ON THE PRODUCT $\alpha_1\beta_1^r\gamma_t$ IN THE STABLE HOMOTOPY GROUPS OF SPHERES

KATSUMI SHIMOMURA AND KIÉ YOSHIZAWA

ABSTRACT. We deduce the essential elements $\alpha_1\beta_1^{p-2}\gamma_t$ for $t > p$ with $t \neq 0, 1$ modulo (p) of the homotopy groups of spheres from result of C-N. Lee[2], which states that $\alpha_1\beta_1^{p-2}\gamma_t$ is essential for $t < p$.

1. Introduction

In the stable homotopy groups $\pi_*(S^0)$ of spheres localized at a prime $p > 5$, we have the Greek letter elements $\alpha_1 \in \pi_{q-1}(S^0)$, $\beta_1 \in \pi_{pq-2}(S^0)$ and $\gamma_t \in \pi_{(tp^2+(t-1)p+t-2)q-3}(S^0)$ for $t \geq 0$, where $q = 2p-2$ (*cf.* [3]). In [2], Chun-Nip Lee showed non-trivial products

$$\alpha_1\beta_1^r\gamma_t \neq 0 \quad \text{if } 2 \leq t < p \text{ and } r \leq p-2.$$

In this paper, we extend this result by elementary computation.

Theorem 1.1. *In the stable homotopy groups $\pi_*(S^0)$,*

$$\alpha_1\beta_1^r\gamma_{up+t} \neq 0 \quad \textit{if } 1 < t < t+u < p \textit{ and } r \leq p-2.$$

In particular, $\beta_1^r\gamma_{up+t} \neq 0$ under the same condition.

The idea we used here will be applied to show other nontriviality of other products of homotopy elements, which requires us additional hard computation. We will study those in forthcoming papers.

The authors would like to thank Ryo Kato and the referee for their useful comments.

2. Computation on the Adams-Novikov E_2-term

In this paper, p denotes a prime greater than five, and every spectrum is localized at the prime p. The Brown-Peterson ring spectrum BP at p gives rise to the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(X)$ of a spectrum X with the E_2-term

$$E_2^{s,t}(X) = \mathrm{Ext}_\Gamma^{s,t}(A, BP_*(X)),$$

where

$$(A, \Gamma) = (BP_*, BP_*(BP)) = (\mathbb{Z}_{(p)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots])$$

is the Hopf algebroid associated to the Brown-Peterson spectrum BP. The generators v_i and t_i of A and Γ have the internal degree

$$|v_i| = |t_i| = 2p^i - 2 = (p^{i-1} + p^{i-2} + \cdots + p + 1)q. \tag{2.1}$$

We consider here the Ext group the cohomology of the cobar complex $\Omega^* BP_*(X)$ (*cf.* [3]).

Consider the Smith-Toda spectrum $V(n)$ defined by the cofiber sequences

$$\begin{aligned}&S^0 \xrightarrow{p} S^0 \xrightarrow{i} V(0) \xrightarrow{j} S^1, \quad \Sigma^q V(0) \xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1}V(0),\\ &\Sigma^{(p+1)q}V(1) \xrightarrow{\beta} V(1) \xrightarrow{i_2} V(2) \xrightarrow{j_2} \Sigma^{(p+1)q+1}V(1) \quad \text{and}\\ &\Sigma^{(p^2+p+1)q}V(2) \xrightarrow{\gamma} V(2) \xrightarrow{i_3} V(3) \xrightarrow{j_3} \Sigma^{(p^2+p+1)q+1}V(2).\end{aligned} \tag{2.2}$$

Here, α, β and γ are the well known maps given in [1], [5] and [6] such that $BP_*(\alpha) = v_1$, $BP_*(\beta) = v_2$ and $BP_*(\gamma) = v_3$. Then, the Greek letter elements α_k, β_k and γ_k for $k > 0$ in the homotopy groups $\pi_*(S^0)$ are defined by

$$\alpha_k = j\alpha^k i, \quad \beta_k = jj_1\beta^k i_1 i \quad \text{and} \quad \gamma_k = jj_1j_2\gamma^k i_2 i_1 i. \tag{2.3}$$

The cofiber sequences (2.2) induce the long exact sequences

$$\begin{aligned}&\cdots \to E_2^{s,*}(S^0) \xrightarrow{p} E_2^{s,*}(S^0) \xrightarrow{i_*} E_2^{s,*}(V(0)) \xrightarrow{\delta} E_2^{s+1,*}(S^0) \to \cdots,\\ &\cdots \to E_2^{s,*}(V(0)) \xrightarrow{v_1} E_2^{s,*}(V(0)) \xrightarrow{(i_1)_*} E_2^{s,*}(V(1)) \xrightarrow{\delta_1} E_2^{s+1,*}(V(0)) \to \cdots, \quad \text{and}\\ &\cdots \to E_2^{s,*}(V(1)) \xrightarrow{v_2} E_2^{s,*}(V(1)) \xrightarrow{(i_2)_*} E_2^{s,*}(V(2)) \xrightarrow{\delta_2} E_2^{s+1,*}(V(1)) \to \cdots,\end{aligned}$$

of the E_2-terms associated to the short exact sequences of Γ-comodules

$$\begin{aligned}&0 \to BP_* \xrightarrow{p} BP_* \xrightarrow{i_*} BP_*(V(0)) \to 0,\\ &0 \to BP_*(V(0)) \xrightarrow{v_1} BP_*(V(0)) \xrightarrow{i_*} BP_*(V(1)) \to 0, \quad \text{and}\\ &0 \to BP_*(V(1)) \xrightarrow{v_2} BP_*(V(1)) \xrightarrow{i_*} BP_*(V(2)) \to 0,\end{aligned} \tag{2.4}$$

respectively. The Greek letter elements α_k, β_k, and γ_k for $k > 0$ in the E_2-term are defined by

$$\alpha_k = \delta(v_1^k) \quad \beta_k = \delta\delta_1(v_2^k) \quad \text{and} \quad \gamma_k = \delta\delta_1\delta_2(v_3^i) \in E_2^{*,*}(S^0). \tag{2.5}$$

We notice that the Geometric Boundary theorem (*cf.* [4, 2.3.4. Th.]) shows that the Greek letter elements of (2.5) in the E_2-term converge to those of (2.3) with the same name in the homotopy group $\pi_*(S^0)$.

Lemma 2.6 ([2, Lemma 4.3]). *The element γ_t is represented by the following*

$$\begin{aligned}&2\binom{t}{2}v_3^{t-2}\left(ct_2^p \otimes b_{11} + t_1^{p^2} \otimes b_{20} - t_1^{p^2} \otimes b_{10}\Delta(t_1^{p^2})\right)\\ &+3\binom{t}{3}v_3^{t-3}\langle t_1^{p^2}, t_1^{p^2}, t_1^{p^2}, k_0\rangle \quad \text{mod } (p, v_1, v_2)\Omega^3 BP_*.\end{aligned}$$

Here, $ct_2 = t_1^{p+1} - t_2$, $pb_{j,k} = \sum_{i=1}^{p-1}\binom{p}{i}t_j^{p^k i} \otimes t_j^{p^k(p-i)}$ and $k_0 = \langle t_1^p, t_1, t_1^p\rangle$.

In the proof of [2, Th. 4.4], Lee showed the following:

Theorem 2.7 ([2, Th. 4.4]). *$(\iota_2)_*(\alpha_1\beta_1^r\gamma_t)$ is nontrivial in $E_2^*(V(2))$ for $2 \le t < p$ and $r \le p-2$. Here, ι_2 denotes the inclusion $i_2 i_1 i\colon S^0 \to V(2)$ to the bottom cell.*

Consider the exact sequence

$$E_2^{s-1,t}(V(3)_{up}) \xrightarrow{\delta} E_2^{s,t-up(p^2+p+1)q}(V(2)) \xrightarrow{v_3^{up}} E_2^{s,t}(V(2)) \tag{2.8}$$

for $0 < u < p$ induced from the cofiber sequence $\Sigma^{up(p^2+p+1)q}V(2) \xrightarrow{\gamma^{up}} V(2) \to V(3)_{up}$.

Lemma 2.9. *Suppose that there are elements $\omega \in E_2^{2p-1,*}(V(3)_{up})$ and $v_3^s\xi \in E_2^{2p,*}(V(2))$ for $s + u < p$ such that $\xi \notin v_3 E_2^{2p,*}(V(2))$ and $\delta_3(\omega) = v_3^s\xi \neq 0$ in (2.8). Then the internal degree of ω is not less than that of $v_3^{up-1}v_4^{s+1}$.*

Proof. Let w denote the cocycle that represents ω. Suppose $|w| < |v_3^{up-1}v_4^{s+1}|$. Since $|v_3^{up-1}v_4^{s+1}| < |v_5|$, $w \in B^{2p-1}/(v_3^{up})$, where $\overline{B}^s = \mathbb{Z}/p[v_4]\otimes\mathbb{Z}/p[t_1,t_2,t_3,t_4]^{\otimes s}$ and $B^s = \mathbb{Z}/p[v_3]\otimes\overline{B}^s$. For a monomial $x = v_3^a v_4^{b_0} x_1 t_4^{b_1} \otimes \ldots \otimes x_{2p-1}t_4^{b_{2p-1}}$ of B^{2p-1} with $x_k \in \mathbb{Z}/p[t_1,t_2,t_3]$, we see that $d(x) = \sum_{k=0}^m v_3^{a+k}x'_k$ for $m = \max\{b_k : 0 \le k < 2p\}$ and $x'_k \in \mathbb{Z}/p[v_3^p]\otimes\overline{B}^{2p}$, since $d(v_4) = v_3 t_1^{p^3} - v_3^p t_1$ and $d(t_4) = -t_1\otimes t_3^p - t_2\otimes t_2^{p^2} - t_3\otimes t_1^{p^3} + v_3 b_{12}$. Consider a monomial x_0 appearing in w, and we have integer k such that $0 \le k \le m$ and $a+k = up+s$. Since $up > a$, we put $a' = up - a > 0$. Then, $a+m \ge a+k = up+s$ implies $m - a' \ge s$. Note that $|v_4| > |v_3^p|$, and we have $|w| = |x_0| \ge |v_3^a v_4^m| = |v_3^{up-a'}v_4^{s+a'}| > |v_3^{up-1}v_4^{s+1}|$. This is a contradiction. □

Proposition 2.10. *$(\iota_2)_*(\alpha_1\beta_1^{p-2}\gamma_{up+t})$ is nontrivial in $E_2^*(V(2))$ for integers t, u with $2 \le t+u < p$.*

Proof. First note that $(\iota_2)_*(\gamma_{up+t}) = v_3^{up}(\iota_2)_*(\gamma_t)$ in $E_2^{3,*}(V(2))$ by Lemma 2.6. Under Theorem 2.7, it suffices to show that the element $\xi = (\iota_2)_*(\alpha_1\beta_1^{p-2}\gamma_t)$ is not in the image of δ_3. By Lemma 2.6, $\xi = v_3^s\xi'$ ($s =$max$\{0, t-3\}$). If ξ is in the image of δ_3, then there exists a cochain $w \in \Omega^{2p-1,*}BP_*(V(3)_{up})$ such that $d(w) = v_3^{up}c = v_3^{up+s}c'$ for representatives c and c' of ξ and ξ', respectively. Furthermore $|w| \ge |v_3^{up-1}v_4^{s+1}|$ by Lemma 2.9. Since $|c| = ((t+1)p^2 + (t-3)p + t - 1)q < ((s+1)p^3 + sp^2 + sp + s)q = |v_3^{-1}v_4^{s+1}| \le |v_3^{-up}w|$ under our assumption, there is no cochain that is cobounded by $v_3^{up}c$. □

Proof of Theorem 1.1. We see that $\xi_u = \alpha_1\beta_1^r\gamma_{up+t} \in E_2^{2r+4,*}(S^0)$ is nontrivial if $r \le p-2$ by Proposition 2.10. Since every factor of ξ_u is a permanent cycle in the spectral sequence, so is ξ_u. Since the elements in the zeroth and the first lines are all permanent cycles, all elements in $E_2^{r,*}(S^0)$ for $r \le 2p$ bounds no element under the differential d_{2p-1}. □

References

1. J. F. Adams, On the groups $J(X)$ IV. Topology **5** (1966), 21–71.
2. Chun-Nip Lee, Detection of some elements in the stable homotopy groups of spheres, Math. Z. **222** (1996), 231-246.
3. H. R. Miller, D. C. Ravenel, and W. S. Wilson, Periodic phenomena in Adams-Novikov spectral sequence, Ann. of Math. **106** (1977), 469–516.
4. D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, AMS Chelsea Publishing, Providence, 2004.
5. L. Smith, On realizing complex bordism modules, IV, Applications to the stable homotopy groups of spheres, Amer. J. Math. **99** (1971), 418–436.
6. H. Toda, On spectra realizing exterior parts of the Steenrod algebra Topology **10**(1971), 53–65.

Department of Mathematics, Faculty of Science, Kochi University, Kochi, 780-8520, Japan

E-mail address: `katsumi@kochi-u.ac.jp`

Department of Mathematics, Faculty of Science, Kochi University, Kochi, 780-8520, Japan