# THE HOMOTOPY GROUPS OF THE $L_2$-LOCALIZATION OF THE MODULO $p$ MOORE SPECTRUM SMASHING WITH THE FIRST RAVENEL SPECTRUM

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Abstract. Let $BP$ be the Brown-Peterson spectrum at an odd prime $p$, and $L_2$ denote the Bousfield localization functor with respect to $v_2^{-1}BP$. The Ravenel spectrum $T(1)$ is characterized by $BP_*(T(1)) = BP_*[t_1]$ on the primitive generator $t_1$. In this paper, we determine the homotopy groups $\pi_*(L_2M \wedge T(1))$ for the mod $p$ Moore spectrum $M$.

## 1. Introduction and statement of results

Let $p$ be a prime number, and $BP$ denote the Brown-Peterson spectrum at $p$. Then we have the Bousfield localization functor $L_2\colon \mathcal{S}_p \to \mathcal{S}_p$ on the category $\mathcal{S}_p$ of $p$-local spectra with respect to $v_2^{-1}BP$ for the generator $v_2$ of the homotopy groups $\pi_*(BP) = BP_* = \mathbb{Z}_{(p)}[v_i\colon i > 0]$. Let $T(1)$ denote the first Ravenel spectrum characterized by $BP_*(T(1)) = BP_*[t_1] \subset BP_*(BP) = BP_*[t_i\colon i > 0]$ as a $BP_*(BP)$-comodule. For the prime two, we studied subgroups of the homotopy groups $\pi_*(L_2T(1))$ in [1] and [3].

Let $M$ and $V$ denote the mod $p$ Moore spectrum and the first Smith-Toda spectrum characterized by $BP_*(M) = BP_*/(p)$ and $BP_*(V) = BP_*/(p, v_1)$. The homotopy groups $\pi_*(L_2V \wedge T(1))$ are determined at every prime by Ravenel in [6]. The second author also determined the homotopy groups $\pi_*(L_2M \wedge T(1))$ at the prime two by use of the $v_1$-Bockstein spectral sequence [7], in which $M \wedge T(1)$ is replaced by the Mahowald spectrum $X\langle 1\rangle$. For the Ravenel spectrum $T(m)$ with $m > 1$, the structure of $\pi_*(L_2M \wedge T(m))$ is determined by Kamiya and the second author if $p > 2$ [4], and by the authors if $p = 2$ [2].

In this paper, we study the homotopy groups $\pi_*(L_2M \wedge T(1))$ at an odd prime $p$. Let $\alpha\colon \Sigma^{2p-2}M \to M$ denote the Adams map and consider the cofiber sequence

$$M \xrightarrow{\eta} \alpha^{-1}M \longrightarrow C \longrightarrow \Sigma M \tag{1.1}$$

for the localization map $\eta$. Then, the Adams map induces a self-map $v_1\colon \Sigma^{2p-2}C \to C$ fitting in the cofiber sequence

$$V \xrightarrow{\varphi} \Sigma^{2p-2}C \xrightarrow{v_1} C \longrightarrow \Sigma V. \tag{1.2}$$

Apply the functor $v_2^{-1}BP_*(-)$ to cofiber sequences (1.1) and (1.2), and we have short exact sequences

$$0 \longrightarrow N_1^0 \xrightarrow{\eta_*} M_1^0 \longrightarrow M_1^1 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow M_2^0 \xrightarrow{\varphi_*} M_1^1 \xrightarrow{v_1} M_1^1 \longrightarrow 0. \tag{1.3}$$

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Note that these modules are $BP_*(BP)$-comodules. We consider the Hopf algebroid $\Gamma(2) = BP_*(BP)/(t_1)$ over $BP_*$ induced from $BP_*(BP)$. Let $E_2^*(X)$ for a spectrum $X$ denote the $E_2$-term of the Adams-Novikov spectral sequence converging to $\pi_*(L_2X)$, and put

$$H^*M \;=\; \mathrm{Ext}^*_{\Gamma(2)}(BP_*, M) \tag{1.4}$$

for a $\Gamma(2)$-comodule $M$. Then, by a change of rings theorem [6],

$$E_2^*(X \wedge T(1)) \;=\; H^*v_2^{-1}BP_*(X)$$

for $X = V$ and $= C$, which equals $H^*M_2^0$ if $X = V$, and $H^*M_1^1$ if $X = C$. Applying the functor $H^*-$ to the second exact sequence of (1.3), we have the long exact sequence

$$0 \longrightarrow H^0M_2^0 \xrightarrow{\varphi_*} H^0M_1^1 \xrightarrow{v_1} H^0M_1^1 \xrightarrow{\delta} H^1M_2^0 \xrightarrow{\varphi_*} H^1M_1^1 \xrightarrow{v_1} \cdots . \tag{1.5}$$

(1.6) (Ravenel *cf.* [6]) *The $E_2$-term $E_2^*(V \wedge T(1)) = H^*M_2^0$ of the Adams-Novikov spectral sequence converging to $\pi_*(L_2V \wedge T(1))$ is isomorphic, as a $K(2)_*[v_3]$-module, to*

$$K(2)_*[v_3] \otimes \wedge(g_0, g_1, g_2, g_3).$$

*Here $K(2)_* = \mathbb{Z}/p[v_2, v_2^{-1}]$, and $g_0 = [t_2^p]$, $g_1 = [t_2]$, $g_2 = [t_3]$ and $g_3 = [t_3^p]$, in which $[x]$ denotes a homology class represented by an element $x$ of the cobar complex $\Omega^1_{\Gamma(2)}BP_*/(p, v_1)$.*

We decompose the module as follows:

**Lemma 1.7.**

$$\begin{array}{rcl}
H^0M_2^0 &=& K(2)_* \oplus A_0^0 \oplus \mathfrak{c}^0 \;=\; K(2)_*[v_3],\\
H^1M_2^0 &=& A_0^0\langle v_3^{-1}g_0\rangle \oplus \mathfrak{c}^0\langle v_3^{p-1}g_0\rangle \oplus A_0^1 \oplus \mathfrak{i}^1 \oplus \mathfrak{c}^1,\\
H^2M_2^0 &=& A_0^1\langle v_3^{-1}g_0\rangle \oplus (\mathfrak{i}^1 \oplus \mathfrak{c}^1)\langle v_3^{p-1}g_0\rangle \oplus A_0^2 \oplus \mathfrak{i}^2 \oplus \mathfrak{c}^2,\\
H^3M_2^0 &=& A_0^2\langle v_3^{-1}g_0\rangle \oplus (\mathfrak{i}^2 \oplus \mathfrak{c}^2)\langle v_3^{p-1}g_0\rangle \oplus K(2)_*[v_3]g_1g_2g_3 \quad \text{and}\\
H^4M_2^0 &=& K(2)_*[v_3]g_0g_1g_2g_3.
\end{array}$$

*Here,*

$$\begin{array}{rcl}
A_0^* &=& K(2)_*\{v_3^s : p \nmid s > 0\} \otimes \wedge(g_1, g_2, g_3),\\
\mathfrak{c}^0 &=& \bigoplus_{l=1}^3 \bigoplus_{n\ge 0} {}^lA_n \;=\; K(2)_*[v_3^p],\\
\mathfrak{i}^1 &=& \bigoplus_{l=1}^3 \bigoplus_{n\ge 0} {}^lB_n, \quad \mathfrak{c}^1 \;=\; \bigoplus_{l=1}^3 \bigoplus_{n\ge 0} {}^lC_{n,1} \oplus {}^lC_{n,2},\\
\mathfrak{i}^2 &=& \bigoplus_{l=1}^3 \bigoplus_{n\ge 0} {}^lD_{n,1} \oplus {}^lD_{n,2}, \quad \mathfrak{c}^2 \;=\; \bigoplus_{l=1}^3 \bigoplus_{n\ge 0} {}^lE_n
\end{array}$$

*for the modules defined by*

$$
(1.8)\qquad
\begin{array}{rcl}
{}^lA_n &=& K(2)_*\{v_3^{sp^{3n+l}} : p\nmid s>0\} \quad (l=1,2,3);\\
{}^lB_n &=& K(2)_*\{v_3^{(s-1)p^{3n+l}}(v_3^{p^lc_n}g_l) : p\nmid s>0\} \quad (l=1,2,3);\\
{}^lC_{n,1} &=& K(2)_*\{v_3^{sp^{3n+l+1}}(v_3^{p^lc_{n+1}}g_l) : p\nmid s>0\} \quad (l=1,2),\\
{}^3C_{n,1} &=& K(2)_*\{v_3^{sp^{3n+1}}(v_3^{p^3c_n}g_3) : p\nmid s>0\},\\
{}^1C_{n,2} &=& K(2)_*\{v_3^{sp^{3n+3}}(v_3^{pc_{n+1}}g_1) : p\nmid s>0\},\\
{}^lC_{n,2} &=& K(2)_*\{v_3^{sp^{3n+l-1}}(v_3^{p^lc_n}g_l) : p\nmid s>0\} \quad (l=2,3);\\
{}^lD_{n,1} &=& K(2)_*\{v_3^{(s-1)p^{3n+l}}(v_3^{p^lc'_n}g^*_{l-1} : p\nmid s>0)\} \quad (l=1,2),\\
{}^3D_{n,1} &=& K(2)_*\{v_3^{(s-1)p^{3n+3}}(v_3^{c'_{n+1}-p+1}g^*_2) : p\nmid s>0\},\\
{}^lD_{n,2} &=& K(2)_*\{v_3^{(sp-1)p^{3n+l}}(v_3^{p^lc'_n}g^*_{l-1}) : p\nmid s>0\} \quad (l=1,2),\\
{}^3D_{n,2} &=& K(2)_*\{v_3^{(sp-1)p^{3n}}(v_3^{c'_n-p+1}g^*_2) : p\nmid s>0\};\\
{}^{l+1}E_n &=& K(2)_*\{v_3^{sp^{3n+l}}(v_3^{p^{l+1}c'_n}g^*_l) : p\nmid s>0\} \quad (l=0,1),\\
{}^3E_n &=& K(2)_*\{v_3^{sp^{3n+2}}(v_3^{c'_{n+1}-p+1}g^*_2) : p\nmid s>0\},
\end{array}
$$

*in which*

$$
(1.9)\qquad
\begin{array}{c}
e_n \;=\; \dfrac{p^{3n}-1}{p^3-1};\quad c_n \;=\; (p-1)e_n,\quad c'_n \;=\; (p^2-1)e_n;\\
g^*_0 = g_1g_2,\quad g^*_1 = g_2g_3 \quad \textit{and} \quad g^*_2 = g_1g_3.
\end{array}
$$

Let $v(n/j:\gamma)\in H^*M^1_1$ denote an element such that

$$v_1^{j-1}v(n/j:\gamma)=\varphi_*(v_3^n\gamma)$$

for $\gamma\in\wedge(g_1,g_2,g_3)$, in which $\varphi_*$ is the homomorphism in (1.5). Note that

$$K(2)_*[v_1]\{v(n/j:\gamma)\}\cong K(2)_*[v_1]/(v_1^j).$$

**Lemma 1.10.** *We have submodules of $H^*M^1_1$:*

$$
\begin{array}{rcl}
{}^l\widetilde{A}_n &=& K(2)_*[v_1]\{v(sp^{3n+l}/a_{3n+l}:1) : p\nmid s>0\} \quad (l=1,2,3);\\
{}^l\widetilde{C}_{n,1} &=& K(2)_*[v_1]\{v(sp^{3n+l+1}+p^lc_{n+1}/a_{3n+l+1}:g_l) : p\nmid s>0\} \quad (l=1,2),\\
{}^3\widetilde{C}_{n,1} &=& K(2)_*[v_1]\{v(sp^{3n+1}+p^3c_n/a_{3n+1}:g_3) : p\nmid s>0\},\\
{}^1\widetilde{C}_{n,2} &=& K(2)_*[v_1]\{v(sp^{3n+3}+pc_{n+1})/a_{3n+3}:g_1) : p\nmid s>0\},\\
{}^l\widetilde{C}_{n,2} &=& K(2)_*[v_1]\{v(sp^{3n+l-1}+p^lc_n/a_{3n+l-1}:g_l) : p\nmid s>0\} \quad (l=2,3);\\
{}^{l+1}\widetilde{E}_n &=& K(2)_*[v_1]\{v(sp^{3n+l}+p^{l+1}c'_n/a_{3n+l}:g^*_l) : p\nmid s>0\} \quad (l=0,1),\\
{}^3\widetilde{E}_n &=& K(2)_*[v_1]\{v(sp^{3n+2}+c'_{n+1}-p+1/a_{3n+2}:g^*_2) : p\nmid s>0\}.
\end{array}
$$

*Here, integers $a_n$ are defined by*

$$
(1.11)\qquad a_0=1,\qquad a_n=\begin{cases}(p+1)e_{k+1}-1 & n=3k+1\\ p^{l-1}(p+1)e_{k+1} & n=3k+l \quad (l=2,3).\end{cases}
$$

Consider the submodules of $H^*M^1_1$:

$$
\begin{array}{rcl}
\widetilde{\mathfrak{c}}^0 &=& \bigoplus_{l=1}^3\bigoplus_{n\ge0}{}^l\widetilde{A}_n;\\
\widetilde{\mathfrak{c}}^1 &=& \bigoplus_{l=1}^3\bigoplus_{n\ge0}{}^l\widetilde{C}_{n,1}\oplus{}^l\widetilde{C}_{n,2};\\
\widetilde{\mathfrak{c}}^2 &=& \bigoplus_{l=1}^3\bigoplus_{n\ge0}{}^l\widetilde{E}_n.
\end{array}
$$

**Theorem 1.12.** *As a $K(2)_*[v_1]$-module, the $E_2$-term $E_2^*(C \wedge T(1)) = H^*M_1^1$ is given by:*

$$\begin{array}{rcl} H^0M_1^1 & = & K(2)_*[v_1]/(v_1^\infty) \oplus A_0^0 \oplus \widetilde{\mathfrak{c}}^0, \\ H^1M_1^1 & = & g\widetilde{\mathfrak{c}}^0 \oplus K(2)_*\{v(s/1 : g_l) : p \nmid s,\ l = 1,2,3\} \oplus \widetilde{\mathfrak{c}}^1, \\ H^2M_1^1 & = & g\left(K(2)_*\{v(s/1 : g_l) : p \nmid s,\ l = 1,2,3\} \oplus \widetilde{\mathfrak{c}}^1\right) \\ & & \quad \oplus K(2)_*\{v(s/1 : g_lg_{l'}) : p \nmid s,\ l, l' = 1,2,3\} \oplus \widetilde{\mathfrak{c}}^2, \\ H^3M_1^1 & = & g\left(K(2)_*\{v(s/1 : g_lg_{l'}) : p \nmid s,\ l, l' = 1,2,3\} \oplus \widetilde{\mathfrak{c}}^2\right) \quad \text{and} \\ H^sM_1^1 & = & 0 \quad \textit{for } s > 3. \end{array}$$

*Here, $g$ denotes an element corresponding to $v_3^{p-1}g_0$.*

The existence of the element $g$ is certified by Lemma 2.8.

This theorem shows that the $E_2$-term $E_2^*(C \wedge T(1))$ has horizontal vanishing line $s = 4$. It follows that the Adams-Novikov differentials $d_r$ are trivial and no extension problem arises.

**Corollary 1.13.** *The homotopy groups $\pi_*(L_2C \wedge T(1))$ are isomorphic to the Adams-Novikov $E_2$-term $E_2^*(C \wedge T(1))$.*

(1.14) (Ravenel *cf.* [6]) *The homotopy groups of $v_1^{-1}M \wedge T(1)$ are isomorphic to $\mathbb{Z}/p[v_1^{\pm 1}, v_2] \otimes \wedge(h_{2,0})$.*

Substitute these results to the exact sequence obtained by applying the functor $\pi_*(L_2 - \wedge T(1))$ to the cofiber sequence (1.1), and we obtain our main result:

**Corollary 1.15.** *As a $K(2)_*[v_1]$-module, the homotopy groups $\pi_*(L_2M \wedge T(1))$ are given as follows:*

$$\begin{array}{rcl} \pi_{*q-1}(L_2M \wedge T(1)) & = & \Sigma^{-1}\mathbb{Z}/p[v_1, v_2]/(v_1^\infty, v_2^\infty) \\ & & \quad \oplus \Sigma^{-q-1}A_0^0 \oplus \Sigma^{-1}\widetilde{\mathfrak{c}}^0 \oplus h_{2,0}\mathbb{Z}/p[v_1^{\pm 1}, v_2], \\ \pi_{*q-2}(L_2M \wedge T(1)) & = & \Sigma^{-2}H^1M_1^1, \\ \pi_{*q-3}(L_2M \wedge T(1)) & = & \Sigma^{-3}H^2M_1^1, \\ \pi_{*q-4}(L_2M \wedge T(1)) & = & \Sigma^{-4}H^3M_1^1, \\ \pi_{*q-t}(L_2M \wedge T(1)) & = & 0 \quad \textit{for } 4 < t < q \quad \text{and} \\ \pi_{*q}(L_2M \wedge T(1)) & = & \mathbb{Z}/p[v_1, v_2]. \end{array}$$

*Here $q = 2p - 2$, and $\Sigma$ denotes a shift of dimension.*

We notice that the last three equalities are replaced by

$$\pi_{4*}(L_2M \wedge T(1)) \ = \ \Sigma^{-4}H^3M_1^1 \oplus \mathbb{Z}/3[v_1, v_2],$$

when $p = 3$.

## 2. Some relations in $\Gamma(2)$

For the Brown-Peterson spectrum $BP$, we have the associated Hopf algebroid

$$(A, \Gamma) = (BP_*, BP_*(BP)) = (\mathbb{Z}_{(p)}[v_i \colon i > 0], A[t_i \colon i > 0]).$$

Here $v_i \in \pi_{2(p^i-1)}(BP)$, the Hazewinkel generators, and $t_i \in BP_{2(p^i-1)}(BP)$. The behavior of the structure maps is read off from the Hazewinkel and the Quillen formulas:

$$\begin{array}{c} v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^{p^i}, \quad \eta_R(m_n) = \sum_{i+j=n} m_i t_j^{p^i} \quad \text{and} \\ \sum_{i+j=n} m_i \Delta(t_j)^{p^i} = \sum_{i+j+k=n} m_i t_j^{p^i} \otimes t_k^{p^{i+j}}. \end{array}$$

A routine computation shows the following two lemmas.

**Lemma 2.1.** *The right unit* $\eta_R : A \to \Gamma(2) = \Gamma/(t_1)$ *behaves on the generators* $v_i$ *as follows:*

$$\begin{aligned}
\eta_R(v_n) &\equiv v_n \mod (p) \quad \textit{for} \;\; n = 0, 1, 2, \\
\eta_R(v_3) &\equiv v_3 + v_1 t_2^p - v_1^{p^2} t_2 \mod (p), \\
\eta_R(v_4) &\equiv v_4 + v_2 t_2^{p^2} + v_1 t_3^p - v_1\omega_{3,1} - v_2^{p^2} t_2 - v_1^{p^3} t_3 \mod (p) \\
&\equiv v_4 + v_2 t_2^{p^2} + v_1 t_{3,1} - v_2^{p^2} t_2 \mod (p) \quad \text{and} \\
\eta_R(v_5) &\equiv v_5 + v_3 t_2^{p^3} + v_2 t_3^{p^2} + v_1 t_4^p - v_2\omega_{3,2} - v_1\omega_{4,1} \\
&\qquad -v_2^{p^3} t_3 - v_1^{p^4} t_4 - t_2 \eta_R(v_3^{p^2}) \mod (p).
\end{aligned}$$

*Here,* $t_{3,1} = t_3^p - \omega_{3,1} - v_1^{p^3-1} t_3$*, and* $p\omega_{i,j} = \eta_R(v_i)^{p^j} - \sum v^{p^j}$ *if* $\eta_R(v_i) \equiv \sum v$ mod $(p)$ *for monomials* $v$*.*

**Lemma 2.2.** *Consider the operation* $D\colon \Gamma(2) \to \Gamma(2) \otimes_A \Gamma(2)$ *defined by* $D(x) = x \otimes 1 + 1 \otimes x - \Delta(x)$ *for the coproduct* $\Delta$*. Then we see that*

$$D(t_2) \;=\; 0 \quad \text{and} \quad D(t_3) \;=\; v_1 b_{2,0}$$

*for* $b_{2,j}$ *defined by* $pb_{2,j} = D(t_2^{p^{j+1}})$*.*

Note that $v_2^{-1}BP_*(X)$ is a $\Gamma(2)$-comodule if $BP_*(X)$ is a $\mathbb{Z}/p$-module, since $v_2$ is a primitive element of a $\Gamma(2)$-comodule $BP_*/(p)$ by Lemma 2.1.

The $E_2$-term of the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(X)$ of a spectrum $X$ is $E_2^{s,t}(X) = \mathrm{Ext}_\Gamma^{s,t}(A, BP_*(X))$. Let $T(1)$ denote the Ravenel ring spectrum characterized by $BP_*(T(1)) = A[t_1]$, in which $t_1$ is a primitive element of the $\Gamma$-comodule. Consider the Hopf algebroid $(A, \Gamma(2)) = (A, \Gamma/(t_1))$ associated with $(A, \Gamma)$. Then, we have an isomorphism

$$E_2^*(X \wedge T(1)) \cong H^* v_2^{-1} BP_*(X) \tag{2.3}$$

(see (1.4) for $H^*-$) by a change of rings theorem (*cf.* [6]) if $BP_*(X)$ is a $\mathbb{Z}/p$-module.

Hereafter, we set $v_2 = 1$ for the sake of simplicity. In fact, we can recover $v_2$ since every equation appearing here is homogeneous. Consider the cobar complex $\Omega^*_{\Gamma(2)} v_2^{-1} BP_*/(p)$ for computing $\mathrm{Ext}^*_{\Gamma(2)}(BP_*, v_2^{-1}BP_*/(p))$.

**Lemma 2.4.** *There are elements* $x_i \in BP_*/(p)$ *for* $i < 3$ *such that* $x_i \equiv v_3^{p^i}$ mod $(p, v_1)$ *and*

$$d\,(x_i) \;\equiv\; \begin{cases} v_1 t_2^p - v_1^{p^2} t_2 \mod (p) & i = 0 \\ v_1^p t_2 - v_1^{p+1} t_{3,1} - v_1^{p^3} t_2^p \mod (p) & i = 1 \\ -v_1^{p^2+p} t_{3,2} + v_1^{2p^2-1} t_2 - v_1^{p^4+p^3-p} t_2^p \mod (p) & i = 2. \end{cases}$$

*Here,* $t_{3,2} = t_{3,1}^p$*.*

*Proof.* Put $x_0 = v_3$, $x_1 = v_3^p - v_1^p v_4$ and $x_2 = x_1^p - v_1^{p^2-1} v_3 + v_1^{p^4-p} v_3^p$. Then, the lemma follows immediately from Lemma 2.1 with the definition of the differential $d$: $d(x) = \eta_R(x) - x$. □

**Lemma 2.5.** *The element* $t_{3,1}$ *is a cocycle.*

*Proof.* Noticing that $d(v_2) \equiv 0 \bmod (p)$, we compute

$$d\left(v_1 t_{3,1}\right) \equiv d(-v_2 t_2^{p^2} + v_2^{p^2} t_2 + d(v_4)) \equiv 0$$

by Lemmas 2.1 and 2.2. Since $v_1$ acts on the cobar complex $\Omega^*_{\Gamma(2)} BP_*/(p)$ monomorphically, we obtain the lemma. □

By virtue of this lemma, we have cocycles

$$t_{3,i} = t_{3,1}^{p^{i-1}}$$

for $i > 0$.

**Lemma 2.6.** *Put*

$$\begin{aligned} w_1 &= v_3^p t_2^{p^4} + t_{3,3} - t_{3,1} - t_2^p \eta_R(v_3^{p^3}) \quad \text{and} \\ w_2 &= v_1^{-1}\omega_{3,1} + v_1^{p-1}\left(\omega_{4,2} - t_4^{p^2}\right) + v_1^{p^3-2} t_3 - v_1^{p^5-p^2-1} t_3^{p^2} + v_1^{p^5-1} t_4^p. \end{aligned}$$

*Then,* $d(v_5^p) \equiv w_1 - v_1 w_2 \bmod (p)$, *and* $d(w_2) \equiv v_1^{p-1} t_2^{p^2} \otimes t_2^{p^4} - v_1^{p^5-1} t_2^p \otimes t_2^{p^3} \bmod (p)$.

*Proof.* The first assertion follows immediately from the congruence on $\eta_R(v_5)$ in Lemma 2.1.

By Lemma 2.5 and the relation $d(x\eta_R(v)) = d(x) \otimes v - x \otimes d(v)$ for $v \in A$ and $x \in \Gamma(2)$, we have

$$\begin{aligned} d(v_1 w_2) &= d(w_1) \\ &\equiv v_1^p t_2^{p^2} \otimes t_2^{p^4} - v_1^{p^3} t_2^p \otimes t_2^{p^4} + v_1^{p^3} t_2^p \otimes t_2^{p^4} - v_1^{p^5} t_2^p \otimes t_2^{p^3} \mod (p) \end{aligned}$$

Since $v_1 \colon \Omega^1_{\Gamma(2)} BP_*/(p) \to \Omega^2_{\Gamma(2)} BP_*/(p)$ is a monomorphism, we have the second. □

**Lemma 2.7.** *The cochain* $t_2^p \otimes t_2^{p^3}$ *cobounds. In other words, there is a cochain* $\sigma$ *such that* $d(\sigma) = t_2^p \otimes t_2^{p^3}$.

*Proof.* The desired element $\sigma$ is read off from the following computation, in which the underlined terms with the same subscript cancel each other out:

$$\begin{aligned}
d\left(v_4^p t_2^{p^3}\right) &\equiv \left(\underline{t_2^{p^3}}_1 - t_2^p + \underline{v_1^p t_{3,2}}_2\right) \otimes t_2^{p^3}, \\
d\left(\frac{1}{2} t_2^{2p^3}\right) &\equiv \underline{-t_2^{p^3} \otimes t_2^{p^3}}_1, \\
d\left(v_1^p t_{3,2} \eta_R(v_4^p)\right) &\equiv -v_1^p t_{3,2} \otimes \left(\underline{t_2^{p^3}}_2 + \underline{v_1^p t_{3,2}}_3 - \underline{t_2^p}_4\right), \\
d\left(-\frac{1}{2} v_1^{2p} t_{3,2}^2\right) &\equiv \underline{v_1^{2p} t_{3,2} \otimes t_{3,2}}_3, \\
d\left(v_1^{p-1} t_{3,2} \eta_R(v_3)\right) &\equiv -v_1^{p-1} t_{3,2} \otimes \left(\underline{v_1 t_2^p}_4 - \underline{v_1^{p^2} t_2}_5\right), \\
d\left(-v_1^{-p-1} d(x_2) \eta_R(x_1)\right) &\equiv v_1^{-p-1} d(x_2) \otimes \left(v_1^p t_2 - v_1^{p+1} t_{3,1} - v_1^{p^3} t_2^p\right) \\
&\equiv v_1^{-p-1}\left(\underline{-v_1^{p^2+p} t_{3,2}}_5 + \underline{v_1^{2p^2-1} t_2}_7 - \underline{v_1^{p^4+p^3-p} t_2^p}_8\right) \otimes v_1^p t_2 \\
&\quad \underline{-v_1^{-p-1}\left(-v_1^{p^2+p} t_{3,2} + v_1^{2p^2-1} t_2 - v_1^{p^4+p^3-p} t_2^p\right) \otimes \left(v_1^{p+1} t_{3,1} + v_1^{p^3} t_2^p\right)}_6, \\
d\left(x_2 (t_{3,1} + v_1^{p^3-p-1} t_2^p)\right) &\equiv \underline{\left(-v_1^{p^2+p} t_{3,2} + v_1^{2p^2-1} t_2 - v_1^{p^4+p^3-p} t_2^p\right) \otimes (t_{3,1} + v_1^{p^3-p-1} t_2^p)}_6, \\
d\left(v_1^{2p^2-p-2} t_2^2\right) &\equiv -\underline{v_1^{2p^2-p-2} t_2 \otimes t_2}_7, \\
d\left(v_1^{p^4+p^3-2p-2} v_3 t_2\right) &\equiv v_1^{p^4+p^3-2p-2}\left(\underline{v_1 t_2^p \otimes t_2}_8 - \underline{v_1^{p^2} t_2 \otimes t_2}_9\right) \quad \text{and} \\
d\left(-v_1^{p^4+p^3+p^2-2p-2} t_2^2\right) &\equiv \underline{v_1^{p^4+p^3+p^2-2p-2} t_2 \otimes t_2}_9.
\end{aligned}$$

□

**Lemma 2.8.** *There exists a cocycle $\tau$ such that $\tau \equiv v_3^{p-1} t_2^p \bmod (p, v_1)$.*

*Proof.* By Lemmas 2.6 and 2.7, we see that $\tau = w_2 - v_1^{p-1}\sigma^p + v_1^{p^5-1}\sigma$ satisfies the desired condition. □

## 3. The elements $x_n$

In this section, we introduce the elements $x_n$ and $g_n$, and observe the differential of them in cobar complex $\Omega^*_{\Gamma(2)} BP_*/(p)$.

The elements $x_n \in BP_*$ for $n \leq 2$ are those given in Lemma 2.4. We define $x_n$ for $n \geq 3$ inductively by

$$x_n = x_{n-1}^p + (-1)^{k+1} v_1^{pa_{n-1}} y_n \tag{3.1}$$

for $n = 3k + l \geq 1$ with $l = 1, 2, 3$, and

$$y_n = \begin{cases} -v_1^{-p^3} v_3^{p^4 c_{k-1}} w^{p^2} + v_1^{-p^2-p} v_3^{p^4 c_{k-1}} x_2 & n = 3k+1 > 4 \\ v_1^{-1} v_3^{p^2 c_k} (v_3 + v_1 v_4^p) & n = 3k+2 \\ 0 & n = 1 \text{ or } n = 3k+3, \end{cases} \tag{3.2}$$

where $w = v_3^{p^2+p} - \frac{1}{2} v_1^{p-1} v_3^2 - v_1^p v_3^{p^2} v_4 - v_1^p v_3 v_4^p + v_1^p v_5$, and the elements $g_n \in \Gamma(2)$:

$$g_n = \begin{cases} t_2^p & n = 0 \\ (-1)^k v_3^{pc_k} t_2^{p^2} & n = 3k+1 \\ (-1)^{k+1} v_3^{p^2 c_k} t_3^{p^2} & n = 3k+2 \\ (-1)^{k+1} v_3^{p^3 c_k} t_3^{p^3} & n = 3k+3. \end{cases} \tag{3.3}$$

Here, the integers $a_n$ and $c_n$ are given in (1.11) and (1.9), respectively.

**Lemma 3.4.** *For the differential $d : \Omega^0_{\Gamma(2)} BP_*/(p) \to \Omega^1_{\Gamma(2)} BP_*/(p)$ of the cobar complex, $d(w) \equiv v_1^p (t_3^{p^2} - t_3) \mod (p, v_1^{p+1})$.*

*Proof.* This follows from the computation by Lemma 2.1:

$$\begin{aligned} d(w) &= d(v_3^{p^2+p} - \frac{1}{2} v_1^{p-1} v_3^2 - v_1^p v_3^{p^2} v_4 - v_1^p v_3 v_4^p + v_1^p v_5) \\ &\equiv \underline{v_1^p v_3^{p^2} t_2^{p^2}}_1 - \underline{v_1^p v_3 t_2^p}_2 - v_1^p v_3^{p^2} \left( \underline{t_2^{p^2}}_1 - \underline{t_2}_3 \right) \\ &\quad - v_1^p v_3 \left( \underline{t_2^{p^3}}_4 - \underline{t_2^p}_2 \right) + v_1^p \left( \underline{v_3 t_2^{p^3}}_4 + t_3^{p^2} - t_3 - \underline{v_3^{p^2} t_2}_3 \right) \\ &\equiv v_1^p (t_3^{p^2} - t_3) \end{aligned}$$

mod $(p, v_1^{p+1})$. □

**Lemma 3.5.** *In the cobar complex $\Omega^1_{\Gamma(2)} BP_*/(p)$, $d(x_n) \equiv v_1^{a_n} g_n \mod (v_1^{a_n+p})$.*

*Proof.* For $n = 0, 1$, these follow from Lemma 2.1 immediately, and for $n = 2$,

$$d(x_2) \equiv -v_1^{p^2+p} t_{3,2} \equiv -v_1^{p^2+p} \left( t_3^{p^2} - v_1^p v_3^{p^2-p} t_2^{p^2} \right) \mod (p, v_1^{p^2+3p}) = (p, v_1^{a_2+2p}),$$

since $\omega_{3,1} \equiv v_1 v_3^{p-1} t_2^p \mod (p, v_1^2)$.

Suppose inductively the congruence on $d(x_{3k+2})$. Then, raising it to the $p$-th power shows the congruence on $d(x_{3k+3})$. By using Lemma 3.4, we compute

$$\begin{aligned} d(x_{3k+3}^p) &\equiv (-1)^{k+1} v_1^{p^2 a_{3k+2}} v_3^{p^4 c_k} t_3^{p^4} \mod (p, v_1^{p^2 a_{3k+2}+p^2}), \\ d((-1)^k & v_1^{p^2 a_{3k+2} - p^3} v_3^{p^4 c_k} w^{p^2}) \\ &\equiv (-1)^k v_1^{p^2 a_{3k+2}} v_3^{p^4 c_k} (t_3^{p^4} - t_3^{p^2}) \mod (p, v_1^{p^2 a_{3k+2}+p^2}), \\ d((-1)^{k+1} & v_1^{p^2 a_{3k+2} - a_2} v_3^{p^4 c_k} x_2) \\ &\equiv (-1)^k v_1^{p^2 a_{3k+2}} v_3^{p^4 c_k} \left( t_3^{p^2} - v_1^p v_3^{p^2-p} t_2^{p^2} \right) \mod (p, v_1^{p^2 a_{3k+2}+2p}) \end{aligned}$$

and obtain the congruence on $d(x_{3k+4})$. We further compute

$$\begin{aligned} d(x_{3k+4}^p) &\equiv (-1)^{k+1} v_1^{p a_{3k+4}} v_3^{p^2 c_{k+1}} t_2^{p^3} \mod (p, v_1^{p a_{3k+4}+p^2}), \\ d((-1)^k & v_1^{p a_{3k+4}-1} v_3^{p^2 c_{k+1}} v_3) \\ &\equiv (-1)^k v_1^{p a_{3k+4}} v_3^{p^2 c_{k+1}} t_2^p \mod (p, v_1^{p a_{3k+4}+p^2-1}), \\ d((-1)^k & v_1^{p a_{3k+4}} v_3^{p^2 c_{k+1}} v_4^p) \\ &\equiv (-1)^k v_1^{p a_{3k+4}} v_3^{p^2 c_{k+1}} \left( t_2^{p^3} - t_2^p + v_1^p t_3^{p^2} \right) \mod (p, v_1^{p a_{3k+4}+p+1}) \end{aligned}$$

to obtain the congruence on $d(x_{3k+5})$, and complete the induction. □

## 4. **Proof of Theorem 1.12**

We begin with

**Lemma 4.1.** *For the modules in (1.8), we have relations*

1) $\bigoplus_{n \geq 0} {}^l B_n \oplus {}^l C_{n,1} \oplus {}^l C_{n,2}$ *equals* $K(2)_*[v_3^p] g_l$ *for* $l = 1, 2, 3$.
2) $\bigoplus_{n \geq 0} {}^l D_{n,1} \oplus {}^l D_{n,2} \oplus {}^l E_n$ *equals* $K(2)_*[v_3^p] g_{l-1}^*$ *for* $l = 1, 2, 3$.

*Proof.* The part 1) based on the fact:

(4.2) *For each non-negative integer $n$, we have unique non-negative integers $m$ and $k$ such that $p^3 \nmid (m+1-p)$ and*

$$n \quad = \quad mp^{3k} + c_k \quad (c_k \;=\; (p-1)e_k).$$

In fact, for each integer $n$, there is an integer $s$ such that $n-(p-1)e_s$ is divisible by $p^{3s}$. Indeed, we may take $s=0$. Let $k$ be the largest integer among such integers, that is the desired one.

Since the condition on $m$ is divided into three conditions

$$a)\ p \nmid (m+1), \quad b)\ p \mid (m+1),\ p^2 \nmid (m+1-p), \quad \text{or} \quad c)\ p^2 \mid (m+1-p), p^3 \nmid (m+1-p),$$

the fact (4.2) divides non-negative integers into three kinds:

$$\begin{array}{ll} a) \quad (s-1)p^{3k}+c_k, \quad b) \quad (sp+p-1)p^{3k}+c_k = sp^{3k+1}+c_{k+1} \quad \text{and} \\ c) \quad (sp^2+p-1)p^{3k}+c_k = sp^{3k+2}+c_{k+1} \end{array}$$

for $p \nmid s > 0$. This shows the desired decomposition.

For the part 2), we consider another expression

$$n \quad = \quad mp^{3k} + c'_k \quad (c'_k = (p+1)c_k = (p^2-1)e_k)$$

*for $m$ with $p^3 \nmid (m+1-p^2)$.* By a similar argument, we obtain the decomposition for $l=1,2$. For $l=2$, just use another expression of a non-negative integer $np$:

$$np \quad = \quad mp^{3k} + c'_k - p + 1 \quad (c'_k = (p+1)c_k = (p^2-1)e_k)$$

for $m$ with $p^3 \nmid (m+1-p^2)$, excluding the case where $k=0$ and $p \nmid (m+1)$. $\square$

*Proof of Lemma 1.7.* Since $\mathfrak{c}^0$ is isomorphic to $K(2)_*[v_3^p]/K(2)_*$, we see the isomorphism on $H^0M_2^0$.

Decompose $H^1M_2^0 = \overline{K(2)_*}[v_3]\langle v_2^{-1}g_0\rangle \oplus K(2)_*[v_3]\{g_1,g_2,g_3\}$ for $\overline{K(2)_*}[v_3] = K(2)_*[v_3]/K(2)_*$. Then the first summand is isomorphic to $A_0^0\langle v_3^{-1}g_0\rangle\oplus K(2)_*[v_3^p]\langle v_3^{p-1}g_0\rangle$. By Lemma 4.1.1), the second summand is isomorphic to $A_0^1 \oplus \mathfrak{i}^1 \oplus \mathfrak{c}^1$. We decompose

$$H^2M_2^0 = A_0^1\langle v_3^{-1}g_0\rangle \oplus K(2)_*[v_3^p]\langle v_3^{p-1}g_0\rangle\{g_1,g_2,g_3\} \oplus K(2)_*[v_3]\{g_0^*,g_1^*,g_2^*\},$$

and obtain the desired decomposition by Lemma 4.1.2). Similarly follow the others. $\square$

*Proof of Lemma 1.10.* By virtue of Lemma 3.5, we may take $v(sp^{3n+l}/a_{3n+l}:1) = x_{3n+l}^s/v_1^{a_{3n+l}}$, and so ${}^l\widetilde{A}_n$ is a submodule of $H^*M_1^1$.

We also put $v(sp^{3n+l+1}+p^lc_{n+1}/a_{3n+l+1}:g_l) = v(sp^{3n+l+1}/a_{3n+l+1}:1)g_{3n+3+l}$ for ${}^l\widetilde{C}_{n,1}$. For the other, it follows similarly. $\square$

*Proof of Theorem 1.12.* Let $D^i$ denote the module appearing on the right hand side of the isomorphism on $H^iM_1^1$ in the theorem. Since the inclusion $D^i \to H^iM_1^1$ defined by Lemma 1.10 satisfies the condition of [5, Remark 3.11], it suffices to show that the sequence

$$0 \longrightarrow H^0M_2^0 \xrightarrow{\varphi_*} D^0 \xrightarrow{v_1} D^0 \xrightarrow{\delta} H^1M_2^0 \xrightarrow{\varphi_*} D^1 \xrightarrow{v_1} \cdots$$

is exact. By Lemma 3.5, we have exact sequences

$$0 \longrightarrow K(2)_* \xrightarrow{\varphi_*} K(2)_*[v_1]/(v_1^\infty) \xrightarrow{v_1} K(2)_*[v_1]/(v_1^\infty) \to 0,$$

$$0 \longrightarrow A_0^0 \xrightarrow{\varphi_*} A_0^0 \xrightarrow{v_1=0} A_0^0 \xrightarrow{\delta} A_0^0\langle v_3^{-1} g_0\rangle \longrightarrow 0,$$

$$0 \longrightarrow \mathfrak{c}^0 \xrightarrow{\varphi_*} \widetilde{\mathfrak{c}}^0 \xrightarrow{v_1} \widetilde{\mathfrak{c}}^0 \xrightarrow{\delta} \mathfrak{i}^1 \longrightarrow 0,$$

$$0 \longrightarrow \mathfrak{c}^1 \xrightarrow{\varphi_*} \widetilde{\mathfrak{c}}^1 \xrightarrow{v_1} \widetilde{\mathfrak{c}}^1 \xrightarrow{\delta} \mathfrak{i}^2 \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow \mathfrak{c}_2 \xrightarrow{\varphi_*} \widetilde{\mathfrak{c}}^2 \xrightarrow{v_1} \widetilde{\mathfrak{c}}^2 \xrightarrow{\delta} K(2)_*[v_3] g_1 g_2 g_3 \longrightarrow 0.$$

These show that the above sequence is exact. □

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