_m \to \Gamma_m \otimes_A \Gamma_m$ act on generators as follows:*

$$\begin{array}{rcl} \eta_R(v_n) & = & v_n \quad \textit{for } n \le m, \\ \eta_R(u_1) & = & u_1 + 2s_1, \\ \eta_R(u_2) & \equiv & u_2 + 2s_2 \mod (v_1), \\ \Delta(s_1) & = & s_1 \otimes 1 + 1 \otimes s_1, \\ \Delta(s_2) & \equiv & s_2 \otimes 1 + 1 \otimes s_2 \mod (v_1). \end{array}$$

3. The Adams-Novikov E_2-term for $\pi_*(L_2C)$

We begin with introducing the cocycles of cobar complexes that represent generators g_{j1} and $\overline{g}_{j0}$.

Lemma (3.1) *The elements $s_j^2+u_js_j$ for $j=1,2$ are cocycles of the cobar complex $\Omega^1_{\Gamma_m}E_m(2)_*/(v_1)$.*

Proof. Since $d(u_j)\equiv 2s_j$ and $d(s_j)\equiv 0$,

$$d(s_j^2+u_js_j)\equiv -2s_j\otimes s_j+2s_j\otimes s_j\equiv 0 \mod (v_1). \qquad \square$$

Lemma (3.2) *There are elements*

$$w_j=\textstyle\sum_{n>0}(-1)^{n-1}\frac{1}{2n}(2u_j^{-1}s_j)^n\in\Omega^1_{\Gamma_m}E_m(2)/(2^k,v_1)$$

for $j=1,2$ and any $k>0$ such that $d(w_j)=0$.

Proof. Note that $\sum_{n>0}(-1)^{n-1}\frac{1}{2n}(2u_j^{-1}s_j)^n=\log(1+2u_j^{-1}s_j)=\log(\eta_R(u_j))-\log(u_j)$. Since

$$\begin{aligned}\log(\eta_R(u_j)) &= \log(1-(1-\eta_R(u_j))) = -\textstyle\sum_{n>0}\frac{1}{n}(1-\eta_R(u_j))^n\\ &= -\textstyle\sum_{n>0}\eta_R(\frac{1}{n}(1-u_j)^n) = -\eta_R(\sum_{n>0}\frac{1}{n}(1-u_j)^n)\\ &= \eta_R(-\textstyle\sum_{n>0}\frac{1}{n}(1-u_j)^n) = \eta_R(\log(1-(1-u_j)))\\ &= \eta_R(\log(u_j)),\end{aligned}$$

we see that $d(\log(1+2u_j^{-1}s_j))=dd(\log(u_j))=0$. $\square$

We denote the homology classes of the cocycles of Lemmas 3.1 and 3.2 by

$$g_{j1}\quad\text{and}\quad \overline{g}_{j0}\in E_2^1(L_2T(m)/v_1\wedge M_k),$$

respectively, for each $k>0$, where M_k denotes the mod 2^k Moore spectrum.

Consider the subalgebras

$$\begin{aligned}F &= K_{m-2}(2)_* = R/(2) = v_2^{-1}\mathbb{Z}/2[v_2,\dots,v_m],\\ F^{(n)} &= F\left[u_1^{2^n},u_2^{2^n}\right]\quad\text{and}\\ F_j^{(n)} &= F\left[u_j^{2^n}\right],\end{aligned}\tag{3.3}$$

and the submodules

$$\begin{aligned}\overline{N}(i) &= \textstyle\bigoplus_{j=1}^2 u_j^{2^i}F_j^{(i+1)}\quad\text{and}\\ N^0(i) &= F^{(i+1)}\left\{u_1^{2^i}u_2^{2^{i+1}},u_2^{2^i}u_1^{2^{i+1}},u_1^{2^i}u_2^{2^i}\right\}\end{aligned}$$

of the polynomial algebra $K_m(2)_*=F[u_1,u_2]$. Then, as an F-module,

$$\begin{aligned}K_m(2)_* &= (F[u_1]+F[u_2])\oplus\textstyle\bigoplus_{i\ge0}N^0(i)\\ &= F\oplus\textstyle\bigoplus_{i\ge0}\left(\overline{N}(i)\oplus N^0(i)\right),\\ u_jK_m(2)_* &= u_jF[u_j]\oplus\textstyle\bigoplus_{i\ge0}N^0(i)\quad\text{and}\\ u_1u_2K_m(2)_* &= \textstyle\bigoplus_{i\ge0}N^0(i)\end{aligned}\tag{3.4}$$

for $j=1,2$. Under these notations, we rewrite (1.4) as follows:

$$E_2^*(L_2D)=\wedge(g_{11},g_{21})\otimes\left(K_m(2)_*\otimes\wedge(u_1\overline{g}_{10},u_2\overline{g}_{20})\right).\tag{3.5}$$

The factor $K_m(2)_*\otimes\wedge(u_1\overline{g}_{10},u_2\overline{g}_{20})$ is decomposed into the direct sum

$$K_m(2)_*\oplus\overline{g}_{10}(u_1K_m(2)_*)\oplus\overline{g}_{20}(u_2K_m(2)_*)\oplus\overline{g}_{10}\overline{g}_{20}(u_1u_2K_m(2)_*).\tag{3.6}$$

We consider the connecting homomorphism $\delta\colon E_2^s(L_2C) \to E_2^{s+1}(L_2D)$ on the factor $K_m(2)_* \otimes \wedge(u_1\overline{g}_{10}, u_2\overline{g}_{20})$. The behavior of δ is read off from the following lemma:

Lemma (3.7) *The connecting homomorphism δ acts as an R-module map on the elements of $E_2^0(L_2C)$ as follows:*

$$\begin{aligned} \delta(1/2^i) &= 0 \quad \textit{and} \\ \delta(u_1^{2^is}u_2^{2^it}/2^{i+1}) &= su_1^{2^is}u_2^{2^it}\overline{g}_{10} + tu_1^{2^is}u_2^{2^it}\overline{g}_{20}, \end{aligned}$$

where s, t and i are non-negative integers.

Proof. Note that $u_j^{a-1}s_j$ represents $u_j^a\overline{g}_{j0}$. Then, the lemma follows immediately from the relations $d(u_j) \equiv 2s_j$ and the binomial coefficient theorem. □

Proposition (3.8) *The Adams-Novikov E_2-term $E_2^*(L_2C)$ is isomorphic to the module given in Theorem 1.6.*

Proof. Put $E^* = K_m(2)_* \otimes \wedge(u_1\overline{g}_{10}, u_2\overline{g}_{20})$, $B^0 = R/(2^\infty) \oplus \bigoplus_{i\ge 0}\left(\overline{M}(i) \oplus M^0(i)\right)$ and $B^1 = \bigoplus_{i\ge 0} M^1(i)$. By [5, Remark 3.11], it suffices to show that the sequence

$$0 \longrightarrow E^0 \longrightarrow B^0 \xrightarrow{2} B^0 \xrightarrow{\delta} E^1 \longrightarrow B^1 \xrightarrow{2} B^1 \xrightarrow{\delta} E^2 \longrightarrow 0 \tag{3.9}$$

is exact. In fact, $B^* \otimes \wedge(g_{11}, g_{21}) \subset E_2^*(L_2C)$ by Lemma 3.7, and the exact sequence (3.9) induces a commutative diagram

$$\begin{array}{ccccccc} (E^*\otimes\Lambda)^s & \longrightarrow & (B^*\otimes\Lambda)^s & \longrightarrow & (B^*\otimes\Lambda)^s & \xrightarrow{\delta} & (E^*\otimes\Lambda)^{s+1} \\ \| & & \downarrow & & \downarrow & & \| \\ E_2^s(L_2D) & \xrightarrow{\iota_*} & E_2^s(L_2C) & \xrightarrow{2} & E_2^s(L_2C) & \xrightarrow{\delta} & E_2^{s+1}(L_2D) \end{array}$$

of exact sequences, where $\Lambda = \wedge(g_{11}, g_{21})$. Then, the middle maps are isomorphisms by [5, Remark 3.11].

By (3.6) and (3.4),

$$\begin{aligned} E^0 &= F \oplus \textstyle\bigoplus_{i\ge 0}\left(\overline{N}(i) \oplus N^0(i)\right), \\ E^1 &= \left(\textstyle\bigoplus_{j=1}^2 \overline{g}_{j0}u_jF[u_j]\right) \oplus \textstyle\bigoplus_{i\ge 0}\left(E^{1,I}(i) \oplus E^{1,C}(i)\right) \\ &= \textstyle\bigoplus_{i\ge 0}\left(\overline{N}^1(i) \oplus E^{1,I}(i) \oplus E^{1,C}(i)\right), \\ E^2 &= \textstyle\bigoplus_{i\ge 0} \overline{g}_{10}\overline{g}_{20}N^0(i), \end{aligned}$$

where

$$\begin{aligned} E^{1,I}(i) &= F^{(i+1)}\left\{u_1^{2^i}u_2^{2^{i+1}}\overline{g}_{10}, u_2^{2^i}u_1^{2^{i+1}}\overline{g}_{20}, u_1^{2^i}u_2^{2^i}\overline{g}_{10}\right\}, \\ E^{1,C}(i) &= F^{(i+1)}\left\{u_1^{2^i}u_2^{2^{i+1}}\overline{g}_{20}, u_2^{2^i}u_1^{2^{i+1}}\overline{g}_{10}, u_1^{2^i}u_2^{2^i}\overline{g}_{20}\right\} \quad \text{and} \\ \overline{N}^1(i) &= \textstyle\bigoplus_{j=1}^2 \overline{g}_{j0}u_j^{2^i}F_j^{(i+1)}. \end{aligned}$$

Note that $u_jF[u_j] = \bigoplus_{i\ge 0} u_j^{2^i}F_j^{(i+1)}$. Each summand of E^0 fits in one of the exact sequences

$$\begin{gathered} 0 \longrightarrow F \longrightarrow F/(2^\infty) \xrightarrow{2} F/(2^\infty) \longrightarrow 0, \\ 0 \longrightarrow \overline{N}(i) \longrightarrow \overline{M}(i) \xrightarrow{2} \overline{M}(i) \xrightarrow{\delta} \overline{N}^1(i) \longrightarrow 0 \quad \text{and} \\ 0 \longrightarrow N^0(i) \longrightarrow M^0(i) \xrightarrow{2} M^0(i) \xrightarrow{\delta} E^{1,I}(i) \longrightarrow 0 \end{gathered}$$

by Lemma 3.7, and the direct sum of these shows the exact sequence

$$0 \longrightarrow E^0 \longrightarrow B^0 \xrightarrow{2} B^0 \xrightarrow{\delta} E^1 \longrightarrow \bigoplus_{i\ge 0} E^{1,C}(i) \longrightarrow 0. \tag{3.10}$$

Lemma 3.7 also shows the exact sequence

$$0 \longrightarrow E^{1,C}(i) \longrightarrow M^1(i) \xrightarrow{2} M^1(i) \xrightarrow{\delta} \overline{g}_{10}\overline{g}_{20}N^0(i) \longrightarrow 0,$$

and the direct sum yields the exact sequence

$$0 \longrightarrow \bigoplus_{i\ge 0} E^{1,C}(i) \longrightarrow B^1 \xrightarrow{2} B^1 \xrightarrow{\delta} E^2 \longrightarrow 0. \tag{3.11}$$

Splice the exact sequences (3.10) and (3.11), and we obtain the desired exact sequence (3.9). □

Since the Adams-Novikov E_2-term $E_2^s(L_2C)$ for $s > 3$ is trivial by Proposition 3.8, every element of $E_2^s(L_2C)$ for $0 < s \le 3$ is a permanent cycle in the Adams-Novikov spectral sequence. For $s = 0$, we have

Lemma (3.12) *Every element of $E_2^0(L_2C)$ is a permanent cycle in the Adams-Novikov spectral sequence.*

Proof. Let $x/2^i \in E_2^0(L_2C)$. Suppose that $d_3(x/2^i) = y/2^j \ne 0$. If $x/2^{i+1} \in R/(2^\infty)$, then there exist elements $y_k = d_3(x/2^k)$ for $k > i$ such that $2y_k = y_{k-1}$ and $2y_{i+1} = y/2^j \ne 0$, and so y_k's generate a module isomorphic to $R/(2^\infty)$ in $E_2^3(L_2C)$. This contradicts to Proposition 3.8. So we may assume that $x/2^{i+1}$ belongs to $\overline{M}(i)$ or $M^0(i)$. Then, $d_3(x/2^l) = y/2 \ne 0$ for $l = i - j + 1$. Since $x \in E_2^0(L_2D)$ is a permanent cycle by Ravenel [8], the integer l is greater than one. Then, the element $x/2^{l-1}$ is a permanent cycle and survives to a homotopy element $[x/2^{l-1}]$ such that $\kappa_*([x/2^{l-1}]) = [y] \in \pi_*(L_2D)$, where κ is the map in (1.3), and $[z]$ denotes the homotopy element detected by an element z in the E_2-term. Since $y \in E_2^3(L_2D)$, there is an element $h \in \{g_{ji} : j = 1, 2, i = 0, 1\}$ such that $yh \ne 0 \in E_2^4(L_2D)$. Note that it detects $[yh] \ne 0 \in \pi_*(L_2D)$. By Proposition 3.8, we see that $xh/2^l \in E_2^1(L_2C)$, which is a permanent cycle since $E_2^s(L_2C) = 0$ for $s > 3$. It implies a contradiction: $0 \ne [yh] = \kappa_*([xh/2^{l-1}]) = \kappa_* 2_*([xh/2^l]) = 0$. We notice here that $xg_{j0}/2^{i+1} \in E_2^1(L_2C)$ since the cochain $xs_j/2^{i+1}$ is a cocycle. □

Lemma (3.13) *In the Adams-Novikov spectral sequence, the extension problem is trivial.*

Proof. Let $\xi \in \pi_*(L_2C)$ be elements detected by $x/2^j \in E_\infty^0(L_2C) = E_2^0(L_2C)$. It suffices to show that $2^j\xi = 0$. Since $x \in E_2^0(L_2D)$ is a permanent cycle (*cf.* [8]) and $2^{j-1}\xi$ is detected by $x/2$, $2^{j-1}\xi$ is in the image of the induced map $\iota_* \colon \pi_*(L_2D) \to \pi_*(L_2C)$ from the map in (1.3). It follows that $2^j(\iota_*([x])) = \iota_*([2x]) = 0$ as desired. □

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