Topological invariants of aperiodic tilings

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Mathematics of Quasi-Periodic Order

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Examples of Topological Invariants

You all know Euler's formula relating the number of faces, edges and vertices of a polyhedron:

$$n_f - n_e + n_v = 2$$

The number 2 is actually a topological invariant of the 2-sphere S^2. It is called the Euler characteristic $\chi(S^2)$.

A polyhedron represents a decomposition of S^2 into cells. A space composed of such cells is called a cell complex. χ does not depend on the decomposition that is chosen.

Homology of Finite Cell Complexes

Given a cell complex, we can consider formal linear combinations of k-cells, forming so-called chain groups C_k under addition. In the polyhedron case, we have $C_2 = \mathbb{Z}^{n_f}$, $C_1 = \mathbb{Z}^{n_e}$, $C_0 = \mathbb{Z}^{n_v}$.

There are natural boundary maps $\partial_k : C_k \rightarrow C_{k-1}$. The boundary of a k-cell is the sum of the cells in its boundary. This gives a sequence of groups and maps

$$0 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

The quotients $H_k = \ker(\partial_k)/\mathrm{im}(\partial_{k+1})$ are called homology groups, and are topological invariants of the cell complex. Their ranks $b_k = \mathrm{rk}(H_k)$ are called Betty numbers, and $\chi = \sum_k (-1)^k b_k$.

Cohomology of Finite Cell Complexes

For finite cell complexes, cohomology is almost the same as homology.

We consider formal linear combinations of k-cells, forming this time so-called co-chain groups C^k under addition. In the polyhedron case, we again have $C^2 = \mathbb{Z}^{n_f}$, $C^1 = \mathbb{Z}^{n_e}$, $C^0 = \mathbb{Z}^{n_v}$.

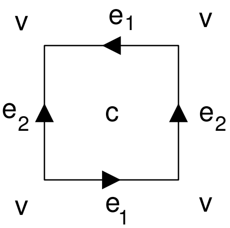
The natural maps between co-chain groups are the co-boundary maps $\delta_k : C^{k-1} \to C^k$. δ_k is simply the transpose of ∂_k.

We now have a sequence of groups and maps

$$0 \xleftarrow{\delta_3} C_2 \xleftarrow{\delta_2} C_1 \xleftarrow{\delta_1} C_0 \xleftarrow{\delta_0} 0$$

The quotients $H^k = \ker(\delta_{k+1})/\mathrm{im}(\delta_k)$ are called co-homology groups, and are topological invariants of the cell complex.

Klein's Bottle and Torsion



For this cell complex, we have $\partial_2 c = 2e_1$, and $\partial_1 \equiv 0$. Thus, we get

$$H_1 = \ker(\partial_1)/\mathrm{im}(\partial_2) = \mathbb{Z}^2/2\mathbb{Z}$$

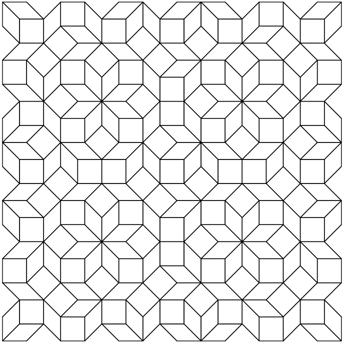
$$= \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_2$$

H_1 contains torsion elements - elements of finite order.

In cohomology, the torsion appears in a different dimension:

$$H^2 = \mathbb{Z}_2, \quad H^1 = H^0 = \mathbb{Z}$$

Properties of Tilings



- finite number of local patterns
  (finite local complexity)
- repetitivity
- well-defined patch frequencies
- translation module
- local isomorphism
  (LI classes)
- mutual local derivability

The Hull of a Tiling

Let $\mathcal{T}$ be a tiling of $\mathbb{R}^d$, of finite local complexity.

We introduce a metric on the set of translates of $\mathcal{T}$:

Two tilings have distance $< \epsilon$, if they agree in a ball of radius $1/\epsilon$ around the origin, up to a translation $< \epsilon$.

The hull $\Omega_{\mathcal{T}}$ is then the closure of $\{\mathcal{T} - x | x \in \mathbb{R}^d\}$.

$\Omega_{\mathcal{T}}$ is a compact metric space, on which $\mathbb{R}^d$ acts by translation.

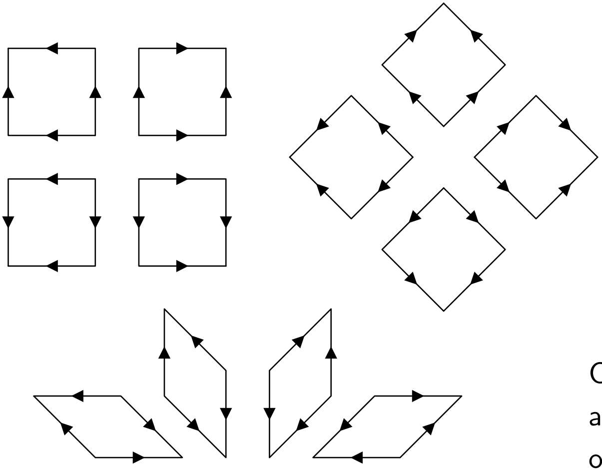
If $\mathcal{T}$ is repetitive, every orbit is dense in $\Omega_{\mathcal{T}}$.

$\Omega_{\mathcal{T}}$ then consists of the LI class of $\mathcal{T}$.

Approximating the Hull by Cell Complexes

We define a sequence of cellular (CW-)spaces Ω_n approximating Ω.

The d-cells of Ω_0 are the interiors of the tiles; two tile boundaries are identified if they are shared somewhere in the tiling.



Cells of first order approximant of the octagonal tiling.

Approximating the Hull by Cell Complexes

We define a sequence of cellular (CW-)spaces Ω_n approximating Ω.

The d-cells of Ω_0 are the interiors of the tiles; two tile boundaries are identified if they are shared somewhere in the tiling.

For Ω_n we proceed as for Ω_0, except that we first label the tiles according to their n^{th} corona (collared tiles).

There are natural, continuous cellular mappings $h : \Omega_n \to \Omega_{n-1}$, and induced homomorphisms $h_* : H^*(\Omega_{n-1}) \to H^*(\Omega_n)$.

Ω then is the inverse limit $\varprojlim \Omega_n$, consisting of all sequences $\{x_k\}_{k=0}^{\infty}$, with $x_k \in \Omega_k$ and $h(x_k) = x_{k-1}$.

The cohomology of Ω is the direct limit $H^*(\Omega) \cong \varinjlim H^*(\Omega_n)$

Cohomology of Substitution Tilings

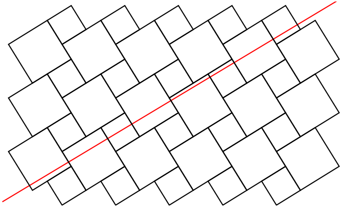
The appromimants Ω_n of the hull were introduced by Anderson and Putnam (AP), Ergod. Th. & Dynam. Sys. 18, 509 (1998).

They used a single CW-space Ω' and the mapping $\Omega' \to \Omega'$ induced by substitution, and take the inverse limit of the iterated mapping. This is equivalent to iterated refinements according to the n^{th} corona, for some n.

This inverse limit using a single Ω_n is easier to control, but is limited to substitution tilings.

Using a sequence of Ω_n is more general, but the limit is hard to control. However, the approach may be of conceptual interest.

Quasiperiodic Projection Tilings



Irrational sections through a periodic klotz tiling.

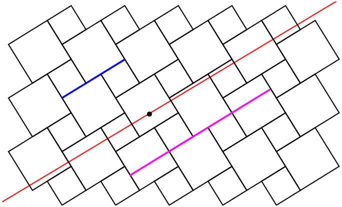
We assume polyhedral acceptance domains with rationally oriented faces.

Such tilings are called canonical projection tilings.

Forrest-Hunton-Kellendonk computed their cohomology for low co-dimensions in terms of acceptance domains.

Here, we shall use a different approach.

Kalugin's Approach



Irrational sections through a periodic klotz tiling.

Disregarding singular cut positions, points in unit cell parametrize tilings.

For proper parametrisation, torus has to be cut up.

This is done is steps $\longrightarrow$ inverse limit construction!

Cohomology of n-torus cut up along set A_r satisfies

$$\longrightarrow H^k(\Omega_r) \longrightarrow H_{n-k-1}(A_r) \longrightarrow H_{n-k-1}(\mathbb{T}^n) \longrightarrow H^{k+1}(\Omega_r) \longrightarrow$$

P. Kalugin, J. Phys. A: Math. Gen. **38**, 3115 (2005).

Simplifying the Set of Cuts

$H_*(A_r)$ and thus $H^*(\Omega_r)$ depends only on homotopy type of A_r.

We assume polyhedral acceptance domains with rationally oriented faces

$\longrightarrow$ with increasing r, pieces of A_r grow together.

For r sufficiently large, A_r is a union of thickened affine tori.

Homotopy type of A_r stabilizes at finite r_0!

Often, we can replace A_r by equivalent arrangement $\tilde{A}$ of thin tori.

For computing $H_*(\tilde{A})$: replace $\tilde{A}$ by its simplicial resolution, A.

For icosahedral tilings, $\tilde{A}$ consists of 4-tori, intersecting in 2-tori and 0-tori.

For codimension-2 tilings, there are only 2-tori and 0-tori.

Kalugins Exact Sequences – 2D Case

Kalugin's long exact sequence can be split; for tilings of dimension 2 and co-dimension 2, it reads:

$$0 \longrightarrow S_k \longrightarrow H^k(\Omega) \longrightarrow H_{4-k-1}(A) \xrightarrow{\alpha^{k+1}} H_{4-k-1}(\mathbb{T}^6) \longrightarrow S_{k+1} \longrightarrow 0$$

$$\begin{array}{ccccccccccccc}
0 & \rightarrow & H_4(\mathbb{T}^4) & \rightarrow & H^0(\Omega) & \rightarrow & 0 & \rightarrow & H_3(\mathbb{T}^4) & \rightarrow & S_1 & \rightarrow & 0 \\
0 & \rightarrow & H_3(\mathbb{T}^4) & \rightarrow & H^1(\Omega) & \rightarrow & H_2(A) & \rightarrow & H_2(\mathbb{T}^4) & \rightarrow & S_2 & \rightarrow & 0 \\
0 & \rightarrow & S_2 & \rightarrow & H^2(\Omega) & \rightarrow & H_1(A) & \rightarrow & H_1(\mathbb{T}^4) & \rightarrow & 0 & & \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & H_0(A) & \rightarrow & H_0(\mathbb{T}^4) & \rightarrow & 0 & &
\end{array}$$

We need to determine $H_*(\mathbb{T}^4)$, $H_*(A)$, $S_k = \operatorname{coker} \alpha^k$, and derive $H^*(\Omega)$ from that.

Mayer-Vietoris Spectral Sequence

First page $E^1_{k,\ell}$ of Mayer-Vietoris double complex for $H_*(A)$:

$\oplus_{\theta\in I_1}\Lambda_2\Gamma^\theta$	
$\oplus_{\theta\in I_1}\Lambda_1\Gamma^\theta$	
$\mathbb{Z}^{L_1}\bigoplus\mathbb{Z}^{L_0}$	$\oplus_{\theta\in I_1}\mathbb{Z}^{L_0^\theta}$

As A is connected, the only differential left has rank $L_1 + L_0 - 1$, so that we get:

$$H_0(A) = \mathbb{Z}$$

$$H_1(A) = \oplus_{\theta\in I_1}\Lambda_1\Gamma^\theta \oplus \mathbb{Z}^f$$

$$H_2(A) = \oplus_{\theta\in I_1}\Lambda_2\Gamma^\theta$$

where $f = \sum_{\theta\in I_1} L_0^\theta - L_1 - L_0 + 1$.

Cohomology of the Hull

Kalugins exact sequences can now be solved:

$$H^0(\Omega) = \mathbb{Z}$$
$$H^1(\Omega) = \Lambda_3\Gamma \oplus \ker \alpha^2$$
$$H^2(\Omega) = \Lambda_2\Gamma / \langle \Lambda_2\Gamma^\theta \rangle_{\theta \in I_1} \oplus \ker \alpha^3$$

The $\ker \alpha^k$ are free groups, whose ranks are computable.

Torsion can only occur in $\operatorname{coker} \alpha^2 = \Lambda_2\Gamma / \langle \Lambda_2\Gamma^\theta \rangle_{\theta \in I_1}$.

Geometrically, $\ker \alpha^k$ consists of closed k-chains which are non-trivial in $H_k(A)$, but are exact in the full torus. Thus, they are boundaries of $(k+1)$-chains of $\mathbb{T}^4$.

Examples

Cohomology of some 2D tilings from the literature:

H^2	H^1	H^0	χ	lines	name
$\mathbb{Z}^8$	$\mathbb{Z}^5$	$\mathbb{Z}$	4	along	Penrose
$\mathbb{Z}^{24} \oplus \mathbb{Z}_5^2$	$\mathbb{Z}^5$	$\mathbb{Z}$	20	between	Tübingen Triangle
$\mathbb{Z}^9$	$\mathbb{Z}^5$	$\mathbb{Z}$	5	along	Ammann-Beenker
$\mathbb{Z}^{14} \oplus \mathbb{Z}_2$	$\mathbb{Z}^5$	$\mathbb{Z}$	10	between	colored Ammann-Beenker
$\mathbb{Z}^{28}$	$\mathbb{Z}^7$	$\mathbb{Z}$	22	along/between	Shield, Socolar

The 3D Case

Similar to the 2D case, except that Kalugin's exact sequences are much more difficult to solve.

In particular, this is so for the torsion part. Only some examples could be solved; for the general case, some extra ideas are required.

$$0 \rightarrow S_k \rightarrow H^k(\Omega) \rightarrow H_{6-k-1}(A) \xrightarrow{\alpha^{k+1}} H_{6-k-1}(\mathbb{T}^6) \rightarrow S_{k+1} \rightarrow 0$$

In all icosahedral examples, we have torsion in $H_2(A)$, and may have torsion in S_3. This leads to group extension problems.

F. Gähler, J. Hunton, J. Kellendonk, Z. Kristallogr. 223, 801-804 (2009).

Icosahedral Examples

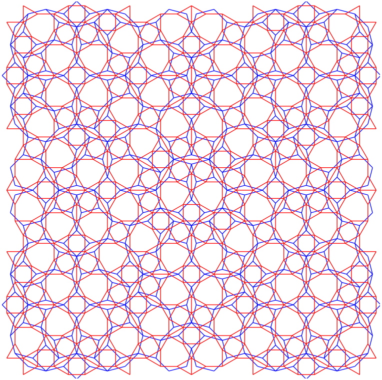
Cohomology of some icosahedral tilings from the literature:

H^3	H^2	H^1	H^0	χ	planes	Γ	
$\mathbb{Z}^{20} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{16}$	$\mathbb{Z}^7$	$\mathbb{Z}$	10	5-fold	F	Danzer
$\mathbb{Z}^{181} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{72} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{12}$	$\mathbb{Z}$	120	mirror	P	Ammann-Kramer
$\mathbb{Z}^{331} \oplus \mathbb{Z}_2^{20} \oplus \mathbb{Z}_4$	$\mathbb{Z}^{102} \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_4$	$\mathbb{Z}^{12}$	$\mathbb{Z}$	240	mirror	F	dual can. D_6
$\mathbb{Z}^{205} \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^{72}$	$\mathbb{Z}^7$	$\mathbb{Z}$	145	3,5-fold	F	canonical D_6

Even the simplest of all icosahedral tilings have torsion!

Formulae have to be evaluated by computer (GAP programs). Combinatorics of intersection tori are determined with (descendants of) programs from the GAP package Cryst (B. Eick, F. Gähler, W. Nickel, Acta Cryst. A53, 467-474 (1997)).

Mutual Local Derivability



One tiling must be locally constructible from the other, and vice versa.

Tilings must have same translation module.

Acceptance domains of one tiling must be constructible by finite unions and intersections of acceptance domains of the other.

MLD induces a bijection between LI classes.

MLD Classification

Both cohomology and MLD class are determined by the arrangement of singular spaces A, and how the lattice Γ acts on it.

To fix an MLD class, we fix a space group and orbit representatives of the singular spaces.

To make MLD classification finite, we consider

- singular spaces in special orientations
- restricted number of orbits
- some non-genericity condition, like
  - closeness condition
  - existence of non-generic intersections
  - singular spaces pass through special points

MLD Relationships

We fix a space group, and compare different singular sets A, generated from "interesting" orbit representatives. Different singular sets may define the same MLD class!

Singular sets may be related by translation, or by inflation. These are local transformations, and so they define same MLD class.

There are also non-local transformations normalizing the space group, like the *-map. This leads to an MD relationship, but not to MLD!

The full translation symmetry $\tilde{\Gamma}$ of the singular set may be larger than the translation symmtery Γ of the tiling.

MLD relationship may be symmetry-preserving (S-MLD) or not. MLD by translation is symmetry-preserving only of translation normalizes the space group.

Cohomology of Octagonal MLD Classes

H^2	H^1	H^0	χ	lines	$\lvert\tilde{\Gamma}/\Gamma\rvert$	mult	cc	gen	remarks
$\mathbb{Z}^9$	$\mathbb{Z}^5$	$\mathbb{Z}$	5	4A	1	2 (tr)	×		Ammann-Beenker
$\mathbb{Z}^{12}$	$\mathbb{Z}^5$	$\mathbb{Z}$	8	4A	1	2 (tr,inf)	×	×	
$\mathbb{Z}^{28}$	$\mathbb{Z}^9$	$\mathbb{Z}$	20	4A+4A	4	2 (tr)	×		1)
$\mathbb{Z}^{33}$	$\mathbb{Z}^9$	$\mathbb{Z}$	25	4A+4A	1	4 (tr,inf)	×		
$\mathbb{Z}^{40}$	$\mathbb{Z}^9$	$\mathbb{Z}$	32	8A	1	∞		×	
$\mathbb{Z}^{14} \oplus \mathbb{Z}_2$	$\mathbb{Z}^5$	$\mathbb{Z}$	10	4B	2	2 (tr)	×		1)
$\mathbb{Z}^{20} \oplus \mathbb{Z}_2$	$\mathbb{Z}^5$	$\mathbb{Z}$	16	4B	2	2 (tr,inf)	×	×	
$\mathbb{Z}^{48} \oplus \mathbb{Z}_2$	$\mathbb{Z}^9$	$\mathbb{Z}$	40	4B+4B	8	2 (tr)	×		1)
$\mathbb{Z}^{58} \oplus \mathbb{Z}_2$	$\mathbb{Z}^9$	$\mathbb{Z}$	50	4B+4B	2	4 (tr,inf)	×		
$\mathbb{Z}^{72} \oplus \mathbb{Z}_2$	$\mathbb{Z}^9$	$\mathbb{Z}$	64	8B	2	∞		×	
$\mathbb{Z}^{23}$	$\mathbb{Z}^8$	$\mathbb{Z}$	16	4A+4B	1	2 (tr)	×		decorated Ammann-Beenker
$\mathbb{Z}^{24}$	$\mathbb{Z}^8$	$\mathbb{Z}$	17	4A+4B	1	2 (tr)	×		
$\mathbb{Z}^{29}$	$\mathbb{Z}^8$	$\mathbb{Z}$	22	4A+4B	1	2 (tr,inf)	×		2)
$\mathbb{Z}^{29}$	$\mathbb{Z}^8$	$\mathbb{Z}$	22	4A+4B	1	2 (tr,inf)	×		2)
$\mathbb{Z}^{35}$	$\mathbb{Z}^8$	$\mathbb{Z}$	28	4A+4B	1	4 (tr,inf)	×		
$\mathbb{Z}^{36}$	$\mathbb{Z}^8$	$\mathbb{Z}$	29	4A+4B	1	4 (tr,inf)	×		

1) MLD class splits in two S-MLD classes   2) inequivalent, different combinatorics

Cohomology of Decagonal MLD Classes

H^2	H^1	H^0	χ	lines	$\lvert\tilde{\Gamma}/\Gamma\rvert$	mult	cc	gen	remarks
$\mathbb{Z}^8$	$\mathbb{Z}^5$	$\mathbb{Z}$	4	5A	1	1	×		Penrose
$\mathbb{Z}^{14}$	$\mathbb{Z}^5$	$\mathbb{Z}$	10	5A	1	3 (inf)	×	×	
$\mathbb{Z}^{33}$	$\mathbb{Z}^{10}$	$\mathbb{Z}$	24	5A+5A	1	3 (inf)	×		
$\mathbb{Z}^{34}$	$\mathbb{Z}^{10}$	$\mathbb{Z}$	25	5A+5A	1	3 (inf)	×		gen. Penrose (γ=1/2)
$\mathbb{Z}^{37}$	$\mathbb{Z}^{10}$	$\mathbb{Z}$	28	10A	1	2 (inf)			
$\mathbb{Z}^{49}$	$\mathbb{Z}^{10}$	$\mathbb{Z}$	40	10A	1	∞		×	
$\mathbb{Z}^{24} \oplus \mathbb{Z}_5^2$	$\mathbb{Z}^5$	$\mathbb{Z}$	20	5B	5	1	×		Tübingen Triangle
$\mathbb{Z}^{54} \oplus \mathbb{Z}_5^2$	$\mathbb{Z}^5$	$\mathbb{Z}$	50	5B	5	3 (inf)	×	×	
$\mathbb{Z}^{129} \oplus \mathbb{Z}_5^2$	$\mathbb{Z}^{10}$	$\mathbb{Z}$	120	5B+5B	5	3 (inf)	×		
$\mathbb{Z}^{134} \oplus \mathbb{Z}_5^2$	$\mathbb{Z}^{10}$	$\mathbb{Z}$	125	5B+5B	5	3 (inf)	×		
$\mathbb{Z}^{149} \oplus \mathbb{Z}_5^2$	$\mathbb{Z}^{10}$	$\mathbb{Z}$	140	10B	5	2 (inf)			
$\mathbb{Z}^{209} \oplus \mathbb{Z}_5^2$	$\mathbb{Z}^{10}$	$\mathbb{Z}$	200	10B	5	∞		×	
$\mathbb{Z}^{49}$	$\mathbb{Z}^{10}$	$\mathbb{Z}$	40	5A+5B	1	1	×		
$\mathbb{Z}^{69}$	$\mathbb{Z}^{10}$	$\mathbb{Z}$	60	5A+5B	1	3 (inf)	×		
$\mathbb{Z}^{79}$	$\mathbb{Z}^{10}$	$\mathbb{Z}$	70	5A+5B	1	3 (inf)	×		
$\mathbb{Z}^{93}$	$\mathbb{Z}^{10}$	$\mathbb{Z}$	84	5A+5B	1	3 (inf)	×		
$\mathbb{Z}^{94}$	$\mathbb{Z}^{10}$	$\mathbb{Z}$	85	5A+5B	1	3 (inf)	×		1)
$\mathbb{Z}^{94}$	$\mathbb{Z}^{10}$	$\mathbb{Z}$	85	5A+5B	1	3 (inf)	×		1)

1) swapped by *-map, which exchanges physical and internal space (non-local equivalence)

Cohomology of Dodecagonal MLD Classes

H^2	H^1	H^0	χ	lines	$\|\tilde{\Gamma}/\Gamma\|$	mult	cc	gen	remarks
$\mathbb{Z}^{28}$	$\mathbb{Z}^{7}$	$\mathbb{Z}$	22	6A	1	1	x		Socolar tiling
$\mathbb{Z}^{33}$	$\mathbb{Z}^{7}$	$\mathbb{Z}$	27	6A	1	1	x		
$\mathbb{Z}^{42}$	$\mathbb{Z}^{7}$	$\mathbb{Z}$	36	6A	1	2 (inf)	x	x	
$\mathbb{Z}^{100}$	$\mathbb{Z}^{13}$	$\mathbb{Z}$	88	6A+6A	4	1	x		
$\mathbb{Z}^{112}$	$\mathbb{Z}^{13}$	$\mathbb{Z}$	100	6A+6A	1	2 (inf)	x		1)
$\mathbb{Z}^{120}$	$\mathbb{Z}^{13}$	$\mathbb{Z}$	108	6A+6A	4	1	x		2)
$\mathbb{Z}^{129}$	$\mathbb{Z}^{13}$	$\mathbb{Z}$	117	6A+6A	1	2 (inf)	x		
$\mathbb{Z}^{112}$	$\mathbb{Z}^{13}$	$\mathbb{Z}$	100	12A	1	2 (inf)			1)
$\mathbb{Z}^{120}$	$\mathbb{Z}^{13}$	$\mathbb{Z}$	108	12A	1	2 (inf)			2)
$\mathbb{Z}^{144}$	$\mathbb{Z}^{13}$	$\mathbb{Z}$	132	12A	1	6 (inf)			
$\mathbb{Z}^{156}$	$\mathbb{Z}^{13}$	$\mathbb{Z}$	144	12A	1	∞		x	
$\mathbb{Z}^{59}$	$\mathbb{Z}^{12}$	$\mathbb{Z}$	48	6A+6B	1	1	x		decorated Socolar tiling
$\mathbb{Z}^{68}$	$\mathbb{Z}^{12}$	$\mathbb{Z}$	57	6A+6B	1	1	x		
$\mathbb{Z}^{69}$	$\mathbb{Z}^{12}$	$\mathbb{Z}$	58	6A+6B	1	2 (inf)	x		
$\mathbb{Z}^{87}$	$\mathbb{Z}^{12}$	$\mathbb{Z}$	76	6A+6B	1	4 (inf)	x		
$\mathbb{Z}^{92}$	$\mathbb{Z}^{12}$	$\mathbb{Z}$	81	6A+6B	1	4 (inf)	x		
$\mathbb{Z}^{95}$	$\mathbb{Z}^{12}$	$\mathbb{Z}$	84	6A+6B	1	4 (inf)	x		

1) not equivalent 2) not equivalent