A RELATION AMONG HOPKINS' PICARD GROUPS OF THE LOCALIZED CATEGORIES WITH RESPECT TO FINITE WEDGES OF THE MORAVA K-THEORIES

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Abstract. We work in the stable homotopy category of p-local spectra for a fixed prime number p. Let E be a spectrum and $\mathcal{L}_E$ denote the stable homotopy category of localized spectra with respected to E in the sense of Bousfield. Then, M. Hopkins introduced a Picard group $\mathrm{Pic}(\mathcal{L}_E)$ of the category $\mathcal{L}_E$. If the spectra E and F satisfy the relation $\langle E\rangle \geq \langle F\rangle$ of the Bousfield classes, then we have a homomorphism $\ell\colon \mathrm{Pic}(\mathcal{L}_E)\to \mathrm{Pic}(\mathcal{L}_F)$. We consider the spectra $K^n_m = E(n)\wedge MJ_m$ for the n-th Johnson-Wilson spectrum $E(n)$ and a type m generalized Moore spectrum MJ_m for $0\leq m\leq n$. For $E = K^n_m$, we have a subgroup $\mathrm{Pic}^0(\mathcal{L}_E)$ of $\mathrm{Pic}(\mathcal{L}_E)$ consisting of exotic elements. In this paper, we study the homomorphism $\ell\colon \mathrm{Pic}^0(\mathcal{L}_{E(n)})\to \mathrm{Pic}^0(\mathcal{L}_{K^n_m})$, and give conditions under which it is an isomorphism. This is a generalization of the result $\mathrm{Pic}^0(\mathcal{L}_2)\cong \kappa_2$ ([3, Remark. 6.5]) for $(p,n,m) = (3,2,2)$.

1. Introduction

Let $\mathcal{S}_{(p)}$ denote the stable homotopy category of p-local spectra for a prime number p. For each spectrum $E\in\mathcal{S}_{(p)}$, we call a spectrum $X\in\mathcal{S}_{(p)}$ *E-local* if $[C,X]_* = 0$ for any C with $C\wedge E = 0$, and denote by $\mathcal{L}_E$ the full subcategory consisting of all E-local spectra. We then have the Bousfield localization functor $L_E\colon \mathcal{S}_{(p)}\to\mathcal{L}_E\subset\mathcal{S}_{(p)}$ along with a natural transformation $\eta\colon id\to L_E$. Let $\langle E\rangle$ for a spectrum E denote the Bousfield class of E. We define an order on Bousfield classes by setting $\langle E\rangle\geq\langle F\rangle$ if $X\wedge F = 0$ whenever $X\wedge E = 0$. Then,

$$L_E = L_F \text{ (or } \mathcal{L}_E = \mathcal{L}_F\text{) if and only if } \langle E\rangle = \langle F\rangle.$$

A spectrum $X\in\mathcal{L}_E$ is called *invertible* if there is a spectrum $Y\in\mathcal{L}_E$ such that $L_E(X\wedge Y)\simeq L_ES^0\in\mathcal{L}_E$. M. Hopkins introduced the Picard group $\mathrm{Pic}(\mathcal{L}_E)$ of a localized category $\mathcal{L}_E$, which consists of equivalence classes of invertible spectra under weak equivalences (*cf.* [26], [5]). We notice that the Picard group needs not be a set. The multiplication $\wedge_E$ of the group is defined by $X\wedge_E Y = L_E(X\wedge Y)$ for $X,Y\in\mathcal{L}_E$, and L_ES^0 is the unit. Hereafter, we abuse notation and write $X\in\mathrm{Pic}(\mathcal{L}_E)$ for the equivalence class of an invertible spectrum X. For spectra E and F with $\langle E\rangle\geq\langle F\rangle$, we have a homomorphism

$$\ell_F\colon \mathrm{Pic}(\mathcal{L}_E)\to\mathrm{Pic}(\mathcal{L}_F) \tag{1.1}$$

defined by $\ell_F(X) = L_FX$ (*cf.* [12, Lemma 2.2]). Moreover, we see easily the following:

(1.2) (*cf.* [12, Lemma 2.5]) *ℓ_F is a monomorphism if $\langle E\rangle\geq\langle F\rangle$ and $L_ES^0 = L_FS^0$.*

Let BP, $E(n)$ and $K(n)$ denote the Brown-Peterson spectrum, the Johnson-Wilson spectrum and the Morava K-theory for each integer $n \geq 0$, respectively, whose coefficient rings are

$$BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots],$$

$E(0)_* = \mathbb{Q} = K(0)_*$, and for $n \geq 1$,

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}] \quad \text{and} \quad K(n)_* = \mathbb{Z}/p[v_n, v_n^{-1}].$$

Consider the spectra

$$K_m^n = \bigvee_{i=m}^{n} K(i) \quad \text{for } 0 \leq m \leq n.$$

Then, the Bousfield classes of these spectra satisfy

$$(1.3) \qquad \langle E(n)\rangle = \langle K_0^n\rangle > \cdots > \langle K_m^n\rangle > \left\langle K_{m+1}^n\right\rangle > \cdots > \langle K_n^n\rangle = \langle K(n)\rangle .$$

Here, the first equality is shown in [20, 2.1.Th.(d)]. We consider the stable homotopy categories localized with respect to these spectra:

$$\mathcal{L}_m^n = \mathcal{L}_{K_m^n} \quad \text{and} \quad \mathcal{L}_n = \mathcal{L}_{E(n)} = \mathcal{L}_0^n,$$

and the Bousfield localization functors

$$L_m^n \colon \mathcal{S}_{(p)} \to \mathcal{L}_m^n \quad \text{and} \quad L_n (= L_0^n) \colon \mathcal{S}_{(p)} \to \mathcal{L}_n$$

for $0 \leq m \leq n$. The smash product $\wedge_m^n$ on $\mathcal{L}_m^n$ is defined by

$$(1.4) \qquad X \wedge_m^n Y = L_{K_m^n}(X \wedge Y) = L_m^n(X \wedge Y)$$

for $X, Y \in \mathcal{L}_m^n$.

We say that a finite spectrum V has *type* m, if $K(i)_*(V) = 0$ for $i < m$ and $K(m)_*(V) \neq 0$. A typical example of a type m finite spectrum is a generalized Moore spectrum MJ for an invariant ideal $J = (p^{e_0}, v_1^{e_1}, \dots, v_{m-1}^{e_{m-1}})$ of BP_*, such that

$$BP_*(MJ) = BP_*/J.$$

For a type m finite spectrum V,

$$(1.5) \qquad \langle K_m^n\rangle = \langle E(n) \wedge V\rangle .$$

Furthermore, for a spectrum W,

$$(1.6)\ (\textit{cf.}\ [7, \text{Cor. } 2.2]) \qquad L_m^n = L_V L_n \quad \text{and} \quad W \wedge_m^n V \simeq L_n W \wedge V.$$

Here, the second follows from the first by $W \wedge_m^n V = L_V L_n W \wedge V \simeq L_n W \wedge V$, since V is finite.

Note that $L_{m+1}^n S^0$ is an $L_m^n S^0$-module spectrum, and we have $\left\langle L_m^n S^0\right\rangle \geq \left\langle L_{m+1}^n S^0\right\rangle$. Since $\left\langle L_{K(n)} S^0\right\rangle = \langle E(n)\rangle = \langle E\rangle$ for

$$E = v_n^{-1} BP$$

by [7, Cor. 2.4] and [20, 2.1.Th.(b)], we see the following:

$$(1.7) \qquad \langle E\rangle = \left\langle L_n S^0\right\rangle = \left\langle L_m^n S^0\right\rangle = \left\langle L_{K(n)} S^0\right\rangle,$$

where $\langle E(n)\rangle = \left\langle L_n S^0\right\rangle$ since $E(n)$ is smashing (*cf.* [21]).

Now we consider the Picard groups of the categories $\mathcal{L}_m^n$. For $\mathcal{L}_n = \mathcal{L}_0^n$ and $\mathcal{L}_{K(n)} = \mathcal{L}_n^n$, we have the followings:

(1.8) ([9, Prop. 1.4, Lemma 1.5], [5, Prop. 7.6]) *Both* $\mathrm{Pic}(\mathcal{L}_n)$ *and* $\mathrm{Pic}(\mathcal{L}_{K(n)})$ *are sets. Furthermore, there is a summand* $\mathrm{Pic}^0(\mathcal{L}_n)$ *such that* $\mathrm{Pic}(\mathcal{L}_n) \cong \mathrm{Pic}^0(\mathcal{L}_n) \oplus \mathbb{Z}$.

(1.9) ([9], [5, Th. 1.3, (15)]) *For an invertible spectrum* X *in* $\mathcal{L}_n$, $E(n)_*(X) \cong E(n)_*$ *as* $E(n)_*$*-modules. For an invertible spectrum* X *in* $\mathcal{L}_{K(n)}$, $E(n)_*(X \wedge V) \cong E(n)_*(V)$ *as* $E(n)_*$*-modules for any generalized Moore spectrum* V *of type* n.

The next theorem is a generalization of (1.9).

Theorem A. *Let* $0 \le m \le n$. *For an invertible spectrum* X *in* $\mathcal{L}^n_m$, *there is an isomorphism* $E(n)_*(X \wedge V) \cong E(n)_*(V)$ *of* $E(n)_*$*-modules for any generalized Moore spectrum* V *of type* m.

By the results of Hopkins and Smith [6] and Devinatz [1], we have a sequence

$$\mathcal{V}_m = \{V_k, \tau_k \colon V_{k+1} \to V_k\}_{k \ge 1} \tag{1.10}$$

of type m generalized Moore spectra V_k for each $m \ge 0$ satisfying the following five properties:

1) Each $V_k \in \mathcal{V}_m$ is a generalized Moore spectrum $MJ_{m,k}$ for an invariant ideal
$$J_{m,k} = (p^{e_{0,k}}, v_1^{e_{1,k}}, \dots, v_{m-1}^{e_{m-1,k}}) \quad \text{with } e_{i,k} \ge 0$$
of BP_*.
2) $J_{m,k} \supset J_{m,k+1}$ and $\bigcap_{k \ge 1} J_{m,k} = 0$.
3) For each $k \ge 1$, $V_k \in \mathcal{V}_m$ is a ring spectrum with multiplication $m_k \colon V_k \wedge V_k \to V_k$ and unit $i_k \colon S^0 \to V_k$, in which i_k is the inclusion to the bottom cell.
4) For each $k \ge 1$, the map τ_k satisfies $\tau_k i_{k+1} = i_k$. In particular, it induces the projection $(\tau_k)_* \colon BP_*/J_{m,k+1} \to BP_*/J_{m,k}$.
5) For each $k \ge 1$, $V_k \in \mathcal{V}_m$ is self-dual: $D(V_k) = \Sigma^{a_k} V_k$ for the Spanier-Whitehead dual $D(X) = F(X, S^0)$ and an integer a_k.

We notice that $\mathcal{V}_0 = \{S^0\}$ and so $V_k = S^0 \in \mathcal{V}_0$ for $k \ge 1$.

(1.11) ([7, Th. 2.1, Cor. 2.2]) $L^n_m X = \mathrm{holim}_{V_k \in \mathcal{V}_m} L_n X \wedge V_k$ for $0 \le m \le n$ and for any spectrum X.

We call an invertible spectrum X in $\mathcal{L}^n_m$ *exotic* if the isomorphism $E(n)_*(X \wedge V) \to E(n)_*(V)$ in Theorem A is the one of $E(n)_*(E(n))$-comodules for each $V \in \mathcal{V}_m$. We have well known subgroups of the Picard groups of $\mathcal{L}_n$ and $\mathcal{L}_{K(n)}$ consisting of exotic elements:

$$\mathrm{Pic}^0(\mathcal{L}_n) \subset \mathrm{Pic}(\mathcal{L}_n) \quad \text{and} \quad \kappa_n \subset \mathrm{Pic}(\mathcal{L}_{K(n)}).$$

(1.12) ([9, Th. 2.4]) *For* $Q \in \mathrm{Pic}^0(\mathcal{L}_n)$, *we have an isomorphism* $E(n)_*(Q) \cong E(n)_*$ *as an* $E(n)_*(E(n))$*-comodule.*

For a given sequence $\mathcal{V}_m$ in (1.10), we consider a collection

$$\begin{aligned} \mathcal{S}^n_m = \{X \in \mathcal{L}^n_m \mid \forall V_k \in \mathcal{V}_m,\ \exists h^X_k \colon E(n)_*(V_k) \cong_{\mathcal{C}(n)} E(n)_*(X \wedge V_k), \\ (\tau_{k-1})_* h^X_k = h^X_{k-1} (\tau_{k-1})_*\}/\simeq, \end{aligned} \tag{1.13}$$

in which $\cong_{\mathcal{C}(n)}$ denotes an isomorphism of $E(n)_*(E(n))$-comodules, and put

$$\mathrm{Pic}^0(\mathcal{L}^n_m) = \mathrm{Pic}(\mathcal{L}^n_m) \cap \mathcal{S}^n_m \subset \mathcal{S}^n_m. \tag{1.14}$$

We see that $\mathcal{S}^n_m$ is a semigroup with multiplication given by the smash product $\wedge^n_m$ (see (3.10)). It looks that $\mathcal{S}^n_m$ depends on the choice of a sequence $\mathcal{V}_m$ of (1.10), and so does $\mathrm{Pic}^0(\mathcal{L}^n_m)$.

Proposition B. *Let $0 \le m \le n$. Then, $\mathcal{S}^n_m$ is defined independently of the choice of $\mathcal{V}_m$. Furthermore, $\mathrm{Pic}^0(\mathcal{L}^n_m)$ is a subgroup of $\mathrm{Pic}(\mathcal{L}^n_m)$.*

We notice that the following:

(1.15) (*cf.* [9], [5]) $\mathrm{Pic}^0(\mathcal{L}^n_0) = \mathrm{Pic}^0(\mathcal{L}_n)$ and $\mathrm{Pic}^0(\mathcal{L}^n_n) = \kappa_n$.

Consider the homomorphism $\ell^n_m \colon \mathrm{Pic}(\mathcal{L}_n) \to \mathrm{Pic}(\mathcal{L}^n_m)$ in (1.1) obtained from the relation $\langle E(n)\rangle \ge \langle K^n_m\rangle$ in (1.3). It follows from (1.6) and (1.12) that $L^n_m Q \in \mathcal{S}^n_m$ for $Q \in \mathrm{Pic}^0(\mathcal{L}_n)$, and so the homomorphism ℓ^n_m is restricted to a homomorphism

$$\ell^n_m \colon \mathrm{Pic}^0(\mathcal{L}_n) \to \mathrm{Pic}^0(\mathcal{L}^n_m). \tag{1.16}$$

We now consider a similar statement to (1.2) on $\ell^n_m \colon \mathrm{Pic}^0(\mathcal{L}_n) \to \mathrm{Pic}^0(\mathcal{L}^n_m)$ with $\langle E(n)\rangle \ge \langle K^n_m\rangle$ and $\langle L_n S^0\rangle = \langle L^n_m S^0\rangle$. Let $\{E^{s,t}_r(X)\}$ for a spectrum $X \in \mathcal{S}_{(p)}$ denote the $E(n)$-based Adams spectral sequence converging to the homotopy groups $\pi_*(L_n X)$:

$$E^{s,t}_2(X) = \mathrm{Ext}^{s,t}_{E(n)_*(E(n))}(E(n)_*, E(n)_*(X)) \Longrightarrow \pi_{t-s}(L_n X). \tag{1.17}$$

We consider a condition:

(C-I)$_m$ There exists a generalized Moore spectrum V of type m such that the inclusion $i_V \colon S^0 \to V$ to the bottom cell induces a monomorphism $(i_V)_* \colon E^{rq+1,rq}_{rq+1}(S^0) \to E^{rq+1,rq}_{rq+1}(V)$ for every $r \ge 1$.

Hereafter, we put

$$q = 2p - 2.$$

Theorem C. *Let m be an integer with $0 \le m \le n$, and suppose* (C-I)$_m$*. Then, $\ell^n_m \colon \mathrm{Pic}^0(\mathcal{L}_n) \to \mathrm{Pic}^0(\mathcal{L}^n_m)$ in* (1.16) *is a monomorphism.*

Next, we consider two conditions, under which ℓ^n_m in (1.16) is an epimorphism:

(C-II) $\mathrm{Pic}^0(\mathcal{L}_n)$ is a finite group.

(C-III)$_m$ There exists a generalized Moore spectrum V of type m such that the inclusion $i_V \colon S^0 \to V$ to the bottom cell induces a monomorphism $(i_V)_* \colon E^{rq+2,rq}_2(S^0) \to E^{rq+2,rq}_2(V)$ for every $r \ge 1$.

Theorem D. *Let m be an integer with $0 \le m \le n$, and suppose* (C-II) *and* (C-III)$_m$*. Then, $\ell^n_m \colon \mathrm{Pic}^0(\mathcal{L}_n) \to \mathrm{Pic}^0(\mathcal{L}^n_m)$ in* (1.16) *is an epimorphism.*

Actually, we show the mapping $\ell^n_m \colon \mathrm{Pic}^0(\mathcal{L}_n) \to \mathcal{S}^n_m$ given by $\ell^n_m(X) = L^n_m X$ surjective in Corollary 5.16. The mapping factors as $\mathrm{Pic}^0(\mathcal{L}_n) \xrightarrow{\ell^n_m} \mathrm{Pic}^0(\mathcal{L}^n_m) \subset \mathcal{S}^n_m$.

Corollary D-1. *Let m be an integer with $0 \le m \le n$, and suppose* (C-II) *and* (C-III)$_m$*. Then, $\mathrm{Pic}^0(\mathcal{L}^n_m) = \mathcal{S}^n_m$. In particular,*

$$\mathcal{S}^n_0 = \mathrm{Pic}^0(\mathcal{L}_n) \quad \textit{and} \quad \mathcal{S}^n_n = \kappa_n.$$

In other words, a spectrum $X \in \mathcal{L}^n_m$ is exotic invertible in $\mathcal{L}^n_m$ if and only if there exists an isomorphism $h^X_k \colon E(n)_*(V_k) \cong E(n)_*(X \wedge V_k)$ of $E(n)_*(E(n))$-comodules for each $V_k \in \mathcal{V}_m$ such that $(\tau_{k-1})_* h^X_k = h^X_{k-1}(\tau_{k-1})_*$.

Corollary D-2. *Let* $\ell^n_{i,m}\colon \mathrm{Pic}^0(\mathcal{L}^n_i) \to \mathrm{Pic}^0(\mathcal{L}^n_m)$ *for* $0 \le i \le m \le n$ *be the homomorphism defined by* $\ell^n_{i,m}(X) = L^n_m X$. *If* (C-II) *and* $(\text{C-III})_m$ *hold, then the homomorphism* $\ell^n_{i,m}$ *for* $0 \le i \le m \le n$ *is an epimorphism.*

We note that

(1.18) *If* $(\text{C-I})_m$ *(resp.* $(\text{C-III})_m$*) holds, then so does* $(\text{C-I})_i$ *(resp.* $(\text{C-III})_i$*) for each* i *with* $0 \le i \le m$.

Corollary D-3. *Let* m *be an integer with* $0 \le m \le n$, *and suppose* $(\text{C-I})_m$, (C-II) *and* $(\text{C-III})_m$. *Then, the functor* L^n_m *defines an isomorphism* $\mathrm{Pic}^0(\mathcal{L}_n) \xrightarrow{\cong} \mathrm{Pic}^0(\mathcal{L}^n_m)$. *Furthermore,* $\mathrm{Pic}^0(\mathcal{L}_n) \xrightarrow{\cong} \mathrm{Pic}^0(\mathcal{L}^n_i)$ *for* $i \le m$.

We also study a generalization of [11] (see Proposition F), and consider a condition:

$(\text{C-IV})_m$ For each spectrum $V_k \in \mathcal{V}_m$, the homotopy group $\pi_0(L_n V_k)$ is finite.

Since $\mathcal{V}_0 = \{S^0\}$, we set $(\text{C-IV})_0$ void.

Proposition E. *Let* m *be an integer with* $0 \le m \le n$ *and* $X \in \mathcal{S}^n_m$. *If* $(\text{C-IV})_m$ *holds and* $X \wedge V_k \simeq L_n V_k$ *for each* $V_k \in \mathcal{V}_m$, *then* $X \simeq L^n_m S^0$.

We fix a spectrum $V_k \in \mathcal{V}_m$ with $k \ge k_X$, in which k_X is the integer in Proposition 4.15. For each m with $1 \le m \le n$ and $s \ge 0$, consider the subsemigroups of $\mathcal{S}^n_m$:

(1.19) $$\mathcal{S}^{n,(s)}_m = \{X \in \mathcal{S}^n_m \mid d_r(1^X_{V_k}) = 0 \in E^{r,r-1}_r(X \wedge V_k) \text{ for } r < sq+1\}.$$

Here, $1^X_{V_k} \in E^{0,0}_2(X \wedge V_k)$ is the generator in (4.12). We notice the existence of an integer s_m such that $E^{rq+1,rq}_{rq+1}(V_k) = 0$ for $r \ge s_m$ and $V_k \in \mathcal{V}_m$ (see (4.14)). Then, Proposition E implies $\mathcal{S}^{n,(s_m)}_m = 0$ (*cf.* [11, Cor. 2.2]). The same argument as [11, §2] works to show the following:

Proposition F. *Let* $0 \le m \le n$. *If* $(\text{C-IV})_m$ *holds,* $\mathcal{S}^n_m$ *has a decreasing finite filtration*

$$\mathcal{S}^n_m = \mathcal{S}^{n,(0)}_m \supset \mathcal{S}^{n,(1)}_m \supset \cdots \supset \mathcal{S}^{n,(s_m-1)}_m \supset \mathcal{S}^{n,(s_m)}_m = 0$$

of subgroups with monomorphisms

$$\varphi_s\colon \mathcal{S}^{n,(s)}_m/\mathcal{S}^{n,(s+1)}_m \to E^{sq+1,sq}_{sq+1}(V_k)$$

for $s \ge 1$. *In particular,* $\mathcal{S}^n_m$ *is an abelian group if* $(\text{C-IV})_m$ *holds, and then* $\mathcal{S}^n_m = \mathrm{Pic}^0(\mathcal{L}^n_m)$.

This is a generalization of [11, Th. 1.2, Lemma 2.8], which is the case for $m = 0$.

The conditions (C-II) and $(\text{C-IV})_m$ are replaced by stronger conditions stated by the $E(n)$-based Adams spectral sequence:

Remark 1.20. The condition (C-II) (resp. $(\text{C-IV})_m$) holds if the E_2-term $E^{s,s-1}_2(S^0)$ (resp. $E^{s,s}_2(V_k)$) is finite for each $s > 0$.

The Picard group $\mathrm{Pic}^0(\mathcal{L}_n)$ is known in the following cases:

- ([9, Th. A, Th. 5.4] (*cf.* [11, Cor. 1.4.(a)])) $\mathrm{Pic}^0(\mathcal{L}_n) = 0$ for $n^2 + n \le q$ except for $(p, n) = (2, 1)$.
- ([3, Th. 1.2] (*cf.* [11, Cor. 1.4.(c)])) $\mathrm{Pic}^0(\mathcal{L}_2) = \mathbb{Z}/3 \oplus \mathbb{Z}/3$ for $(p, n) = (3, 2)$
- ([9, Th. 6.1] (*cf.* [11, Cor. 1.4.(b)])) $\mathrm{Pic}^0(\mathcal{L}_1) = \mathbb{Z}/2$ for $(p, n) = (2, 1)$

We notice that the condition $n^2+n<q$ in [11, Cor. 1.4.(a)] and [11, (1.3)(a)] may be replaced by $n^2+n\le q$ with $(p,n)\ne(2,1)$, since $\bigoplus_{r\ge1}E_{rq+1}^{rq+1,rq}(S^0)=0$ if $n^2+n\le q$ with $(p,n)\ne(2,1)$ by [20, (10.10)].

Theorem G. *In the above cases, the conditions* (C-I)$_m$*,* (C-II) *and* (C-III)$_m$ *hold. Furthermore,* (C-IV)$_m$ *holds.*

Corollary G-1.

1) If $n^2+n\le q$ and $(p,n)\ne(2,1)$, then $\mathrm{Pic}^0(\mathcal{L}_m^n)=0$ for $0\le m\le n$.
2) If $(p,n)=(3,2)$, then $\mathrm{Pic}^0(\mathcal{L}_2)\cong\mathrm{Pic}^0(\mathcal{L}_1^2)\cong\kappa_2$.
3) If $(p,n)=(2,1)$, then $\mathrm{Pic}^0(\mathcal{L}_1)\cong\kappa_1$.

We notice that $\mathrm{Pic}^0(\mathcal{L}_m^n)$ is the kernel of a homomorphism from $\mathrm{Pic}(\mathcal{L}_m^n)$ to an algebraic Picard group, and so the homomorphism is a monomorphism in the first case. Pstragowski [18] shows the monomorphism is an isomorphism for $\mathcal{L}_n^n=\mathcal{L}_{K(n)}$ with $q>n^2+n$.

This paper is organized as follows: In the next section, we study invertible spectra and show Theorem A. A converse of Theorem A is also studied under a stronger condition (see Proposition 2.6). In section three, we study the condition of $\mathcal{S}_m^n$ and set up Lemma 3.3, by which we show Proposition B, and also construct a map of geometric resolutions (*cf.* (4.6)) in Lemma 5.1.

In order to prove Theorem D, we construct an invertible spectrum of $\mathcal{L}_n$ by setting up an infinite tower. For this sake, we recall terminology, notions and results on invertible spectra and the E-based Adams spectral sequence for $E=v_n^{-1}BP$ from previous papers in section four. We also prove Theorem C and Proposition E in this section.

Over a map between geometric resolutions given in Lemma 5.1, we construct an infinite tower (*cf.* (4.17)) along with a map of towers, and then show Theorem D in section five. The last section is devoted to proving Theorem G.

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2. Invertible spectra in $\mathcal{L}_m^n$

In the following, we fix non-negative integers m and n with $0\le m\le n$. In this section, we characterize an invertible spectrum in $\mathcal{L}_m^n(\subset\mathcal{L}_n)$ by the $E(n)_*$-homology. Let thick $\langle L_nS^0\rangle$ denote the thick subcategory of $\mathcal{L}_n$ generated by L_nS^0.

Lemma 2.1. *Let $X\in\mathcal{L}_m^n$ and V be a type m finite spectrum. Then, X is strongly dualizable in $\mathcal{L}_m^n$ if and only if $X\wedge V\in$ thick $\langle L_nS^0\rangle$. In particular, for an invertible spectrum X of $\mathcal{L}_m^n$, $X\wedge V\in$ thick $\langle L_nS^0\rangle$.*

Proof. Since an invertible spectrum is strongly dualizable by [8, Prop. A.2.8], the latter statement follows from the former.

We turn to the former statement. Since $\mathcal{L}_n$ is a monogenic stable homotopy category, a spectrum $X\in\mathcal{L}_n$ is strongly dualizable if and only if $X\in$ thick $\langle L_nS^0\rangle$ (*cf.* [8, Th. 2.1.3]). Thus, it suffices to show that X is strongly dualizable in $\mathcal{L}_m^n$ if and only if $X\wedge V$ is strongly dualizable in $\mathcal{L}_n$.

Suppose X strongly dualizable in $\mathcal{L}_m^n$. Then, $D(X)\wedge_m^n U=F(X,U)$ for $U\in\mathcal{L}_m^n$, where $D(X)=F(X,S^0)$. For $W\in\mathcal{L}_n$, we compute

$$D(X\wedge V)\wedge W=D(V)\wedge D(X)\wedge_m^n W=D(V)\wedge F(X,L_m^nW)=F(X\wedge V,W)$$

in $\mathcal{L}_n$ by (1.6). Thus, $X \wedge V$ is strongly dualizable in $\mathcal{L}_n$.

Conversely, suppose $X \wedge V$ strongly dualizable in $\mathcal{L}_n$. Consider the natural map $\bar{c}\colon F(X, S^0) \wedge W \simeq F(X, S^0) \wedge F(S^0, W) \xrightarrow{\circ} F(X, W)$ for a spectrum $W \in \mathcal{L}_n$. Then, similarly as above, we see $\bar{c} \wedge D(V)$ to be an equivalence, and so is $L^n_m \bar{c}$ by (1.6). □

Proof of Theorem A. Let V be a generalized Moore spectrum of type m such that $BP_*(V) = BP_*/J_m$ for an invariant ideal $J_m = (p^{e_0}, v_1^{e_1}, \dots, v_{m-1}^{e_{m-1}})$. Then, for $m \le i \le n$, we have ideals $J_i = (p^{e_0}, v_1^{e_1}, \dots, v_{i-1}^{e_{i-1}})$ of BP_* and spectra MJ_i such that $BP_*(MJ_i) = BP_*/J_i$. By downword induction on i, we show the theorem for m. For $i = n$, it follows from (1.9).

In general, we verify easily the following:

(2.2) *Let M be a finitely generated $E(n)_*$-module. If $x \in M$ is infinitely divisible by an element $v \in \mathbb{Z}_{(p)}[v_1, \cdots, v_{n-1}] \subset E(n)_*$, then $x = 0$.*

Suppose that the theorem holds true for $i + 1 > m$. Let X be an invertible spectrum in $\mathcal{L}^n_m$. Then, $L^n_i X$ is an invertible spectrum in $\mathcal{L}^n_i$ and $X \wedge MJ_i = L^n_i X \wedge MJ_i$. For MJ_i,

$$E(n)_*(X \wedge MJ_i) \text{ is a finitely generated } E(n)_*\text{-module} \tag{2.3}$$

by Lemma 2.1. Consider the cofiber sequence

$$\Sigma^{|\mathfrak{v}_i|} MJ_i \xrightarrow{\mathfrak{v}_i} MJ_i \xrightarrow{\mathfrak{i}_i} MJ_{i+1} \xrightarrow{\mathfrak{j}_i} \Sigma^{|\mathfrak{v}_i|+1} MJ_i \tag{2.4}$$

for a map $\mathfrak{v}_i$ with $BP_*(\mathfrak{v}_i) = v_i^{e_i}$. Since $L^n_{i+1} X$ is invertible in $\mathcal{L}^n_{i+1}$, we have an isomorphism $\mathfrak{h}\colon E(n)_*(X \wedge MJ_{i+1}) \cong E(n)_{*+a}(MJ_{i+1})$ for an integer a by the inductive hypothesis. Note that the degree $|\mathfrak{v}_i|$ is a multiple of q. Apply $E(n)_t(X \wedge -)$ to the cofiber sequence (2.4) to obtain the exact sequence

$$\begin{aligned} E(n)_{t-|\mathfrak{v}_i|}(X \wedge MJ_i) &\xrightarrow{(\mathfrak{v}_i)_*} E(n)_t(X \wedge MJ_i) \\ &\xrightarrow{(\mathfrak{i}_i)_*} E(n)_t(X \wedge MJ_{i+1}) \xrightarrow{(\mathfrak{j}_i)_*} E(n)_{t-|\mathfrak{v}_i|-1}(X \wedge MJ_i). \end{aligned} \tag{2.5}$$

Since $E(n)_t(X \wedge MJ_{i+1}) \cong E(n)_{t+a}(MJ_{i+1}) = 0$ unless $q \mid (t + a)$, the self map $\mathfrak{v}_i$ induces an epimorphism $(\mathfrak{v}_i)_*\colon E(n)_{t-|\mathfrak{v}_i|}(X \wedge MJ_i) \to E(n)_t(X \wedge MJ_i)$ for t with $q \nmid (t + a)$. Then, by (2.2) with (2.3), $E(n)_t(X \wedge MJ_i) = 0$ unless $q \mid (t + a)$. It follows that

$$0 \to E(n)_{*-|\mathfrak{v}_i|}(X \wedge MJ_i) \xrightarrow{(\mathfrak{v}_i)_*} E(n)_*(X \wedge MJ_i) \xrightarrow{(\mathfrak{i}_i)_*} E(n)_*(X \wedge MJ_{i+1}) \to 0$$

is short exact. Thus, we obtain a generator $g \in E(n)_{-a}(X \wedge MJ_i)$ such that $(\mathfrak{i}_i)_*(g) = \mathfrak{h}^{-1}(1) \in E(n)_{-a}(X \wedge MJ_{i+1})$ for the generator $1 \in E(n)_0(MJ_{i+1})$. Since $E(n)_*(X \wedge MJ_i)$ is an $E(n)_*(MJ_i)$-module, we define a homomorphism $f\colon E(n)_*(MJ_i) \to E(n)_{*-a}(X \wedge MJ_i)$ by $f(1) = g$. Then it fits in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & E(n)_{*-|\mathfrak{v}_i|}(MJ_i) \arrow[r, "(\mathfrak{v}_i)_*"] \arrow[d, "f"'] & E(n)_*(MJ_i) \arrow[r, "(\mathfrak{i}_i)_*"] \arrow[d, "f"] & E(n)_*(MJ_{i+1}) \arrow[r] & 0 \\ 0 \arrow[r] & E(n)_{*-|\mathfrak{v}_i|-a}(X \wedge MJ_i) \arrow[r, "(\mathfrak{v}_i)_*"] & E(n)_{*-a}(X \wedge MJ_i) \arrow[r, "(\mathfrak{i}_i)_*"] & E(n)_{*-a}(X \wedge MJ_{i+1}) \arrow[r] \arrow[u, "\cong"', "\mathfrak{h}"] & 0 \end{tikzcd}$$

of short exact sequences. Since $E(n)_*$ is noetherian, the kernel of f is a finitely generated $E(n)_*$-module. Moreover, the cokernel of f is also finitely generated by

(2.3). Therefore, the snake lemma together with (2.2) shows f to be an isomorphism. □

With an additional condition, we obtain a converse of Theorem A:

Proposition 2.6. *Suppose that a spectrum $X \in \mathcal{L}_m^n$ is strongly dualizable and there is a generalized Moore spectrum V of type m such that $E(n)_*(X \wedge V) \cong E(n)_*(V)$ and $E(n)_*(D(X) \wedge V) \cong E(n)_*(V)$ as an $E(n)_*$-module. Then, X is an invertible spectrum in $\mathcal{L}_m^n$. Its inverse is $L_m^n D(X)$.*

Proof. Consider the cofiber sequence

$$D(X) \wedge X \xrightarrow{\varepsilon} L_n S^0 \xrightarrow{c} C \tag{2.7}$$

for the evaluation map ε, and a commutative diagram

$$\begin{array}{ccccc}
[D(X) \wedge X, L_n S^0]_0 & \xrightarrow{ad} & [D(X), D(X)]_0 & \xrightarrow{(-\wedge V)_*} & [D(X) \wedge V, D(X) \wedge V]_0 \\
\downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle F(X,c)_*} & & \downarrow{\scriptstyle (1\wedge c\wedge 1)_*} \\
[D(X) \wedge X, C]_0 & \xrightarrow{ad} & [D(X), F(X,C)]_0 & \xrightarrow{(-\wedge V)_*} & [D(X) \wedge V, D(X) \wedge C \wedge V]_0
\end{array}$$

in which c is a map of (2.7), and ad denotes an adjunction. Here, $D(X) \wedge C \wedge V = F(X, C) \wedge V$ by (1.6), since X is strongly dualizable. We see that $D(X) \wedge c \wedge V = (1 \wedge c \wedge 1)_*(- \wedge V)_*(ad(\varepsilon)) = (- \wedge V)_*(ad(c_*(\varepsilon))) = 0$, since $ad(\varepsilon) = id_{D(X)}$ and $c\varepsilon = 0$. It follows that the cofiber sequence $D(X)\wedge(2.7)\wedge V$ give rise to a decomposition

$$D(X) \wedge D(X) \wedge X \wedge V \simeq (D(X) \wedge V) \vee \left(\Sigma^{-1} D(X) \wedge C \wedge V\right). \tag{2.8}$$

By the hypothesis, we have equivalences $E(n) \wedge X \wedge V \simeq E(n) \wedge V$ and $E(n) \wedge D(X) \wedge V \simeq E(n) \wedge V$ up to suspension, and so

$$E(n) \wedge D(X) \wedge D(X) \wedge X \wedge V \simeq E(n) \wedge D(X) \wedge X \wedge V \simeq E(n) \wedge X \wedge V \simeq E(n) \wedge V$$

up to suspension. Apply $E(n)_*(-)$ to (2.8), and we have an epimorphism $E(n)_*/J \cong E(n)_*(D(X) \wedge D(X) \wedge X \wedge V) \to E(n)_*(D(X) \wedge V) \cong E(n)_*/J$ for the ideal J such that $E(n)_*(V) \cong E(n)_*/J$. By Nakayama's Lemma (*cf.* [13, Th. 2.4]), the epimorphism is an isomorphism, and so we obtain $E(n)_*(C \wedge V) = E(n)_*(D(X) \wedge C \wedge V) = 0$ by (2.8). Thus, C is $E(n) \wedge V$-acyclic, and hence $L_m^n C$ is trivial by (1.5). Thus the evaluation map ε induces the desired equivalence $D(X) \wedge_m^n X \underset{\simeq}{\xrightarrow{L_m^n \varepsilon}} L_m^n S^0$. □

3. $\mathrm{Pic}^0(\mathcal{L}_m^n)$ is a subgroup of $\mathrm{Pic}(\mathcal{L}_m^n)$

In this section, we give a paraphrase of the condition $E(n)_*(V) \cong_{\mathcal{C}(n)} E(n)_*(X \wedge V)$ on $\mathcal{S}_m^n$ in Lemma 3.3 by using $E = v_n^{-1}BP$ instead of $E(n)$, and verify that $\mathcal{S}_m^n$ is depends only on the integers m and n, and that $\mathrm{Pic}^0(\mathcal{L}_m^n)$ is a subgroup of $\mathrm{Pic}(\mathcal{L}_m^n)$, which is the claim of Proposition B. We also use Lemma 3.3 in section five to construct a map between geometric resolutions (Lemma 5.1).

Let E denote the ring spectrum $v_n^{-1}BP$ for a fixed integer $n \geq 0$. Then, we obtain a Hopf algebroid

$$(E_*, E_*(E)) = (v_n^{-1}BP_*, E_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E_*),$$

which inherits the Hopf algebroid structure from the well known Hopf algebroid

$$(BP_*, BP_*(BP)) = (\mathbb{Z}_{(p)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots]).$$

We write the multiplication and the unit of the ring spectrum E as

$$\mu_E\colon E\wedge E\to E \quad\text{and}\quad i_E\colon S^0\to E. \tag{3.1}$$

Since the category of $E(n)_*(E(n))$-comodules is equivalent to the category of $E_*(E)$-comodules by [10, Th. 4.2], the isomorphisms h_k^X in the definition of $\mathcal{S}_m^n$ is replaced by the isomorphisms

$$\widetilde{h}_k^X\colon E_*(V_k)\cong E_*(X\wedge V_k) \tag{3.2}$$

of $E_*(E)$-comodules satisfying $(\tau_{k-1})_*\widetilde{h}_k^X=\widetilde{h}_{k-1}^X(\tau_{k-1})_*$. Hereafter, we consider every $X\in\mathcal{S}_m^n$ to be a spectrum satisfying this.

Lemma 3.3. *For $X\in\mathcal{S}_m^n$, there exists a map $\widetilde{h}_X\colon S^0\to E\wedge_m^n X$ such that the induced map*

$$\widehat{h}_X\colon E\xrightarrow{E\wedge\widetilde{h}_X} E\wedge E\wedge_m^n X\xrightarrow{\eta_m^n} L_m^n(E\wedge E\wedge X)\xrightarrow{L_m^n(\mu_E\wedge X)} E\wedge_m^n X \tag{3.4}$$

for the map μ_E in (3.1) satisfies

1) *$\pi_*(\widehat{h}_X\wedge V_k)\colon E_*(V_k)\to E_*(X\wedge V_k)$ for $V_k\in\mathcal{V}_m$ is the isomorphism $\widetilde{h}_k^X$ of $E_*(E)$-comodules in (3.2), and*
2) *$\widehat{h}_X$ sits in the commutative diagram*

$$\begin{tikzcd} E\wedge V_k \arrow[r, "\widehat{h}_X\wedge V_k"] \arrow[d, "E\wedge i_E\wedge V_k"'] & E\wedge X\wedge V_k \arrow[d, "E\wedge i_E\wedge X\wedge V_k"] \\ E\wedge E\wedge V_k \arrow[r, "E\wedge\widehat{h}_X\wedge V_k"] & E\wedge E\wedge X\wedge V_k \end{tikzcd}$$

for $V_k\in\mathcal{V}_m$ and the map i_E in (3.1). Here, note that $E\wedge_m^n X\wedge V_k\simeq E\wedge X\wedge V_k$ by (1.6).

Proof. Let $X\in\mathcal{S}_m^n$. The limit of the isomorphisms $\{\widetilde{h}_k^X\}_k$ gives rise to an isomorphism

$$\widetilde{h}_*^X\colon \lim_{V\in\mathcal{V}_m} E_*(V)\cong\lim_{V\in\mathcal{V}_m} E_*(X\wedge V). \tag{3.5}$$

We begin with defining a map $\widetilde{h}_X\colon S^0\to E\wedge_m^n X$ such that $\pi_*(\widehat{h}_X\wedge V_k)=\widetilde{h}_k^X$. By (1.4) and (1.11), the Milnor sequence admits an epimorphism

$$\mathfrak{p}^Y\colon\pi_*(E\wedge_m^n Y)\to\lim_{V\in\mathcal{V}_m} E_*(Y\wedge V)$$

for a spectrum Y, and we obtain a commutative diagram

$$\begin{tikzcd} \pi_*(E\wedge_m^n X) \arrow[r, two heads, "\mathfrak{p}^X"] \arrow[d, "(i_k)_*"'] & \lim_{V\in\mathcal{V}_m} E_*(X\wedge V) \arrow[d, "(i_k)_*"] \\ \pi_*(E\wedge_m^n X\wedge V_k) \arrow[r, "\mathfrak{p}^{X\wedge V_k}", "="'] & \lim_{V\in\mathcal{V}_m} E_*(X\wedge V\wedge V_k)=E_*(X\wedge V_k) \end{tikzcd}$$

for the inclusion $i_k\colon S^0\to V_k$ in (1.10) 3). For the generators

$$\widetilde{1}_k=(i_E\wedge i_k)\in E_0(V_k),$$

we have an element $(\widetilde{1}_k)_k\in\lim_{V\in\mathcal{V}_m}E_*(V)$. Let

$$\widetilde{h}_X\colon S^0\to E\wedge_m^n X\in\pi_*(E\wedge_m^n X)$$

be a map such that $\mathfrak{p}^X(\widetilde{h}_X) = \widetilde{h}^X_*((\widetilde{1}_k)_k) \in \lim_{V\in\mathcal{V}_m} E_*(X\wedge V)$ for $\widetilde{h}^X_*$ in (3.5). Note that $\widetilde{h}^X_*((\widetilde{1}_k)_k) = (\widetilde{h}^X_k(\widetilde{1}_k))_k$ by definition. Then,

$$\widetilde{h}^X_k(\widetilde{1}_k) = (i_k)_*\mathfrak{p}^X(\widetilde{h}_X) = \mathfrak{p}^{X\wedge V_k}(i_k)_*(\widetilde{h}_X) = \widetilde{h}_X\wedge i_k \in E_0(X\wedge V_k). \tag{3.6}$$

On the other hand, the induced homomorphism $\pi_*(\widehat{h}_X\wedge V_k)\colon E_*(V_k)\to E_*(X\wedge V_k)$ acts on the generator $\widetilde{1}_k\in E_0(V_k)$ by

$$\begin{aligned}\pi_*(\widehat{h}_X\wedge V_k)(\widetilde{1}_k) &= (\mu_E\wedge X\wedge V_k)(E\wedge\widetilde{h}_X\wedge V_k)(i_E\wedge i_k)\\ &= (\mu_E\wedge X\wedge V_k)(i_E\wedge E\wedge X\wedge V_k)(\widetilde{h}_X\wedge i_k) = \widetilde{h}_X\wedge i_k \underset{(3.6)}{=} \widetilde{h}^X_k(\widetilde{1}_k).\end{aligned} \tag{3.7}$$

Since $E_*(V_k)$ is a monogenic E_*-module, we see $\pi_*(\widehat{h}_X\wedge V_k) = \widetilde{h}^X_k$, which implies (3.4) 1).

Next, we turn to show the commutativity of the diagram in (3.4) 2). The $E_*(E)$-comodule structure $\psi_W\colon E_*(W)\to E_*(E)\otimes_{E_*}E_*(W)$ on $E_*(W)$ for a spectrum W is given by the composite $(\mu_E)_*^{-1}E_*(i_E\wedge W)$, where the isomorphism $(\mu_E)_*\colon E_*(E)\otimes_{E_*}E_*(W)\to E_*(E\wedge W)$ is given by $(\mu_E)_*(x\otimes y) = (E\wedge\mu_E\wedge W)(x\wedge y)$. Consider the diagram

$$\begin{tikzcd}
 & E_*(V_k) \arrow[dl, "E_*(i_E\wedge 1)"'] \arrow[d, "\psi_{V_k}"] \arrow[rr, "\widetilde{h}^X\wedge V_k"] & & E_*(X\wedge V_k) \arrow[d, "\psi_{X\wedge V_k}"] \arrow[dr, "E_*(i_E\wedge 1)_*"] & \\
E_*(E\wedge V_k) \arrow[rrrr, bend right, "E_*(\widehat{h}_X\wedge 1)"'] & E_*(E)\otimes_{E_*}E_*(V_k) \arrow[l, "\mu_*", "\simeq"'] \arrow[rr, "1\otimes\widetilde{h}^X_k"] & & E_*(E)\otimes_{E_*}E_*(X\wedge V_k) \arrow[r, "\mu_*", "\simeq"'] & E_*(E\wedge X\wedge V_k).
\end{tikzcd}$$

Here, μ_* denotes $(\mu_E)_*$. We begin with showing the diagram to be commutative, in other words,

$$E_*(i_E\wedge X\wedge V_k)\widetilde{h}^X_k = E_*(\widehat{h}_X\wedge V_k)E_*(i_E\wedge V_k). \tag{3.8}$$

Since $\widetilde{h}^X_k$ is a homomorphism of comodules, the middle rectangle commutes. The triangles on both sides commute by definition. Thus, it suffices to verify

$$(\mu_E)_*(1\otimes\widetilde{h}^X_k) = E_*(\widehat{h}_X\wedge V_k)(\mu_E)_*. \tag{3.9}$$

For $x\otimes y\in E_*(E)\otimes_{E_*}E_*(V_k)$, we compute, by 1) of the lemma,

$$\begin{aligned}(\mu_E)_*(1\otimes\widetilde{h}^X_k)(x\otimes y) &= (\mu_E)_*(x\otimes\widetilde{h}^X_k(y)) \underset{1)}{=} (\mu_E)_*(x\otimes(\widehat{h}_X\wedge V_k)y)\\ &= (E\wedge\mu_E\wedge X\wedge V_k)(E\wedge E\wedge\widehat{h}_X\wedge V_k)(x\wedge y) \quad\text{and}\end{aligned}$$

$$E_*(\widehat{h}_X\wedge V_k)(\mu_E)_*(x\otimes y) = (E\wedge\widehat{h}_X\wedge V_k)(E\wedge\mu_E\wedge V_k)(x\wedge y).$$

Both of the right hand sides of the above equalities agree by the commutative diagram

$$\begin{tikzcd}
S \arrow[r, "x\wedge y"] & E\wedge E\wedge E\wedge V_k \arrow[r, "1^3\wedge\widetilde{h}_X\wedge 1"'] \arrow[d, "1\wedge\mu_E\wedge 1"] \arrow[rr, bend left, "1^2\wedge\widehat{h}_X\wedge 1"] & E\wedge E\wedge E\wedge E\wedge X\wedge V_k \arrow[r, "1^2\wedge\mu_E\wedge 1\wedge 1"'] \arrow[d, "1\wedge\mu_E\wedge 1\wedge 1\wedge 1"'] & E\wedge E\wedge E\wedge X\wedge V_k \arrow[d, "1\wedge\mu_E\wedge 1\wedge 1"] \\
 & E\wedge E\wedge V_k \arrow[r, "1^2\wedge\widetilde{h}_X\wedge 1"] \arrow[rr, bend right, "1\wedge\widehat{h}_X\wedge 1"'] & E\wedge E\wedge E\wedge X\wedge V_k \arrow[r, "1\wedge\mu_E\wedge 1\wedge 1"] & E\wedge E\wedge X\wedge V_k.
\end{tikzcd}$$

Thus, the equality (3.9) holds, and the relation (3.8) follows.

To see the diagram (3.4) 2) commutative, we verify $x = y$ for $x = (E \wedge i_E \wedge X \wedge V_k)(\widehat{h}_X \wedge V_k)$ and $y = (E \wedge \widehat{h}_X \wedge V_k)(E \wedge i_E \wedge V_k)$ in $[E \wedge V_k, E \wedge E \wedge X \wedge V_k]_0$. The homomorphism

$$(\widetilde{1}_k)^* \colon [E \wedge V_k, E \wedge E \wedge X \wedge V_k]_0 \to \pi_0(E \wedge E \wedge X \wedge V_k).$$

induced from the generator $\widetilde{1}_k \in E_0(V_k) = \pi_0(E \wedge V_k)$ acts on the elements by

$$(\widetilde{1}_k)^*(x) \underset{1)}{=} E_*(i_E \wedge 1)_* \widetilde{h}_k^X(\widetilde{1}_k) \underset{(3.8)}{=} E_*(\widehat{h}_X \wedge V_k) E_*(i_E \wedge V_k)(\widetilde{1}_k) = (\widetilde{1}_k)^*(y).$$

We verify easily that x and y are $E \wedge V_k$-module maps, and obtain $x = y$ from [17, Lemma 1.3]. □

Proof of Proposition B. Let $\mathcal{S}(\mathcal{V}_m)$ for a sequence $\mathcal{V}_m$ of (1.10) denote the collection given in (1.13). Let $\mathcal{V}_m$ and $\mathcal{V}'_m$ be sequences given in (1.10). For $X \in \mathcal{S}(\mathcal{V}'_m)$, there exists $\widehat{h}_X \colon E \to E \wedge X$ inducing an isomorphism $E_*(V') \cong E_*(X \wedge V')$ for $V' \in \mathcal{V}'_m$ of $E_*(E)$-comodules by Lemma 3.3. Therefore, $\widehat{h}_X$ is a V'-equivalence. Note that $\langle V' \rangle = \langle V \rangle$ for any type m finite spectrum V. It follows that $\widehat{h}_X \colon E \to E \wedge X$ induces an isomorphism $(\widehat{h}_X \wedge V)_* \colon E_*(V) \cong E_*(X \wedge V)$ for $V \in \mathcal{V}_m$. For each $V \in \mathcal{V}_m$, there exist a spectrum $V' \in \mathcal{V}'_m$ and a map $\tau \colon V' \to V$ inducing a canonical projection $(E \wedge \tau)_* \colon E_*(V') \to E_*(V)$ of comodules. Consider the diagram

$$\begin{tikzcd}
E_*(V') \arrow[r, twoheadrightarrow, "(E\wedge\tau)_*"] \arrow[d, "(\widehat{h}_X\wedge V')_*"', "\cong"] & E_*(V) \arrow[d, "(\widehat{h}_X\wedge V)_*"] \\
E_*(X \wedge V') \arrow[r, twoheadrightarrow, "(E\wedge X\wedge\tau)_*"'] & E_*(X \wedge V).
\end{tikzcd}$$

Since the left vertical arrow is an isomorphism of $E_*(E)$-comodules, so is the right vertical arrow. It follows that $\mathcal{S}(\mathcal{V}'_m) \subset \mathcal{S}(\mathcal{V}_m)$. Exchange $\mathcal{V}_m$ and $\mathcal{V}'_m$, and we see the converse.

We set out to verify the claim

(3.10) *The collection $\mathcal{S}^n_m$ is closed under the smash product $\wedge^n_m$. That is, for $X, Y \in \mathcal{S}^n_m$, $X \wedge^n_m Y \in \mathcal{S}^n_m$.*

Let $\widehat{h}_X$ and $\widehat{h}_Y$ be the maps in Lemma 3.3. Then, the composite $E \wedge V_k \xrightarrow{\widehat{h}_Y \wedge V_k} E \wedge^n_m Y \wedge V_k \xrightarrow{\widehat{h}_X \wedge Y \wedge V_k} E \wedge X \wedge^n_m Y \wedge V_k$ induces an isomorphism $\widetilde{h}_k^{X \wedge^n_m Y} \colon E_*(V_k) \cong E_*(X \wedge^n_m Y \wedge V_k)$ of the comodules by Lemma 3.3. Furthermore, the relation $(\tau_{k-1})_* \widetilde{h}_k^{X \wedge^n_m Y} = \widetilde{h}_{k-1}^{X \wedge^n_m Y}(\tau_{k-1})_*$ follows trivially.

Next we show $D(X) \in \mathcal{S}^n_m$ if $X \in \operatorname{Pic}^0(\mathcal{L}^n_m)$. Since X is invertible in $\mathcal{L}^n_m$, we have an equivalence $\varepsilon \colon D(X) \wedge X \to S^0$. Then, Lemma 3.3 2) yields a commutative diagram

$$\begin{tikzcd}
E \wedge D(X) \wedge V_k \arrow[r, "\widehat{h}_X\wedge 1\wedge 1"] \arrow[d, "1\wedge i_E\wedge 1\wedge 1"'] & E \wedge X \wedge D(X) \wedge V_k \arrow[r, "1\wedge\varepsilon\wedge 1", "\simeq"'] \arrow[d, "1\wedge i_E\wedge 1\wedge 1"] & E \wedge V_k \arrow[d, "1\wedge i_E\wedge 1"] \\
E \wedge E \wedge D(X) \wedge V_k \arrow[r, "1\wedge\widehat{h}_X\wedge 1\wedge 1"'] & E \wedge E \wedge X \wedge D(X) \wedge V_k \arrow[r, "\simeq", "1\wedge 1\wedge\varepsilon\wedge 1"'] & E \wedge E \wedge V_k
\end{tikzcd}$$

for $V_k \in \mathcal{V}_m$. The upper composite gives rise to an isomorphism $\widetilde{h}_k^{D(X)}$ of comodules satisfying $(\tau_{k-1})_* \widetilde{h}_k^{D(X)} = \widetilde{h}_{k-1}^{D(X)}(\tau_{k-1})_*$. □

4. Recollections on Adams towers

In this section, we recollect some notation and some facts from [11, §§3-4] and [25] on a geometric resolution (4.6) and an s-tower $(4.20)_s$ relevant to the E-based Adams tower, in order to construct an invertible spectrum in $\mathcal{L}_n$ by [25, Prop. 2.13] (see (4.19)). Under the notation, we also show Theorem C and Proposition E (or Proposition 4.16).

As the previous section, E denotes the ring spectrum $v_n^{-1}BP$ for a fixed integer $n \geq 0$. The unit map i_E in (3.1) induces the cofiber sequence

$$S^0 \xrightarrow{i_E} E \xrightarrow{j_E} \overline{E} \xrightarrow{k_E} S^1. \tag{4.1}$$

Lemma 4.2. *Let $V_k \in \mathcal{V}_m$. Suppose that there exists a map $h\colon S^0 \to W$ for a spectrum W inducing an epimorphism (resp. monomorphism) $h_*\colon E_* \to E_*(W)$. Then, it induces an epimorphism (resp. monomorphism) $h_*\colon E_*(\overline{E}^s) \to E_*(\overline{E}^s \wedge W)$ for $s \geq 0$. Here, $\overline{E}^s$ denotes the s-fold smash product $\overline{E} \wedge \ldots \wedge \overline{E}$ of the spectrum $\overline{E}$ in* (4.1).

Proof. The map h induces a commutative diagram

$$\begin{tikzcd}
E_*(\overline{E}^s) \arrow[r, shift left, "E_*(i_E\wedge 1)"] \arrow[d, "h_*"'] & E_*(E\wedge \overline{E}^s) \arrow[l, shift left, "(\mu_E\wedge 1)_*"] \arrow[r, "E_*(j_E\wedge 1)"] \arrow[d, "h_*"] & E_*(\overline{E}^{s+1}) \arrow[r] \arrow[d, "h_*"] & 0 \\
E_*(\overline{E}^s\wedge W) \arrow[r, shift left, "E_*(i_E\wedge 1)"] & E_*(E\wedge \overline{E}^s\wedge W) \arrow[l, shift left, "(\mu_E\wedge 1)_*"] \arrow[r, "E_*(j_E\wedge 1)"] & E_*(\overline{E}^{s+1}\wedge W) \arrow[r] & 0
\end{tikzcd} \tag{4.3}$$

of split exact sequences. Since $E_*(E)$ is flat over E_* (*cf.* [14, Remark 3.7]), we have a natural isomorphism $E_*(E \wedge U) \cong E_*(E) \otimes_{E_*} E_*(U)$ for a spectrum U, and a commutative diagram

$$\begin{tikzcd}
E_*(E)\otimes_{E_*} E_*(\overline{E}^s) \arrow[r, phantom, "\cong"] \arrow[d, "1\otimes h_*"'] & E_*(E\wedge \overline{E}^s) \arrow[d, "h_*"] \\
E_*(E)\otimes_{E_*} E_*(\overline{E}^s\wedge W) \arrow[r, phantom, "\cong"] & E_*(E\wedge \overline{E}^s\wedge W).
\end{tikzcd}$$

Therefore, the middle h_* in the diagram (4.3) is an epimorphism (resp. monomorphism) if so is the left h_*. Thus, the lemma follows from the diagram (4.3) by induction. □

The cofiber sequence (4.1) yields the E-based Adams tower

$$\begin{tikzcd}
S^0 \arrow[d, "i_E"'] & \overline{E} \arrow[l, dotted, "k_E"'] \arrow[d, "1\wedge i_E"] & \cdots \arrow[l, dotted, "1\wedge k_E"'] & \overline{E}^s \arrow[l, dotted, "1\wedge k_E"'] \arrow[d, "1\wedge i_E"] & \overline{E}^{s+1} \arrow[l, dotted, "1\wedge k_E"'] \arrow[d, "1\wedge i_E"] & \cdots \arrow[l, dotted, "1\wedge k_E"'] \\
E \arrow[ur, "j_E"] \arrow[r, "d^0"'] & \overline{E}\wedge E \arrow[r, "d^1"'] & \cdots \arrow[r, "d^{s-1}"'] & \overline{E}^s\wedge E \arrow[ur, "1\wedge j_E"] \arrow[r, "d^s"'] & \overline{E}^{s+1}\wedge E \arrow[r, "d^{s+1}"'] & \cdots
\end{tikzcd}$$

in which dotted arrows denote degree -1 maps, and

$$d^s = (\overline{E}^s \wedge d)\colon \overline{E}^s \wedge E \to \overline{E}^{s+1} \wedge E \tag{4.4}$$

for

$$d = d^0 = j_E \wedge i_E\colon E = E \wedge S^0 \to \overline{E} \wedge E. \tag{4.5}$$

We call the sequence

$$
\begin{aligned}
(4.6)\qquad E\wedge W \xrightarrow{d^0\wedge W} \overline{E}\wedge E\wedge W \xrightarrow{d^1\wedge W} \cdots \xrightarrow{d^{s-1}\wedge W} \overline{E}^s\wedge E\wedge W \\
\xrightarrow{d^s\wedge W} \overline{E}^{s+1}\wedge E\wedge W \xrightarrow{d^{s+1}\wedge W} \cdots
\end{aligned}
$$

for a spectrum W obtained from the bottom sequence of the above diagram the *geometric resolution* of W. Let $k^s\colon \overline{E}^s \to S^s$ denote the composite $k_E(\overline{E}\wedge k_E)\cdots(\overline{E}^{s-1}\wedge k_E)$ of the upper sequence in the tower, and let $\overline{E}_s$ denote a fiber of k^s sitting in the cofiber sequence

$$
(4.7)\qquad \overline{E}^s \xrightarrow{k^s} S^s \xrightarrow{\hat{i}_s} \Sigma\overline{E}_s \xrightarrow{\hat{j}_s} \Sigma\overline{E}^s
$$

for each $s\geq 1$. Note that

$$
\hat{i}_1 = i_E\colon S^0 \to \overline{E}_1 = E.
$$

This gives rise to a commutative diagram

$$
(4.8)\qquad
\begin{tikzcd}
S^s \arrow[r, equal] \arrow[d, "\hat{i}_{s+1}"'] & S^s \arrow[r] \arrow[d, "\hat{i}_s"] & 0 \arrow[r] \arrow[d] & S^{s+1} \arrow[d, "\hat{i}_{s+1}"] \\
\overline{E}_{s+1} \arrow[r, "k_s^S"] \arrow[d, "\hat{j}_{s+1}"'] & \Sigma\overline{E}_s \arrow[r, "i_s^S"] \arrow[d, "\hat{j}_s"] & \Sigma E\wedge\overline{E}^s \arrow[r, "j_s^S"] \arrow[d, equal] & \Sigma\overline{E}_{s+1} \arrow[d, "\hat{j}_{s+1}"] \\
\overline{E}^{s+1} \arrow[r, "1\wedge k_E"] & \Sigma\overline{E}^s \arrow[r, "1\wedge i_E"] & \Sigma E\wedge\overline{E}^s \arrow[r, "1\wedge j_E"] & \Sigma\overline{E}^{s+1}
\end{tikzcd}
$$

in which rows and columns are cofiber sequences. The middle row of the diagram (4.8) yields another E-based Adams tower

$$
(4.9)\qquad
\begin{tikzcd}
0 \arrow[d] & \overline{E}_1 \arrow[l, dotted] \arrow[d, "d=i_1^S"] & \cdots \arrow[l, dotted, "k_1^S"'] & \overline{E}_s \arrow[l, dotted, "k_{s-1}^S"'] \arrow[d, "i_s^S"] & \overline{E}_{s+1} \arrow[l, dotted, "k_s^S"'] \arrow[d, "i_{s+1}^S"] & \cdots \arrow[l, dotted, "k_{s+1}^S"'] \\
E \arrow[r, "d^0"'] \arrow[ur, equal] & \overline{E}\wedge E \arrow[r, "d^1"'] & \cdots \arrow[r, "d^{s-1}"'] & \overline{E}^s\wedge E \arrow[r, "d^s"'] \arrow[ur, "j_s^S"] & \overline{E}^{s+1}\wedge E \arrow[r, "d^{s+1}"'] & \cdots
\end{tikzcd}
$$

for d^s in (4.4). We notice that homotopy groups of the smash product of the tower and a spectrum W define an exact couple, which yields the E-based Adams spectral sequence

$$
(4.10)\qquad E_2^{s,t}(W) = \operatorname{Ext}^{s,t}_{E_*(E)}(E_*, E_*(W)) \Longrightarrow \pi_{t-s}(L_nW),
$$

where $\operatorname{holim}_s(\Sigma^{1-s}\overline{E}_s\wedge W) = L_nW$. We notice that the canonical map $E\to E(n)$ inducing the projection $E_*\to E(n)_*$ gives rise to an isomorphism of the spectral sequences (1.17) and (4.10). Indeed, we have the isomorphism of the E_2-terms:

(4.11) ([11, Th. 3.3], [10, Cor. 4.8])

$$
\operatorname{Ext}^{*,*}_{E_*(E)}(E_*, M) \cong \operatorname{Ext}^{*,*}_{E(n)_*(E(n))}(E(n)_*, E(n)_*\otimes_{E_*} M)
$$

for an $E_*(E)$-comodule M, on which v_n acts isomorphically.

Let V denote a generalized Moore spectrum of type m with $0\leq m\leq n$. The generator $1_V^X\in E_0(X\wedge V)$ is also the generator

$$
(4.12)\qquad 1_V^X \in E_2^{0,0}(X\wedge V) \quad (\subset E_0(X\wedge V)).
$$

The generator plays a key role in the proof of Theorem C, the definition of integers k_X and r_X in Proposition 4.15 and the definition of the subsemigroups $\mathcal{S}_m^{n,(s)}$ in (1.19).

Note that $E_2^{s,t}(V) = 0$ unless $q \mid t$, which implies

$$E_s^{*,*}(V) = E_{rq+1}^{*,*}(V) \quad \text{for } r \geq 1 \text{ and } (r-1)q+1 < s \leq rq+1. \tag{4.13}$$

Moreover, there exists an integer s_m such that

(4.14) (*cf.* [21, Th. 8.2.6]) $\qquad E_{s_m q+1}^{s,*}(V) = 0 \quad \text{if } s \geq s_m q + 1.$

We notice that s_m depends only on m, and is independent of the choice of V.

In the following, we write $d_r(x)$ for $x \in E_2^{s,t}(W)$ without mentioning $d_s(x) = 0$ for $s < r$.

Proof of Theorem C. Suppose that $L_m^n X = L_m^n S^0$ for $X \in \mathrm{Pic}^0(\mathcal{L}_n)$. Then, for the spectrum V of (C-I)$_m$, $X \wedge V \simeq L_m^n X \wedge V \simeq L_n V$ by (1.6). It follows that the element 1_V^X in (4.12) is a permanent cycle. If the generator $1^X \in E_2^{0,0}(X)$ survives to E_{rq+1}-term $E_{rq+1}^{0,0}(X)$, then $(i_V)_*(d_{rq+1}(1^X)) = d_{rq+1}((i_V)_*(1^X)) = d_{rq+1}(1_V^X) = 0 \in E_{rq+1}^{rq+1,rq}(X \wedge V)$. By the hypothesis (C-I)$_m$, we obtain $d_{rq+1}(1^X) = 0$, and $1^X \in E_{(r+1)q+1}^{0,0}(X)$. Thus, we deduce inductively that 1^X is a permanent cycle. An element $i^X \in \pi_0(X)$ detecting 1^X yields the desired equivalence $i^X \colon L_n S^0 \simeq X$. □

In the following, 1_k^X denotes $1_{V_k}^X$ in (4.12) for $V_k \in \mathcal{V}_m$.

Proposition 4.15. *For each $X \in \mathcal{S}_m^n$, there exist integers k_X and r_X such that*

$$d_{r_X q+1}(1_k^X) \neq 0 \in E_{r_X q+1}^{r_X q+1, r_X q}(X \wedge V_k)$$

for $V_k \in \mathcal{V}_m$ with $k \geq k_X$ unless 1_k^X are permanent cycles for all $k \geq 1$ (we set $k_X = 1$ and $r_X = \infty$ in this case).

Proof. Suppose that $1_\ell^X \in E_2^{0,0}(X \wedge V_\ell)$ for some integer ℓ is not a permanent cycle. Then, there is an integer r such that $d_{rq+1}(1_\ell^X) \neq 0$. By the naturality of the differentials of the spectral sequences, we deduce that for every integer $k > \ell$, there exists an integer $s \leq r$ such that $d_{sq+1}(1_k^X) \neq 0$. This shows the existence of the integers k_X and r_X. □

The following proposition is a restatement of Proposition E:

Proposition 4.16. *Let $X \in \mathcal{S}_m^n$ and r_X be the integer given in Proposition 4.15. If (C-IV)$_m$ holds and $r_X = \infty$, then $X \simeq L_m^n S^0$.*

Proof. Put $U_k = i_k + K_k \subset \pi_0(L_n V_k)$ for $K_k = \mathrm{Ker}\,((i_E)_* \colon \pi_0(L_n V_k) \to E_0(V_k))$. Here, $i_k \in \pi_0(L_n V_k)$ denotes the element corresponding to the inclusion $i_k \colon S^0 \to V_k$ in (1.10) 3). Since $r_X = \infty$, 1_k^X is a permanent cycle, and then an element $i_k^X \colon S^0 \to X \wedge V_k$ detected by 1_k^X induces an equivalence $e_k^X \colon L_n V_k \simeq X \wedge V_k$. Indeed, e_k^X induces the isomorphism $\widetilde{h}_k^X$ in (3.2). Let $\sigma_k \colon L_n V_{k+1} \to L_n V_k$ be the map fitting in the commutative diagram

$$\begin{array}{ccc} L_n V_{k+1} & \xrightarrow[\simeq]{e_{k+1}^X} & X \wedge V_{k+1} \\ {\scriptstyle \sigma_k}\downarrow & & \downarrow{\scriptstyle 1 \wedge \tau_k} \\ L_n V_k & \xrightarrow[\simeq]{e_k^X} & X \wedge V_k. \end{array}$$

Since $X \in \mathcal{S}^n_m$, this induces a commutative diagram

$$\begin{tikzcd} E_*(V_{k+1}) \arrow[r, "\widetilde{h}^X_{k+1}"] \arrow[d, "(\sigma_k)_*"'] & E_*(X \wedge V_{k+1}) \arrow[d, "(\tau_k)_*"] & E_*(V_{k+1}) \arrow[l, "\widetilde{h}^X_{k+1}"'] \arrow[d, "(\tau_k)_*"] \\ E_*(V_k) \arrow[r, "\widetilde{h}^X_k"] & E_*(X \wedge V_k) & E_*(V_k). \arrow[l, "\widetilde{h}^X_k"'] \end{tikzcd}$$

Then, the induced homomorphism $(\sigma_k)_*\colon \pi_0(L_nV_{k+1}) \to \pi_0(L_nV_k)$ satisfies $(\sigma_k)_*(i_{k+1}) \equiv i_k$ modulo K_k, since

$$(i_E)_*(\sigma_k)_*(i_{k+1}) = (\sigma_k)_*(i_E)_*(i_{k+1}) = (\sigma_k)_*(\widetilde{1}_{k+1}) = (\tau_k)_*(\widetilde{1}_{k+1}) = \widetilde{1}_k = (i_E)_*(i_k)$$

for the generators $\widetilde{1}_s = (i_E)_*(i_s) \in E_0(V_s)$. It gives rise to an inverse system $\{U_k, (\sigma_k)_*\}$ of sets. Consider mappings $(\sigma_{j,k})_* = (\sigma_k)_*(\sigma_{k+1})_* \cdots (\sigma_{j-1})_*\colon U_j \to U_k$ for $j > k$. Then, by the condition (C-IV)$_m$, we have a finite filtration

$$U_k \supset \mathrm{Im}\,(\sigma_k)_* \supset \mathrm{Im}\,(\sigma_{k+2,k})_* \supset \cdots \supset \mathrm{Im}\,(\sigma_{j_k,k})_* = \mathrm{Im}\,(\sigma_{j_k+1,k})_* = \cdots$$

for some integer j_k. Put $\overline{U}_k = \mathrm{Im}\,(\sigma_{j_k,k})_*$. The relation $\mathrm{Im}\,(\sigma_{j,k})_* = (\sigma_k)_*(\mathrm{Im}\,(\sigma_{j,k+1})_*)$ for $j > \max\{j_k, j_{k+1}\}$ implies $\overline{U}_k = (\sigma_k)_*(\overline{U}_{k+1})$. Thus, $(\sigma_k)_*$ induces a surjection $(\sigma_k)_*\colon \overline{U}_{k+1} \to \overline{U}_k$. Therefore, we have an element $\iota \in \lim_{(\sigma_k)_*} \overline{U}_k \subset \lim_{(\sigma_k)_*} \pi_0(L_nV_k)$. Since $X = \mathrm{holim}_k(X \wedge V_k)$, we have an epimorphism $\pi_0(X) \to \lim_k \pi_0(L_nV_k)$. Then, we also denote by $\iota \in \pi_0(X)$ an element corresponding to ι. Hence, we obtain an equivalence $L^n_m\iota\colon L^n_mS^0 \simeq X$ since ι is an $(E \wedge V_k)_*$-equivalence. □

We also consider a tower

$$\begin{tikzcd} 0 \arrow[d] & Q_1 \arrow[l, dotted] \arrow[d, "d=i^Q_1"] \arrow[dl, equal] & \cdots \arrow[l, dotted, "k^Q_1"'] & Q_s \arrow[l, dotted, "k^Q_{s-1}"'] \arrow[d, "i^Q_s"] & Q_{s+1} \arrow[l, dotted, "k^Q_s"'] \arrow[d, "i^Q_{s+1}"] & \cdots \arrow[l, dotted, "k^Q_{s+1}"'] \\ E \arrow[r, "d^0"'] & \overline{E} \wedge E \arrow[r, "d^1"'] & \cdots \arrow[r, "d^{s-1}"'] & \overline{E}^s \wedge E \arrow[r, "d^s"'] \arrow[ur, "j^Q_s"] & \overline{E}^{s+1} \wedge E \arrow[r, "d^{s+1}"'] & \cdots \end{tikzcd} \tag{4.17}$$

with the same bottom sequence (geometric resolution of S^0) as (4.9). In the same manner as (4.10), the tower (4.17) defines a spectral sequence

$$E^{s,t}_2 = \mathrm{Ext}^{s,t}_{E_*(E)}(E_*, E_*) \Longrightarrow \pi_{t-s}(Q), \tag{4.18}$$

where $Q = \mathrm{holim}_{k^Q_s} \Sigma^{1-s}Q_s$.

(4.19) ([25, Prop. 2.13]) *If a tower* (4.17) *exists, then* $Q = \mathrm{holim}_{k^Q_s} \Sigma^{1-s}Q_s$ *is an exotic invertible spectrum of* $\mathcal{L}_n$.

We consider a sub-tower of (4.17):

$$\begin{tikzcd} 0 \arrow[d] & Q_1 \arrow[l, dotted] \arrow[d, "d=i^Q_1"] \arrow[dl, equal] & \cdots \arrow[l, dotted, "k^Q_1"'] & Q_s \arrow[l, dotted, "k^Q_{s-1}"'] \arrow[d, "i^Q_s"] & Q_{s+1} \arrow[l, dotted, "k^Q_s"'] \\ E \arrow[r, "d^0"'] & \overline{E} \wedge E \arrow[r, "d^1"'] & \cdots \arrow[r, "d^{s-1}"'] & \overline{E}^s \wedge E \arrow[ur, "j^Q_s"'] & \end{tikzcd} \tag{4.20$_s$}$$

for each integer $s \geq 1$, which we call an *s-tower*.

(4.21) ([11, Lemma 4.5]) *Suppose that an s-tower* $(4.20)_s$ *for* $s > 1$ *exists and let* G *be an* E*-module spectrum with action* $\nu_G\colon G \wedge E \to G$. *Then, we have a split short* *exact sequence*

$$0 \to \pi_{s+t-1}(G) \xrightarrow{\psi_s} [Q_s, G]_t \xrightarrow{(j^Q_{s-1})^*} (\operatorname{Im}(d^{s-1})^*)_t \to 0.$$

Here, ψ_s is defined by

$$\psi_s(x) = \nu_G(x \wedge E)(k^Q)^{s-1} \quad \textit{for } (k^Q)^{s-1} = k^Q_1 \cdots k^Q_{s-1}, \tag{4.22}$$

and $(d^{s-1})^*\colon [\overline{E}^s \wedge E, G]_t \to [\overline{E}^{s-1} \wedge E, G]_t$ is induced from $d^{s-1}\colon \overline{E}^{s-1} \wedge E \to \overline{E}^s \wedge E$.

5. Construction of an invertible spectrum in $\mathcal{L}_n$

In this section, we prove Theorem D, that is, the localization L^n_m induces an epimorphism $\operatorname{Pic}^0(\mathcal{L}_n) \to \operatorname{Pic}^0(\mathcal{L}^n_m)$, in the following steps.

1) For $X \in \operatorname{Pic}^0(\mathcal{L}^n_m)$ and $V_k \in \mathcal{V}_m$, consider the E-based Adams tower $\{\overline{E}_s \wedge X \wedge V_k, i^{X\wedge V_k}_s, j^{X\wedge V_k}_s, k^{X\wedge V_k}_s\}$ $(= (4.9) \wedge X \wedge V_k)$ over the geometric resolution $\{\overline{E}^s \wedge E \wedge X \wedge V_k\}_s$.
2) Set up a map $\{\widetilde{h}^s_{X,k}\}\colon \{\overline{E}^s \wedge E\}_s \to \{\overline{E}^s \wedge E \wedge X \wedge V_k\}_s$ of geometric resolutions of S^0 and $X \wedge V_k$ (Lemma 5.1).
3) Inductively, construct an ∞-tower $\{Q_s, i^Q_s, j^Q_s, k^Q_s\}$ over the geometric resolution $\{\overline{E}^s \wedge E\}_s$ along with a map $\{(\mathfrak{f}^s_k, \widehat{h}^s_{X,k})\}\colon \{(Q_s, \overline{E}^s \wedge E)\} \to \{(\overline{E}_s \wedge X \wedge V_k, \overline{E}^s \wedge E \wedge X \wedge V_k)\}$ of towers under the condition (C-III)$_m$. For this sake, we set up Lemmas 5.6 and 5.9.
4) Show that $Q = \operatorname{holim}_s Q_s$ for Q_s given in step 3) is an invertible spectrum of $\mathcal{L}_n$ such that $Q \wedge V_k \simeq X \wedge V_k$. (Lemma 5.11).

These are summerized in Theorem 5.13, and Theorem D in Introduction follows from Corollary 5.16 as explained in Introduction.

Lemma 5.1. *For each $X \in \mathcal{S}^n_m$ and $V_k \in \mathcal{V}_m$, there exist maps $\widehat{h}^s_{X,k}\colon \overline{E}^s \wedge E \to \overline{E}^s \wedge E \wedge X \wedge V_k$ for $s \geq 0$ in the commutative diagram*

$$\begin{tikzcd}
E \arrow[r, "d^0"] \arrow[d, "\widehat{h}^0_{X,k}"'] & \overline{E} \wedge E \arrow[r, "d^1"] \arrow[d, "\widehat{h}^1_{X,k}"] & \cdots \arrow[r, "d^{s-1}"] & \overline{E}^s \wedge E \arrow[r, "d^s"] \arrow[d, "\widehat{h}^s_{X,k}"] & \overline{E}^{s+1} \wedge E \arrow[r, "d^{s+1}"] \arrow[d, "\widehat{h}^{s+1}_{X,k}"] & \cdots \\
E \wedge X \wedge V_k \arrow[r, "d^0\wedge 1"] & \overline{E} \wedge E \wedge X \wedge V_k \arrow[r, "d^1\wedge 1"] & \cdots \arrow[r, "d^{s-1}\wedge 1"] & \overline{E}^s \wedge E \wedge X \wedge V_k \arrow[r, "d^s\wedge 1"] & \overline{E}^{s+1} \wedge E \wedge X \wedge V_k \arrow[r, "d^{s+1}\wedge 1"] & \cdots
\end{tikzcd}$$

such that $\widehat{h}^s_{X,k}$ induces the same map as the projection $(i_k)_*\colon \pi_*(\overline{E}^s \wedge E) \to \pi_*(\overline{E}^s \wedge E \wedge X \wedge V_k) \cong \pi_*(\overline{E}^s \wedge E \wedge V_k)$.

Proof. For spectra $X \in \mathcal{S}^n_m$ and $V_k \in \mathcal{V}_m$, Lemma 3.3 yields a commutative diagram

$$\begin{tikzcd}
E \arrow[r, "1\wedge i_E"'] \arrow[d, "1\wedge i_k"] \arrow[rr, bend left, "d^0"] \arrow[dd, dotted, bend right=60, "\widehat{h}^0_{X,k}"'] & E \wedge E \arrow[r, "j_E\wedge 1"'] \arrow[d, "1\wedge i_k"] & \overline{E} \wedge E \arrow[d, "1\wedge i_k"'] \arrow[dd, dotted, bend left=60, "\widehat{h}^1_{X,k}"] \\
E \wedge V_k \arrow[r, "1\wedge i_E\wedge 1"] \arrow[d, "\widehat{h}_X\wedge 1"] & E \wedge E \wedge V_k \arrow[r, "j_E\wedge 1"] \arrow[d, "1\wedge\widehat{h}_X\wedge 1"] & \overline{E} \wedge E \wedge V_k \arrow[d, "1\wedge\widehat{h}_X\wedge 1"'] \\
E \wedge X \wedge V_k \arrow[r, "1\wedge i_E\wedge 1"] \arrow[rr, bend right, "d^0\wedge 1"'] & E \wedge E \wedge X \wedge V_k \arrow[r, "j_E\wedge 1"] & \overline{E} \wedge E \wedge X \wedge V_k
\end{tikzcd}$$

Put

$$\widehat{h}^s_{X,k} = \overline{E}^s \wedge \widehat{h}_X \wedge i_k\colon \overline{E}^s \wedge E \to \overline{E}^s \wedge E \wedge X \wedge V_k \quad \text{for } s \geq 0, \tag{5.2}$$

and the above diagram gives rise to the commutative diagram

$$(5.3)\qquad \begin{array}{ccc} \overline{E}^s \wedge E & \xrightarrow{d^s} & \overline{E}^{s+1} \wedge E \\ {\scriptstyle \widehat{h}^s_{X,k}}\downarrow & & \downarrow{\scriptstyle \widehat{h}^{s+1}_{X,k}} \\ \overline{E}^s \wedge E \wedge X \wedge V_k & \xrightarrow[d^s \wedge 1]{} & \overline{E}^{s+1} \wedge E \wedge X \wedge V_k \end{array}$$

for d^s in (4.4).

The induced homomorphism $(\widehat{h}^s_{X,k})_*$ is the same as $(i_k)_*$ by Lemma 3.3 1). □

For spectra $X \in \mathcal{S}^n_m$ and $V_k \in \mathcal{V}_m$, we consider the existence of a map of towers

$$(5.4)_s \quad \{(\mathfrak{f}^t_k, \widehat{h}^t_{X,k})\}_{t \le s} \colon \{(Q_t, \overline{E}^t \wedge E)\}_{t \le s} \to \{(\overline{E}_t \wedge X \wedge V_k, \overline{E}^t \wedge E \wedge X \wedge V_k)\}_{t \le s}$$

of s-towers: from the tower $(4.20)_s$ to the tower $(4.9) \wedge X \wedge V_k$. Here, the maps $\widehat{h}^t_{X,k}$ are the ones in Theorem 5.1. A map $(5.4)_s$ means the maps $\{(\mathfrak{f}^t_k, \widehat{h}^t_{X,k})\}_{t \le s}$ satisfying

$$(5.5)_t \qquad \begin{gathered} \widehat{h}^t_{X,k} i^Q_t = (i^S_t \wedge X \wedge V_k)\mathfrak{f}^t_k, \quad \mathfrak{f}^t_k j^Q_{t-1} = (j^S_{t-1} \wedge X \wedge V_k)\widehat{h}^{t-1}_{X,k} \quad \text{and} \\ \mathfrak{f}^{t-1}_k k^Q_{t-1} = (k^S_{t-1} \wedge X \wedge V_k)\mathfrak{f}^t_k \end{gathered}$$

for $1 \le t \le s$.

Lemma 5.6. *Let $X \in \mathcal{S}^n_m$ and k_X be the integer in Proposition 4.15. Suppose that there exist an s-tower $\{Q_t, i^Q_t, j^Q_t, k^Q_t\}$ lying in $(4.20)_s$ and a map in $(5.4)_{s-1}$ of towers for an integer $s \ge 2$ and a spectrum $V_k \in \mathcal{V}_m$ with $k \ge k_X$. Then, we have a map $(5.4)_s$ of towers for a replaced $i^Q_s \colon Q_s \to \overline{E}^s \wedge E$ fitting in $(4.20)_s$.*

Proof. The relation $\widehat{h}^{s-1}_{X,k} i^Q_{s-1} = (i^S_{s-1} \wedge X \wedge V_k)\mathfrak{f}^{s-1}_k$ in $(5.5)_{s-1}$ defines a map $\mathfrak{f}^s_k$ fitting in the commutative diagram

(5.7)

$$\begin{array}{ccccccc} Q_{s-1} & \xrightarrow{i^Q_{s-1}} & \overline{E}^{s-1} \wedge E & \xrightarrow{j^Q_{s-1}} & Q_s & \xrightarrow{k^Q_{s-1}} & \Sigma Q_{s-1} \\ {\scriptstyle \mathfrak{f}^{s-1}_k}\downarrow & & \downarrow{\scriptstyle \widehat{h}^{s-1}_{X,k}} & & \downarrow{\scriptstyle \mathfrak{f}^s_k} & & \downarrow{\scriptstyle \mathfrak{f}^{s-1}_k} \\ \overline{E}_{s-1} \wedge X \wedge V_k & \xrightarrow{i^S_{s-1}\wedge 1} & \overline{E}^{s-1} \wedge E \wedge X \wedge V_k & \xrightarrow{j^S_{s-1}\wedge 1} & \overline{E}_s \wedge X \wedge V_k & \xrightarrow{k^S_{s-1}\wedge 1} & \Sigma\overline{E}_{s-1} \wedge X \wedge V_k \end{array}$$

of cofiber sequences. This implies the second and the third equalities of $(5.5)_s$.

Put $\mathfrak{o}^s_k = (i^S_s \wedge 1)\mathfrak{f}^s_k - \widehat{h}^s_{X,k} i^Q_s \in [Q_s, \overline{E}^s \wedge E \wedge X \wedge V_k]_0$, and consider the commutative diagram

$$\begin{array}{ccccc} \pi_{s-1}(\overline{E}^s \wedge E) & \xrightarrow{\psi_s} & [Q_s, \overline{E}^s \wedge E]_0 & \xrightarrow{(j^Q_{s-1})^*} & (\operatorname{Im}(d^{s-1})^*)_0 \\ \downarrow{\scriptstyle (\widehat{h}^s_{X,k})_*} & & \downarrow{\scriptstyle (\widehat{h}^s_{X,k})_*} & & \downarrow{\scriptstyle (\widehat{h}^s_{X,k})_*} \\ \pi_{s-1}(\overline{E}^s \wedge E \wedge X \wedge V_k) & \xrightarrow{\psi_s} & [Q_s, \overline{E}^s \wedge E \wedge X \wedge V_k]_0 & \xrightarrow{(j^Q_{s-1})^*} & (\operatorname{Im}(d^{s-1})^*)_0 \end{array}$$

of the exact sequences in (4.21). Then,

$$\begin{aligned} (j^Q_{s-1})^*(\mathfrak{o}^s_k) &= ((i^S_s \wedge 1)\mathfrak{f}^s_k - \widehat{h}^s_{X,k} i^Q_s) j^Q_{s-1} \underset{(4.20)}{=} (i^S_s \wedge 1)\mathfrak{f}^s_k j^Q_{s-1} - \widehat{h}^s_{X,k} d^{s-1} \\ &\underset{(5.7)}{=} (i^S_s \wedge 1)(j^S_{s-1} \wedge 1)\widehat{h}^{s-1}_{X,k} - \widehat{h}^s_{X,k} d^{s-1} \underset{(4.9)}{=} (d^{s-1} \wedge 1)\widehat{h}^{s-1}_{X,k} - \widehat{h}^s_{X,k} d^{s-1} \underset{(5.3)}{=} 0. \end{aligned}$$

Thus, $\mathfrak{o}^s_k \in \operatorname{Im}\psi_s$. Since the left homomorphism $(\widehat{h}^s_{X,k})_*$ in the above diagram is an epimorphism by Lemmas 4.2 and 3.3, we have an element $\overline{\mathfrak{o}}^s_k \in \pi_{s-1}(\overline{E}^s \wedge E)$

such that $\psi_s(\widehat{h}^s_{X,k})_*(\overline{\mathfrak{o}}^s_k) = \mathfrak{o}^s_k$. Replace i^Q_s by $\mathfrak{i} = i^Q_s + \psi_s(\overline{\mathfrak{o}}^s_k) \in [Q_s, \overline{E}^s \wedge E]_0$, and we obtain the lemma by computation

$$\mathfrak{i}j^Q_{s-1} = (i^Q_s + \psi_s(\overline{\mathfrak{o}}^s_k))j^Q_{s-1} = d^{s-1} + (j^Q_{s-1})^*(\psi_s(\overline{\mathfrak{o}}^s_k)) = d^{s-1}, \quad \text{and}$$
$$\begin{aligned}(i^S_s \wedge 1)\mathfrak{f}^s_k - \widehat{h}^s_{X,k}\mathfrak{i} &= (i^S_s \wedge 1)\mathfrak{f}^s_k - \widehat{h}^s_{X,k}(i^Q_s + \psi_s(\overline{\mathfrak{o}}^s_k)) = \mathfrak{o}^s_k - (\widehat{h}^s_{X,k})_*\psi_s(\overline{\mathfrak{o}}^s_k)\\ &= \mathfrak{o}^s_k - \psi_s(\widehat{h}^s_{X,k})_*(\overline{\mathfrak{o}}^s_k) = \mathfrak{o}^s_k - \mathfrak{o}^s_k = 0.\end{aligned}$$
□

We note that for the spectrum V in the condition (C-III)$_m$, we have a spectrum $V_k \in \mathcal{V}_m$ and a map $\tau\colon V_k \to V$ such that $i_V = \tau i_k$. Let k^V denote the minimum integer of such integers k. Then, we have a monomorphism

$$(i_k)_*\colon E_2^{rq+2,rq}(S^0) \to E_2^{rq+2,rq}(V_k) \tag{5.8}$$

for $k \ge k^V$, if the condition (C-III)$_m$ holds.

Lemma 5.9. *Let $X \in \mathcal{S}^n_m$ and suppose the condition* (C-III)$_m$. *Suppose further that there exists a tower $\{Q_t, i^Q_t, j^Q_t, k^Q_t\}$ in* (4.20)$_s$ *along with a map* (5.4)$_s$ *for positive integers s and $k \ge \max\{k_X, k^V\}$. Then, the tower extends to* (4.20)$_{s+1}$ *after replacing i^Q_s by a suitable map fitting in* (4.20)$_s$.

Proof. It suffices to show the existence of a map $\mathfrak{i}\colon Q_s \to \overline{E}^s \wedge E$ such that $\mathfrak{i}j^Q_{s-1} = d^{s-1}$ and $d^s\mathfrak{i} = 0$. Indeed, replace i^Q_s by $\mathfrak{i}$ and Q_{s+1} by the one in the cofiber sequence $Q_s \xrightarrow{\mathfrak{i}} \overline{E}^s \wedge E \xrightarrow{j^Q_s} Q_{s+1} \xrightarrow{k^Q_s} \Sigma Q_s$, and we obtain a map $i^Q_{s+1}\colon Q_{s+1} \to \overline{E}^{s+1} \wedge E$ such that $d^s = i^Q_{s+1}j^Q_s$, and then we may take Q_{s+2} to be the cofiber of i^Q_{s+1}.

Put $o_s = d^s i^Q_s \in [Q_s, \overline{E}^{s+1} \wedge E]_0$, and consider the diagram

$$\begin{array}{ccccc}
\pi_{s-1}(\overline{E}^{s+1} \wedge E) & \xrightarrow{\psi_s} & [Q_s, \overline{E}^{s+1} \wedge E]_0 & \xrightarrow{(j^Q_{s-1})^*} & (\mathrm{Im}\,(d^{s-1})^*)_0\\
\downarrow{\scriptstyle (\widehat{h}^{s+1}_{X,k})_*} & & \downarrow{\scriptstyle (\widehat{h}^{s+1}_{X,k})_*} & & \downarrow{\scriptstyle (\widehat{h}^{s+1}_{X,k})_*}\\
\pi_{s-1}(\overline{E}^{s+1} \wedge E \wedge X \wedge V_k) & \xrightarrow{\psi_s} & [Q_s, \overline{E}^{s+1} \wedge E \wedge X \wedge V_k]_0 & \xrightarrow{(j^Q_{s-1})^*} & (\mathrm{Im}\,(d^{s-1})^*)_0
\end{array}$$

of exact sequences of (4.21) for $V_k \in \mathcal{V}_m$ with $k \ge \max\{k_X, k^V\}$. Since $(j^Q_{s-1})^*(o_s) = d^s i^Q_s j^Q_{s-1} = d^s d^{s-1} = 0$, we have an element $\widetilde{o}_s \in \pi_{s-1}(\overline{E}^{s+1} \wedge E) = E_1^{s+1,s-1}(S^0)$ such that $\psi_s(\widetilde{o}_s) = o_s$. We compute

$$\begin{aligned}\psi_s(\widehat{h}^{s+1}_{X,k})_*(\widetilde{o}_s) &= (\widehat{h}^{s+1}_{X,k})_*\psi_s(\widetilde{o}_s) = (\widehat{h}^{s+1}_{X,k})_*(o_s) = \widehat{h}^{s+1}_{X,k} d^s i^Q_s \underset{(5.3)}{=} (d^s \wedge 1)\widehat{h}^s_{X,k} i^Q_s\\ &\underset{(5.4)_s}{=} (d^s \wedge 1)(i^S_s \wedge 1)\mathfrak{f}^s_k = 0.\end{aligned}$$

It follows that

$$(\widehat{h}^{s+1}_{X,k})_*(\widetilde{o}_s) = 0, \tag{5.10}$$

since ψ_s is a monomorphism. Consider a commutative diagram

$$\begin{array}{ccccc}
\pi_{s-1}(\overline{E}^{s} \wedge E) & \xrightarrow{\psi_s} & [Q_s, \overline{E}^{s} \wedge E]_0 & \xrightarrow{(j^Q_{s-1})^*} & (\mathrm{Im}\,(d^{s-1})^*)_0\\
\downarrow{\scriptstyle d^s_*} & & \downarrow{\scriptstyle d^s_*} & & \downarrow{\scriptstyle d^s_*}\\
\pi_{s-1}(\overline{E}^{s+1} \wedge E) & \xrightarrow{\psi_s} & [Q_s, \overline{E}^{s+1} \wedge E]_0 & \xrightarrow{(j^Q_{s-1})^*} & (\mathrm{Im}\,(d^{s-1})^*)_0\\
\downarrow{\scriptstyle d^{s+1}_*} & & \downarrow{\scriptstyle d^{s+1}_*} & & \downarrow{\scriptstyle d^{s+1}_*}\\
\pi_{s-1}(\overline{E}^{s+2} \wedge E) & \xrightarrow{\psi_s} & [Q_s, \overline{E}^{s+2} \wedge E]_0 & \xrightarrow{(j^Q_{s-1})^*} & (\mathrm{Im}\,(d^{s-1})^*)_0.
\end{array}$$

We compute

$$\psi_s d_*^{s+1}(\widetilde{o}_s) = d_*^{s+1}\psi_s(\widetilde{o}_s) = d_*^{s+1} o_s = d^{s+1} d^s i_s^Q = 0,$$

and we see $[\widetilde{o}_s] \in E_2^{s+1,s-1}(S^0)$, since ψ_s is a monomorphism. Furthermore, $(\widehat{h}_X)_*(i_k)_*(\widetilde{o}_s) \underset{(5.2)}{=} (\widehat{h}^{s+1}_{X,k})_*(\widetilde{o}_s) \underset{(5.10)}{=} 0 \in \pi_{s-1}(\overline{E}^{s+1} \wedge E \wedge X \wedge V_k)$, and so $(i_k)_*(\widetilde{o}_s) = 0 \in \pi_{s-1}(\overline{E}^{s+1} \wedge E \wedge V_k)$ by (3.4) 1) and Lemma 4.2. Indeed, $(\widehat{h}_X)_* = \pi_*(\overline{E}^{s+1} \wedge \widehat{h}_X \wedge V_k)$. It follows that $(i_k)_*([\widetilde{o}_s]) = 0 \in E_2^{s+1,s-1}(V_k)$. Thus, $[\widetilde{o}_s] = 0 \in E_2^{s+1,s-1}(S^0)$ by (5.8), and there exists an element $w \in \pi_{s-1}(\overline{E}^s \wedge E) = E_1^{s,s-1}(S^0)$ such that $d^s w = \widetilde{o}_s$. Put now $\mathfrak{i} = i_s^Q - \psi_s(w)$. Then

$$d^s\mathfrak{i} = d^s i_s^Q - d^s\psi_s(w) = o_s - \psi_s d^s(w) = o_s - \psi_s\widetilde{o}_s = o_s - o_s = 0, \quad \text{and}$$

$$\mathfrak{i} j_{s-1}^Q = i_s^Q j_{s-1}^Q - \psi_s(w) j_{s-1}^Q = d^{s-1} - (j_{s-1}^Q)^*\psi_s(w) = d^{s-1}.$$

Thus, this $\mathfrak{i}$ is the desired one. □

Lemma 5.11. *Suppose that there exists a tower* (4.17) *along with a map* $(5.4)_\infty$ *of ∞-towers. Then, Q in* (4.18) *is an invertible spectrum of $\mathcal{L}_n$ such that $Q \wedge V_k \simeq X \wedge V_k$.*

Proof. By (4.19), Q is an invertible spectrum. Furthermore, the maps $\mathfrak{f}_k^s\colon Q_s \to \overline{E}_s \wedge X \wedge V_k$ yield a map $\mathfrak{f}_k\colon Q \to X \wedge V_k$. This induces an E_*-equivalence $Q \wedge V_k \to X \wedge V_k$, which gives an equivalence $Q \wedge V_k \simeq X \wedge V_k$. □

Since $1 = [i_E] \in E_2^{0,0}(S^0)$ is a permanent cycle of the E-based Adams spectral sequences for computing $\pi_*(L_n S^0)$, there exist elements $x_t \in \pi_{t-1}(\overline{E}_t)$ such that

$$x_1 = i_E \quad \text{and} \quad k_{t-1}^S x_t = x_{t-1} \tag{5.12}$$

for $t \geq 1$.

Theorem 5.13. *Suppose* $(\text{C-III})_m$*. For spectra $X \in \mathcal{S}_m^n$ and $V_k \in \mathcal{V}_m$ with $k \geq \max\{k_X, k^V\}$, there exists an invertible spectrum $Q_k^X \in \mathcal{S}_0^n$ such that $Q_k^X \wedge V_k \simeq X \wedge V_k$. Furthermore, we have $r_X = r_{L_m^n Q_k^X}$ for the integer r_X in Proposition 4.15.*

Proof. For spectra X in $\mathcal{S}_m^n$ and $V_k \in \mathcal{V}_m$, we inductively construct a tower (4.17) satisfying the supposition of Lemma 5.11. In other words, we show $(5.14)_s$ below for each integer $s \geq 2$ inductively.

$(5.14)_s$ *There exist an s-tower $\{Q_t, i_t^Q, j_t^Q, k_t^Q\}$ in $(4.20)_s$ and a map $\{(\mathfrak{f}_k^t, \widehat{h}_{X,k}^t)\}\colon \{(Q_t, \overline{E}^t \wedge E)\} \to \{(\overline{E}_t \wedge X \wedge V_k, \overline{E}^t \wedge E \wedge X \wedge V_k)\}$ of s-towers in $(5.4)_s$ for an integer $k \geq k_X$. Furthermore, $Q_t = \overline{E}_t$ for $t \leq r_X q + 1$.*

Put $Q_0 = 0$, $Q_t = \overline{E}_t$ for $t \in \{1, 2\}$, $i_1^Q = i_1^S$, $j_1^Q = j_t^S$, $k_1^Q = k_1^S$ (see (4.9)), and $\mathfrak{f}_k^1 = \widehat{h}_{X,k}^0\colon Q_1 = \overline{E}_1 = E \to E \wedge X \wedge V_k = \overline{E}_1 \wedge X \wedge V_k$, and we obtain $(5.14)_1$.

Suppose inductively that for $t < s$ $(\leq r_X q)$, there exist maps $\mathfrak{f}_k^t\colon \overline{E}_t \to \overline{E}_t \wedge X \wedge V_k$ satisfying $(5.5)_t$ with $Q = S$. In the same manner as the proof of Lemma 5.6, we define $\mathfrak{f}_k^s\colon \overline{E}_s \to \overline{E}_s \wedge X \wedge V_k$ by the commutative diagram (5.7) with $Q_t = \overline{E}_t$, and see $(5.5)_s$ with $Q = S$ except for the first equality.

We turn to the first equation in $(5.5)_s$. As in the proof of Lemma 5.6, we have an element

$$\mathfrak{o}_s = (i_s^S \wedge X \wedge V_k)\mathfrak{f}_k^s - \widehat{h}_{X,k}^s i_s^S \in [\overline{E}_s, \overline{E}^s \wedge E \wedge X \wedge V_k]_0$$

such that $(j^S_{s-1})^*(\mathfrak{o}_s) = 0$. Therefore, we have an element $\widetilde{\mathfrak{o}_s} \in \pi_s(\overline{E}^s \wedge E \wedge X \wedge V_k)$ such that $\psi_s(\widetilde{\mathfrak{o}}_s) = \mathfrak{o}_s$ for the homomorphism ψ_s in (4.21). Then,

$$\mathfrak{o}_s x_s = \psi_s(\widetilde{\mathfrak{o}}_s) x_s \underset{(4.22)}{=} \nu(\widetilde{\mathfrak{o}}_s \wedge E)(k^S)^{s-1} x_s \underset{(5.12)}{=} \nu(\widetilde{\mathfrak{o}_s} \wedge E) i_E = \widetilde{\mathfrak{o}}_s \tag{5.15}$$

for the action $\nu\colon \overline{E}^s \wedge E \wedge X \wedge V_k \wedge E \to \overline{E}^s \wedge E \wedge X \wedge V_k$ given by μ_E in (3.1). Therefore,

$$\begin{aligned}
\widetilde{\mathfrak{o}}_s &\underset{(5.15)}{=} \mathfrak{o}_s x_s = ((i^S_s \wedge X \wedge V_k)\mathfrak{f}^s_k - \widehat{h}^s_{X,k} i^S_s) x_s \\
&= (i^S_s \wedge X \wedge V_k)\mathfrak{f}^s_k x_s \quad (\text{since } i^S_s x_s = i^S_s k^S_s x_{s+1} = 0) \\
&\underset{(5.5)_s}{=} \widehat{h}^s_{X,k} i^S_s x_s \underset{(5.12)}{=} \widehat{h}^s_{X,k} i^S_s k^S_s x_{s+1} \underset{(4.8)}{=} 0.
\end{aligned}$$

Thus, $\mathfrak{o}_s = \psi_s(\widetilde{\mathfrak{o}}_s) = 0$ implies the first equation in $(5.5)_s$ with $Q = S$. Therefore, $(5.5)_t$ holds for each $t \le r_X q$ inductively.

Suppose that $(5.14)_s$ holds true for $s \ge r_X q$. Then, the s-tower extends to an $(s+1)$-tower by Lemma 5.9. By Lemma 5.6, the $(s+1)$-tower admits a map $\mathfrak{f}^{s+1}_k$ satisfying $(5.14)_{s+1}$. It follows inductively that $(5.14)_s$ holds for all positive integers s. Therefore, we have an invertible spectrum Q in $\mathcal{L}_n$ such that $Q \wedge V_k \simeq X \wedge V_k$ by Lemma 5.11. Furthermore, $Q_t = \overline{E}_t$ for $t \le r_X q + 1$ implies that $r_X = r_{L^n_m Q}$. □

Corollary 5.16. *Suppose that* (C-II) *and* (C-III)$_m$. *Then, the mapping* $\ell^n_m\colon \mathrm{Pic}^0(\mathcal{L}_n) \to \mathcal{S}^n_m$ *is a surjection.*

Proof. Let $X \in \mathcal{S}^n_m$. For every spectrum $Q \in \mathrm{Pic}^0(\mathcal{L}_n)$, consider a set $S(Q) = \{k \mid Q^X_k \simeq Q\} \subset \mathbb{Z}$ for spectra Q^X_k given in Theorem 5.13. Since $\mathrm{Pic}^0(\mathcal{L}_n)$ is a finite group, there exists a spectrum $Q^X \in \mathrm{Pic}^0(\mathcal{L}_n)$ such that $|S(Q^X)| = \aleph_0$. Then, $L^n_m Q^X = \mathrm{holim}_{k \in S(Q^X)} Q^X \wedge V_k \simeq \mathrm{holim}_{k \in S(Q^X)} X \wedge V_k \simeq X$. □

6. The cases for small n

In this section, we verify the conditions (C-I)$_m$, (C-II), (C-III)$_m$ and (C-IV)$_m$ for the cases where $(p,n) = (2,1)$, $(3,2)$ or $n^2 + n \le q$. For (C-I)$_m$ and (C-III)$_m$ with $m \le n$, it suffices to show (C-I)$_n$ and (C-III)$_n$ by (1.18). Furthermore, we verify (C-IV)$_m$ for $m > 0$, since (C-IV)$_0$ is void.

In general, we have the following lemma on (C-IV)$_m$:

Lemma 6.1. *If $\pi_s(L_n M(p))$ is finite for each integer s, then* (C-IV)$_m$ *holds for* $1 \le m \le n$.

Proof. Consider the subcategory

$$\mathcal{T} = \{F \in \mathrm{thick}\left\langle S^0 \right\rangle \mid \pi_s(L_n F) \text{ is finite for each } s \in \mathbb{Z}\} \subset \mathcal{L}_n$$

Then, it is thick. Since $\mathcal{T}$ contains $M(p)$, the thick subcategory theorem in [6] implies that $\pi_s(L_n V)$ is finite for any type m (≥ 1) finite spectrum V and for any integer s. In particular, (C-IV)$_m$ holds for $m \ge 1$. □

We note that if $E^{s,*}_2(X)$ is a $\mathbb{Z}/p^2$-module, then it is also a $\mathbb{Z}/p^2[v^p_1]$-module, since $\eta_R(v_1) \equiv v_1 \bmod (p)$. By [16], we may set $V_k = M(p^k, v_1^{p^k}) \in \mathcal{V}_2$ for $k \ge 1$.

Lemma 6.2. *Let $V_k = M(p^k, v_1^{p^k}) \in \mathcal{V}_2$ for $k \geq 1$, and suppose the existence of an integer $\overline{s}$ such that $E_2^{s,t}(S^0)$ is a $\mathbb{Z}/p$-module and $v_1^p E_2^{s,t}(S^0) = 0$ for $s \geq \overline{s}$. Then, $(i_k)_*\colon E_2^{s,t}(S^0) \to E_2^{s,t}(V_k)$ is a monomorphism for $k \geq 2$ and $s \geq \overline{s}$.*

Proof. Consider the cofiber sequence $S^0 \xrightarrow{p^r} S^0 \xrightarrow{\iota_r} M(p^r)$, which induces an exact sequence

$$E_2^{s,t}(S^0) \xrightarrow{p^r=0} E_2^{s,t}(S^0) \xrightarrow{(\iota_r)_*} E_2^{s,t}(M(p^r)) \xrightarrow{\delta} E_2^{s+1,t}(S^0). \tag{6.3}$$

Let $s \geq \overline{s}$. Since $E_2^{s,t}(S^0)$ is a $\mathbb{Z}/p$-module, the homomorphism $(\iota_r)_*$ is a monomorphism. This further indicates that $E_2^{s,t}(M(p^r))$ is a $\mathbb{Z}/p^2$-module, and then a $\mathbb{Z}/p^2[v_1^p]$-module. Therefore, the above exact sequence is the one of $\mathbb{Z}/p^2[v_1^p]$-modules. We consider a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & E_2^{s,t}(S^0) & \xrightarrow{(\iota_r)_*} & E_2^{s,t}(M(p^r)) & \xrightarrow{\delta} & E_2^{s+1,t}(S^0) & \to 0 \\
& {\scriptstyle v_1^p}\downarrow & & \downarrow{\scriptstyle v_1^p} & & \downarrow{\scriptstyle v_1^p} & \\
0 \to & E_2^{s,t}(S^0) & \xrightarrow{(\iota_r)_*} & E_2^{s,t}(M(p^r)) & \xrightarrow{\delta} & E_2^{s+1,t}(S^0) & \to 0
\end{array}$$

of short exact sequences. A diagram chasing with the hypothesis $v_1^p E_2^{s,t}(S^0) = 0$ shows $v_1^{2p} E_2^{s,t}(M(p^r)) = 0$. Thus, we have $v_1^{p^r} = 0\colon E_2^{s,t-|bq|}(M(p^r)) \to E_2^{s,t}(M(p^r))$ for $r \geq 2$. Apply this to the exact sequence $E_2^{s,t-|bq|}(M(p^r)) \xrightarrow{v_1^{p^r}} E_2^{s,t}(M(p^r)) \xrightarrow{(\widetilde{\iota}_r)_*} E_2^{s,t}(M(p^r, v_1^{p^r}))$ induced from the cofiber sequence $\Sigma^{p^r q} M(p^r) \xrightarrow{v_1^{p^r}} M(p^r) \xrightarrow{\widetilde{\iota}_r} M(p^r, v_1^{p^r})$, and we see $(\widetilde{\iota}_r)_*\colon E_2^{s,t}(M(p^r)) \to E_2^{s,t}(M(p^r, v_1^{p^r}))$ a monomorphism for $r \geq 2$. Therefore, $(i_r)_* = (\widetilde{\iota}_r)_*(\iota_r)_*$ is a monomorphism for $r \geq 2$. □

From now, we give a proof of Theorem G.

6.1. **The case $n^2 + n \leq q$.** We exclude the case $(p, n) = (2, 1)$. In this case, $E_2^{s,*}(S^0) = 0$ for $s > n^2 + n$ (*cf.* [20, (10.10)], and hence (C-I)$_m$, (C-II), (C-III)$_m$ and (C-IV)$_m$ follow trivially (*cf.* Remark 1.20).

6.2. **The case $(p, n) = (2, 1)$.** The condition (C-II) holds by [9, Th. 6.1]. For (C-III)$_1$, consider a short exact sequence $0 \to E(2)_* \to M_0^0 \to M_0^1 \to 0$ of comodules for $M_0^0 = 2^{-1}E(2)_*$. We use an abbreviation of the Ext group:

$$H_{(n)}^{s,t} M = \mathrm{Ext}_{E(n)_*(E(n))}^{s,t}(E(n)_*, M) \tag{6.4}$$

for an $E(n)_*(E(n))$-comodule M. It is well known that $H_{(1)}^s M_0^0 = 0$ for $s > 0$. Therefore, the connecting homomorphism associated to the above short exact sequence is an isomorphism $H_{(1)}^s M_0^1 \cong H_{(1)}^{s+1} E(2)_* = E_2^{s+1}(S^0)$. Note that $H_{(1)}^s M_0^1$ for $s \geq 2$ is a $\mathbb{Z}/2$-module by [15, Th. 4.16]. Then, we have (C-III)$_1$, that is, $(i_k)_*\colon E_2^{2s+2,2s}(S^0) \to E_2^{2s+2,2s}(M(2^k))$ is a monomorphism, since we have an exact sequence

$$E_2^{2s+2,2s}(S^0) \xrightarrow{2^k} E_2^{2s+2,2s}(S^0) \xrightarrow{(i_k)_*} E_2^{2s+2,2s}(M(2^k)).$$

Furthermore, we deduce $E_2^{s,s}(M(2^k))$ finite by [15, Th. 4.16]. Thus (C-IV)$_1$ follows from Remark 1.20.

Turn to (C-I)$_1$. By [15, Th. 4.16], we have the E_2-term

$$E_2^{2k+1,2k}(S^0) = H^{2k,2k}M_0^1 = \begin{cases} \mathbb{Z}/2\{v_1^{-k}h_0^{2k}/2\} & \text{if } k \text{ is odd} \\ \mathbb{Z}/2\{v_1^{1-k}\rho_1 h_0^{2k-1}/2\} & \text{if } k \text{ is even} \end{cases}.$$

From [19, §5], we deduce the differential:

$$\begin{cases} d_3(v_1^{3-k}\rho_1 h_0^{2k-4}/2) = v_1^{1-k}\rho_1 h_0^{2k-1}/2 & \text{if } k \equiv 0 \bmod 4 \\ d_3(v_1^{-k}h_0^{2k}/2) = v_1^{-k-2}h_0^{2k+3}/2 & \text{if } k \equiv 1 \bmod 4 \\ d_3(v_1^{1-k}\rho_1 h_0^{2k-1}/2) = v_1^{-1-k}\rho_1 h_0^{2k+2}/2 & \text{if } k \equiv 2 \bmod 4 \\ d_3(v_1^{2-k}h_0^{2k-3}/2) = v_1^{-k}h_0^{2k}/2 & \text{if } k \equiv 3 \bmod 4 \end{cases}$$

These show $E_{2k+1}^{2k+1,2k}(S^0) = 0$ for $k \ge 1$, which implies the condition (C-I)$_1$.

6.3. **The case** $(p,n) = (3,2)$**.** By [11, Cor. 1.4 (c)], $\mathrm{Pic}^0(\mathcal{L}_2)$ is a finite group, and so the condition (C-II) holds.

We read off from [23, Th. 2.11] (see also [4]) that $\pi_s(L_2M(3))$ is finite for each degree s. Lemma 6.1 together with this implies the condition (C-IV)$_m$ for $m \in \{1,2\}$.

Consider the comodules N_0^1 and M_0^2 defined by the short exact sequences $0 \to E(2)_* \to 3^{-1}E(2)_* \to N_0^1 \to 0$ and $0 \to N_0^1 \to v_1^{-1}N_0^1 \to M_0^2 \to 0$. Then, they induce the connecting homomorphisms $\delta\colon H_{(2)}^{s,t}N_0^1 \to H_{(2)}^{s+1,t}E(2)_* = E_2^{s+1,t}(S^0)$ and $\delta'\colon H_{(2)}^{s,t}M_0^2 \to H_{(2)}^{s+1,t}N_0^1$ for $H_{(2)}^*$ in (6.4), which are isomorphisms if $s \ge 1$ and $s \ge 2$, respectively by [15]. By [24, Cor. 2.5, Prop. 4.7], we see that $E_2^{s,t}(S^0) \cong H_{(2)}^{s-2,t}M_0^2$ is a $\mathbb{Z}/3$-module and $v_1^2 H_{(2)}^{s-2,t}M_0^2 = 0$ for $s \ge 6 = q+2$. That is,

$$E_2^{s,t}(S^0) \text{ is a } \mathbb{Z}/3\text{-module and } v_1^3 E_2^{s,t}(S^0) = 0 \text{ for } s \ge 6. \tag{6.5}$$

Therefore, Lemma 6.2 implies (C-III)$_2$.

Lemma 6.6. *The E_{rq+1}-term of the $E(2)$-based Adams spectral sequence for $\pi_{-1}(L_2S^0)$ is given by*

$$E_5^{5,4}(S^0) = \mathbb{Z}/3\{v_2^{-2}h_{11}b_{10}^2, v_2^{-1}\xi b_{10}\zeta_2\} \quad \textit{and} \quad E_{4r+1}^{4r+1,4r}(S^0) = 0 \quad \textit{for } r \ge 2.$$

Proof. Let M^2 denotes a spectrum such that $E(2)_*(M^2) = M_0^2$. Actually, we define N^1 and M^2 to be cofibers of the natural maps $L_2S^0 \to L_0S^0$ and $N^1 \to L_1N^1$. Note that $E_2^{s,t}(M^2) = H_{(2)}^{s,t}M_0^2$. By [24, Prop. 4.7, Th. 6.4], we read off

$$H_{(2)}^{7,8}M_0^2 = 0 \quad \text{and} \quad E_{4r+1}^{4r-1,4r}(M^2) = 0 \quad \text{for } r \ge 3.$$

Furthermore, we have an exact sequence $H_{(2)}^{3,4}M_1^1 \xrightarrow{\varphi} H_{(2)}^{3,4}M_0^2 \xrightarrow{3} H_{(2)}^{3,4}M_0^2$ with $\varphi(x) = x/3$ ([15, §3]) and $H_{(2)}^{3,4}M_1^1 = \mathbb{Z}/p\{v_2^{-1}h_1b_0/v_1, \xi\zeta_2/v_1\}$ by [24, Th. 2.3]. Therefore, [24, Prop. 5.3] implies

$$H_{(2)}^{3,4}M_0^2 = \mathbb{Z}/3\{v_2^{-1}h_1b_0/3v_1, \xi\zeta_2/3v_1\}.$$

Now the lemma follows from the isomorphism $\delta\delta'\colon H_{(2)}^{s,t}M_0^2 \to E_2^{s+2,t}(S^0)$. $\square$

Lemma 6.7. *The condition* (C-I)$_2$ *holds. In other words, The unit map $i_k\colon S^0 \to V_k$ induces a monomorphism $(i_k)_*\colon E_{4r+1}^{4r+1,4r}(S^0) \to E_{4r+1}^{4r+1,4r}(V_k)$ for $V_k = M(3^k, v_1^{3^k}) \in \mathcal{V}_2$.*

Proof. By Lemma 6.6, the homomorphism $(i_k)_*$ is a monomorphism for $r \geq 2$. For $r = 1$, the E_5-term is the same as the E_2-term. Let $V(1) = M(p, v_1)$. The E_2-term of $L_2V(1)$ is given in [22, Th. 5.8] (see also [2]), and we see that the inclusion $inc\colon S^0 \to V(1)$ induces a monomorphism $E_2^{5,4}(S^0) \to E_2^{5,4}(V(1))$ by Lemma 6.6. Since inc factors through $i_k\colon S^0 \to V_k$, we obtain a monomorphism $(i_k)_*\colon E_2^{5,4}(S^0) \to E_2^{5,4}(V_k)$. □

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