Some observations on a substitution rule with singular vertex configrations

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1. Preliminary

We recall several basic definitions.

A patch of a tiling is a finite subset of the tiles in a tiling. A vertex configuration of a vertex in a tiling is the patch consisting of all tiles which contains the vertex.

Let $\mathcal{S} = \{S_1, S_2, \cdots, S_l\}$ be a finite set of polygons S_j. When each tile T in a tiling $\mathcal{T}$ is congruent to some $S_i \in \mathcal{S}$, $\mathcal{S}$ is called a prototile set of $\mathcal{T}$.

A set of matching rules for a prototile set $\mathcal{S}$ is a finite set of patches that may appear in the tilings admitted by $\mathcal{S}$.

Fix $\lambda(> 1)$. For $\mathcal{S} = \{S_1, S_2, \cdots, S_l\}$, any S_k $(\in \mathcal{S})$ is decomposed into λ^{-1} scale-down copies $\lambda^{-1}\mathcal{S} = \{\lambda^{-1}S_1, \lambda^{-1}S_2, \cdots, \lambda^{-1}S_l\}$ of $\mathcal{S}$. This decomposition is called a substitution rule of $\mathcal{S}$ if such a decomposition is possible. Let Φ denote a substitution rule of $\mathcal{S}$, and $\Phi[S_k]$ denote the decomposition of $S_k \in \mathcal{S}$. Similarly, let $\sigma(S_k)$ denote the decomposition of λS_k into $\mathcal{S}$. A patch in a tiling corresponding to $\sigma^n(S_k)$ is called a supertile with level n of S_k.

A substitution is primitive if the substitution rule is a linear map that can be represented by a primitive matrix. An $n \times n$ matrix A is said to be primitive if its entries are nonnegative integers and if there exists a positive integer k such that all the entries of A^k are positive.

A substitution is said to force the border if there exists a positive integer n such that any two level n supertiles of the same type have the same pattern of neighboring tiles.

The predecessor P for a tiling T under a substitution σ is a tiling such that $\sigma(P) = T$. That is the tiling that substitutes to T. A substitution tiling T is a tiling that has a infinite sequence of predecessor tilings P_n such that $\sigma^n(P_n) = T$. A singular vertex configuration is one in a substitution tiling which does not appear in supertiles.

The prototiles of Penrose tilings are four types of triangles with arrows on the edges (two triangles a, b in Figure 1 and their mirror images). The substitution rule of prototiles is given as in Figure 2 and their mirror images ([1]).

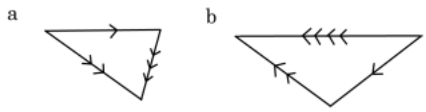

Figure 1: Two prototiles with arrows of Penrose tilings

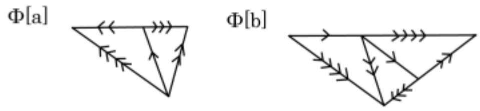

Figure 2: The substitution rule of Penrose tilings

The prototiles of Danzer tiling are six types of triangles with arrows on the edges (three triangles a, b, c in Figure 3 and their reflections). The substitution rule of prototiles is given as in Figure 4 and their mirror images ([5]).

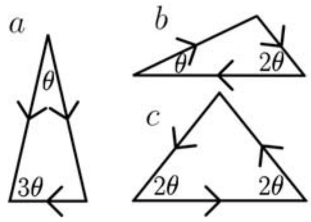

Figure 3: Three prototiles with arrows of Danzer tiling ($\theta = \pi/7$)

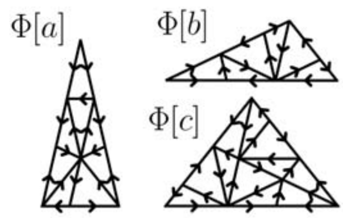

Figure 4: The substitution rule of Danzer tiling

One of the purpose of this note is to analyze construction of tilings with rotational symmetry obtained by substitution rules (See [4]). We can construct nonperiodic tilings with a given prototile set by the up-down generation using substitution rule (cf.[1]). The whole plane cannot always be covered by the up-down generation. In particular, Penrose tiling and Danzer tiling with rotational symmetry cannot be constructed only by the up-down generation procedure. It is necessary to extend the tilings to the whole plane by arranging unbounded configurations obtained by the up-down generation procedure.

We have the following questions:

(1) For which n, can tilings with n-fold symmetry be constructed only by the up-down generation procedure ?

(2) To construct tilings, how can we arrange unbounded configurations obtained by the up-down generation procedure ?

It seems that theses questions are open.

2. Some observations

2.1. Question (1)

If we do not need primitivity, we have the following answer for Question (1).

"If there is a substitution with n-fold tiling, we can make a substitution that can generate an n-fold tiling from up-down generation."

Substitutions with n-fold tiling are constructed in [2] and [5]. In order to present this answer, we use Penrose tiling as an example. In Figure 5 above four triangle tiles are the original Penrose tiles. We add two tiles of star shape. These are two shapes of vertex configurations at the center of 5-fold symmetry. We see that this substitution is primitive.

By the similar way, we can make a substitution from the substitution of Danzer tiling. This new substitution is not primitive, because the vertex configuration at the center of 7-fold symmetry never appear in any supertile of triangle tiles (cf.[3]).

So we have the following new question:

(1)' For which n, can tilings with n-fold symmetry be constructed only by the up-down generation procedure of some primitive substitution?

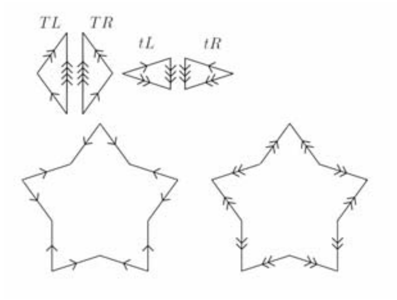

Figure 5: Prototile

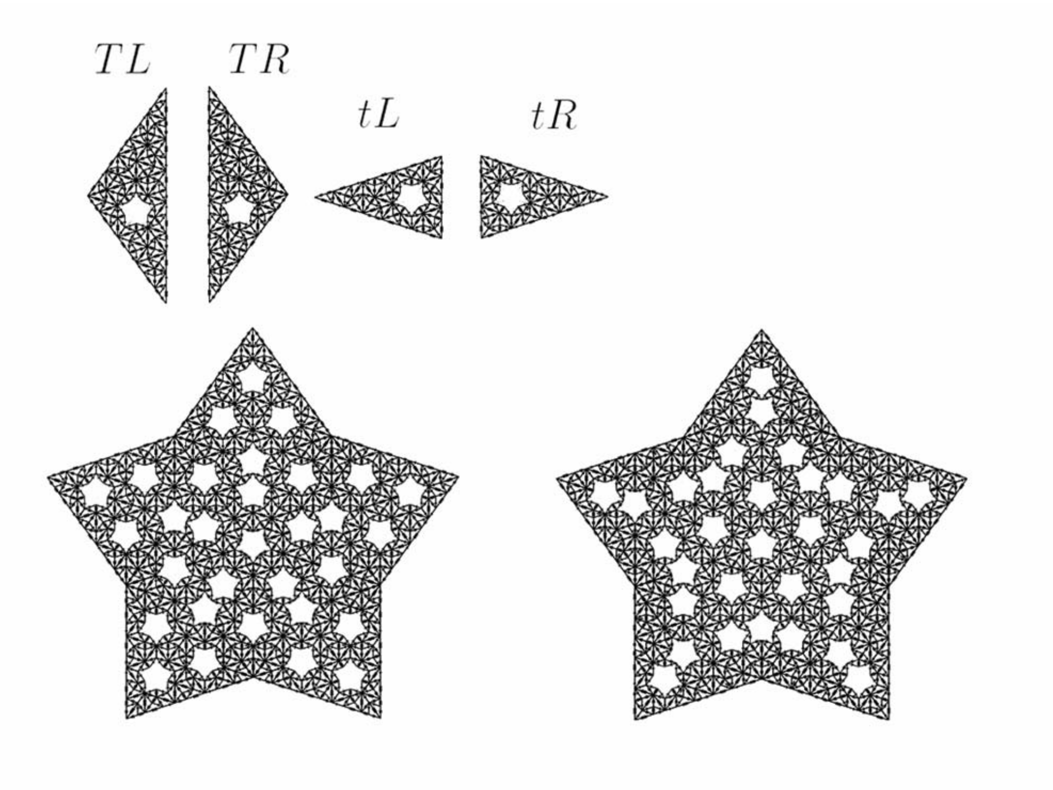

Figure 6: Substitution

2.1. Question (2)

As is known, the substitution of Penrose tilings forces the border. So the arrangement of unbouded configurations is unique in the case of Penrose tilings.

We find some singular vertex configurations in Danzer tilings(There is no singular vertex configuration in the substitution of Penrose tilings). (i), (ii) of Figure 7 is singular vertex configurations. These do not exist in the list of 39 vertex configurations which appear in supertiles (cf.[3]).

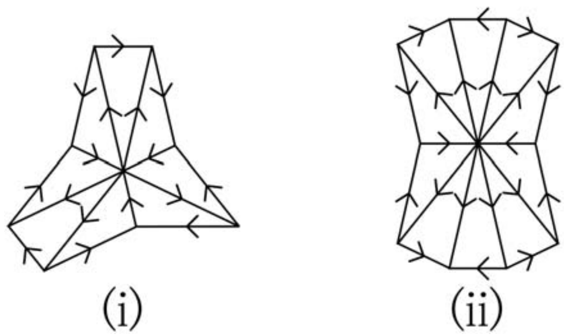

Figure 7: Vertex configurations (i), (ii)

When we subdivides vertex configuration (i) (resp. (ii)) and rescaling, we have patch (I) (resp. (II)) in Figure 8 (resp. 9) which includes (i) (resp. (ii)). By repeating subdivision and rescaling, we get a tiling with vertex configuration (i) (resp. (ii)). Vertex configuration (i) (resp. (ii)) appears only in one place, and other vertex configurations are in 39 vertex configurations. In addition, this tiling is invariant under the substitution.

Note that the above results remains true even if some of a's and reflections of a in vertex configurations (i), (ii) are changed as you like.

We observe the tiling with only one vertex configuration (i). Then we can see that there are different neighboring tiles in supertiles of prototile a type. Hence the substitution of Danzer tilings doesn't force the border. The arrangement of unbouded configurations is not unique in the case of Danzer tilings,

So we have the following new question:

(2)' Nonexistence of singular vertex configurations is the necessary and sufficient condition for that a substitution forces the border.

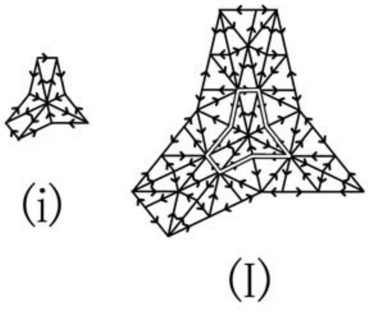

Figure 8: The Patch (I)

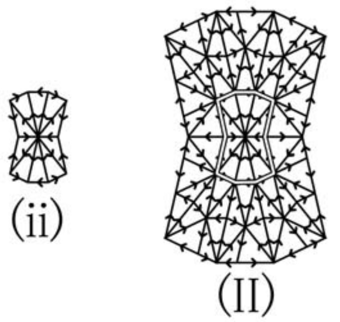

Figure 9: The Patch (II)

References

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