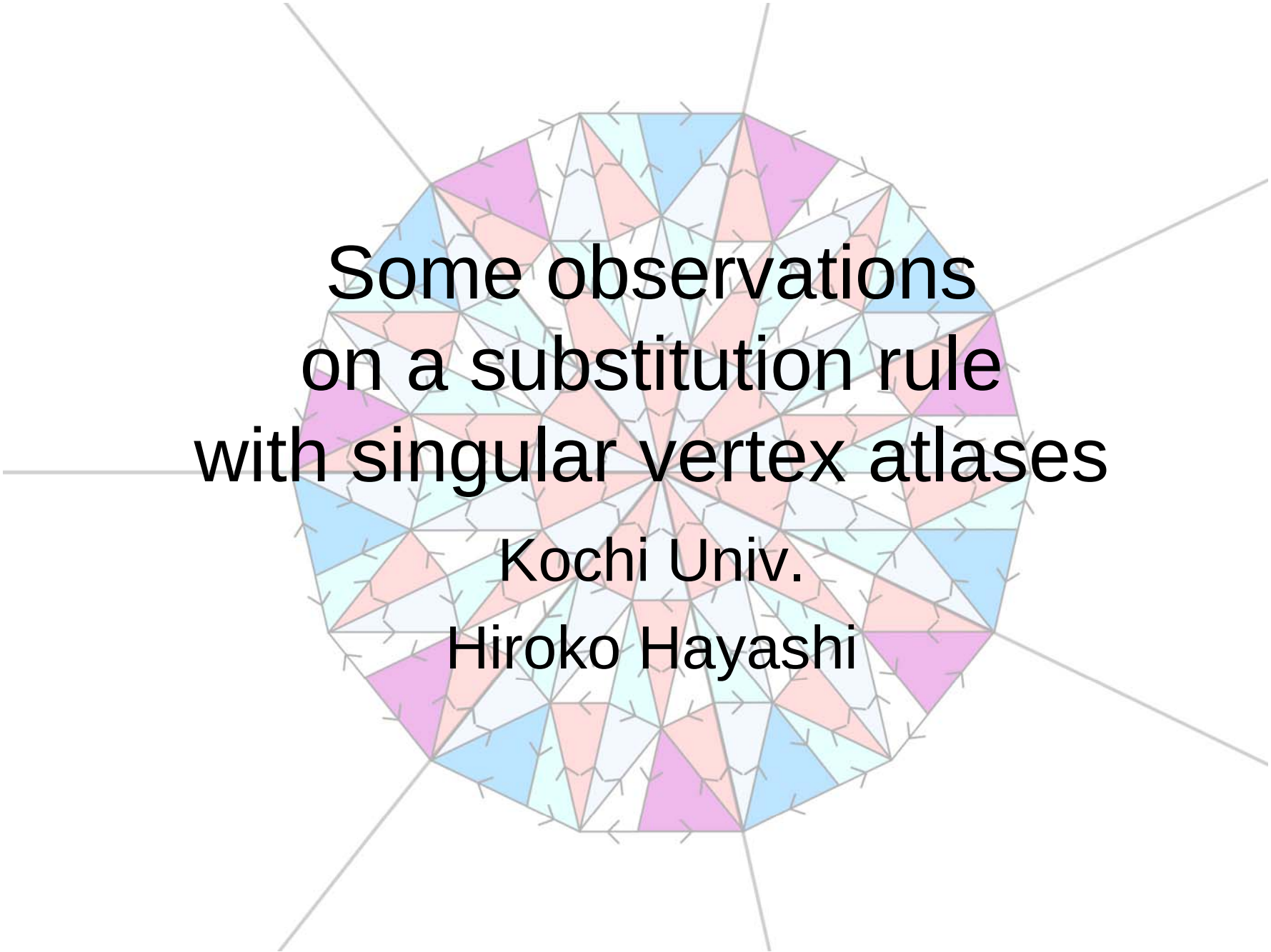


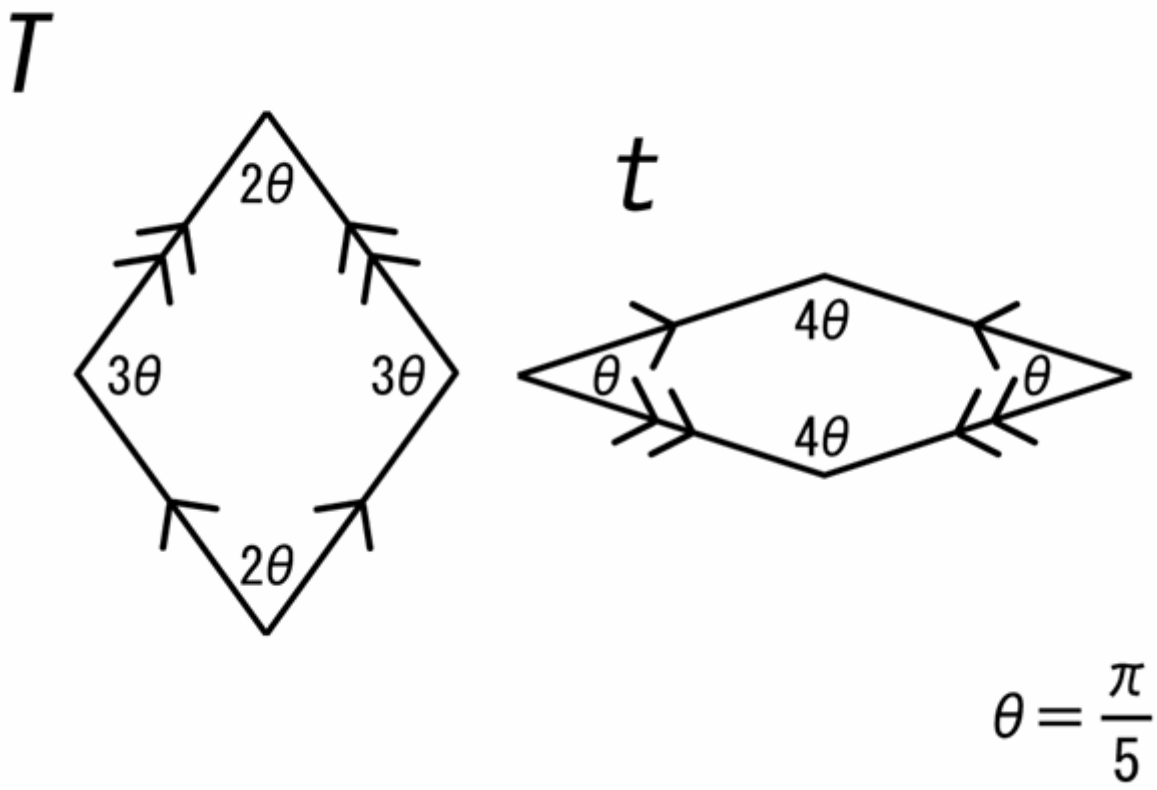
Some observations on a substitution rule with singular vertex atlases

Kochi Univ.

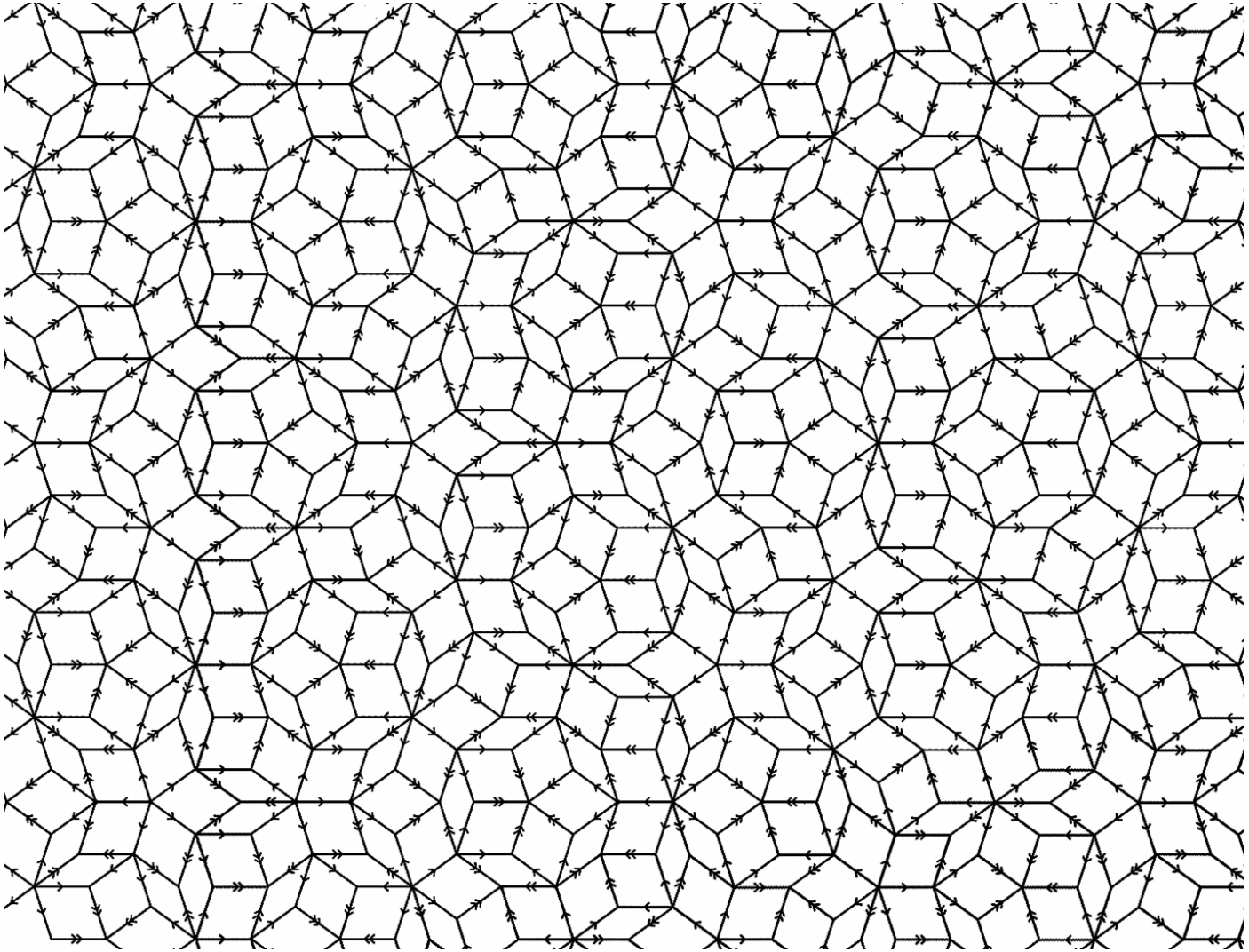
Hiroko Hayashi

【1】Preliminary

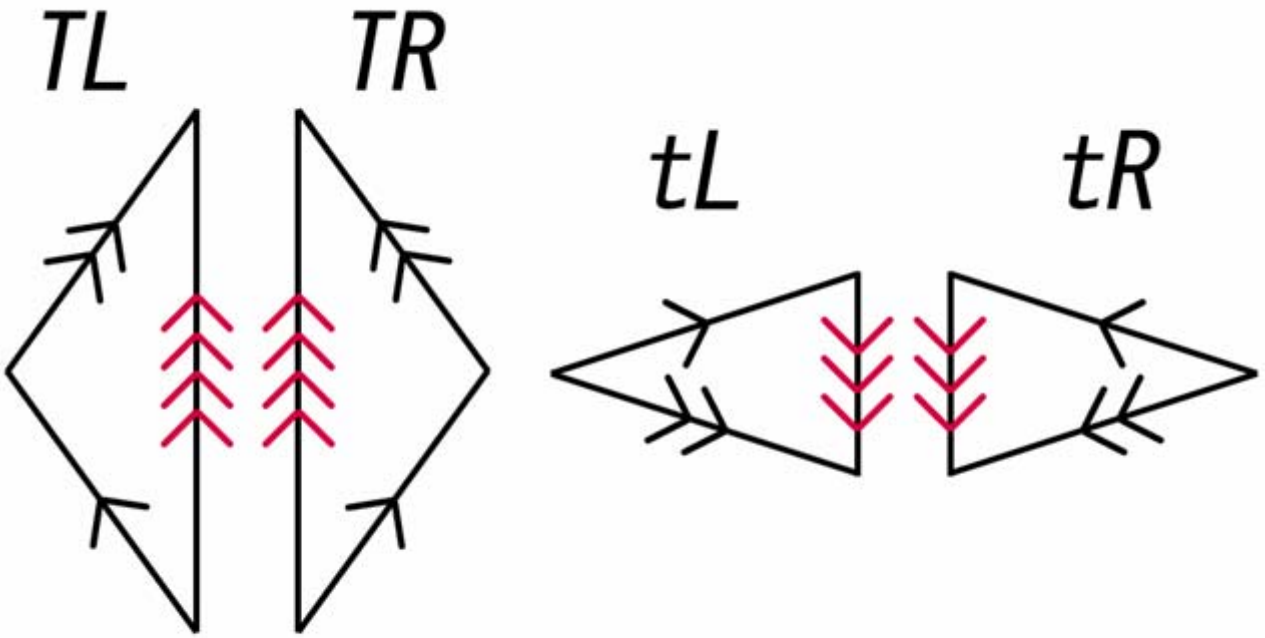
Penrose tiles



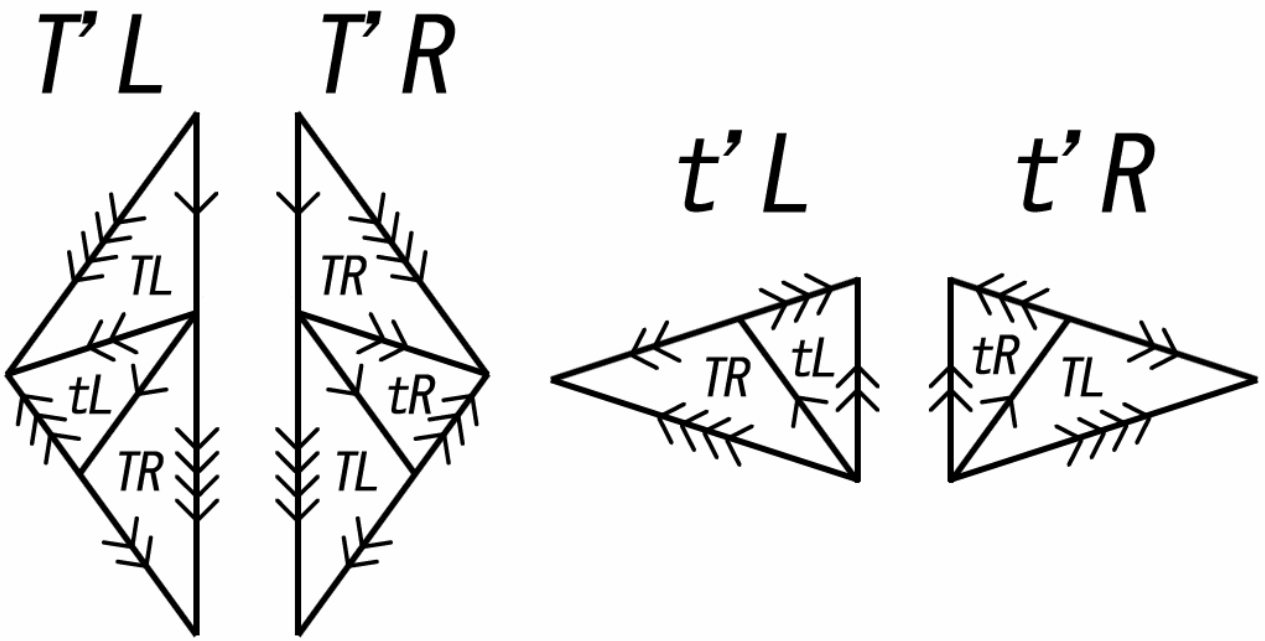
Penrose tiling



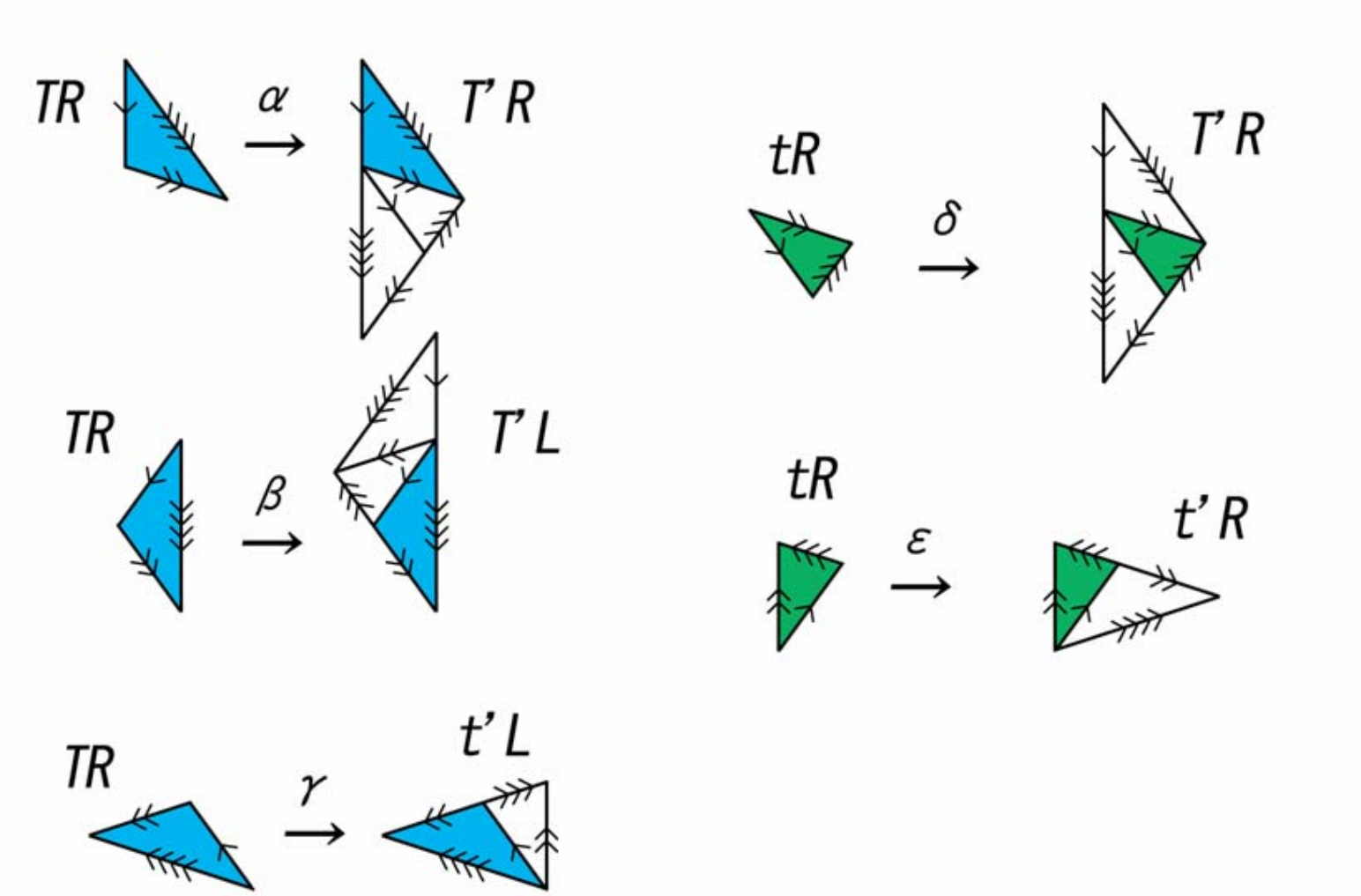
Divided Penrose tiles



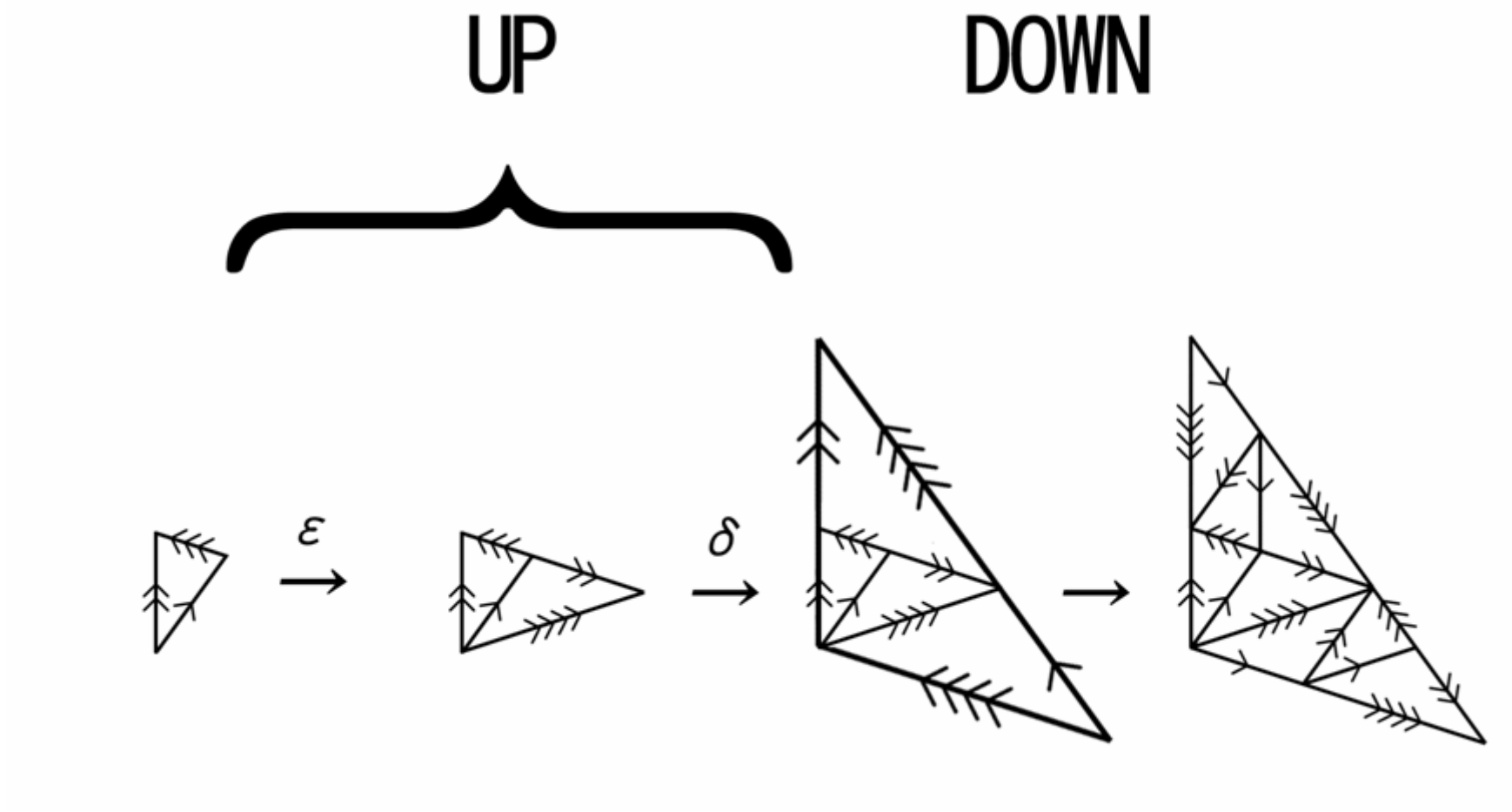
The substitution rule



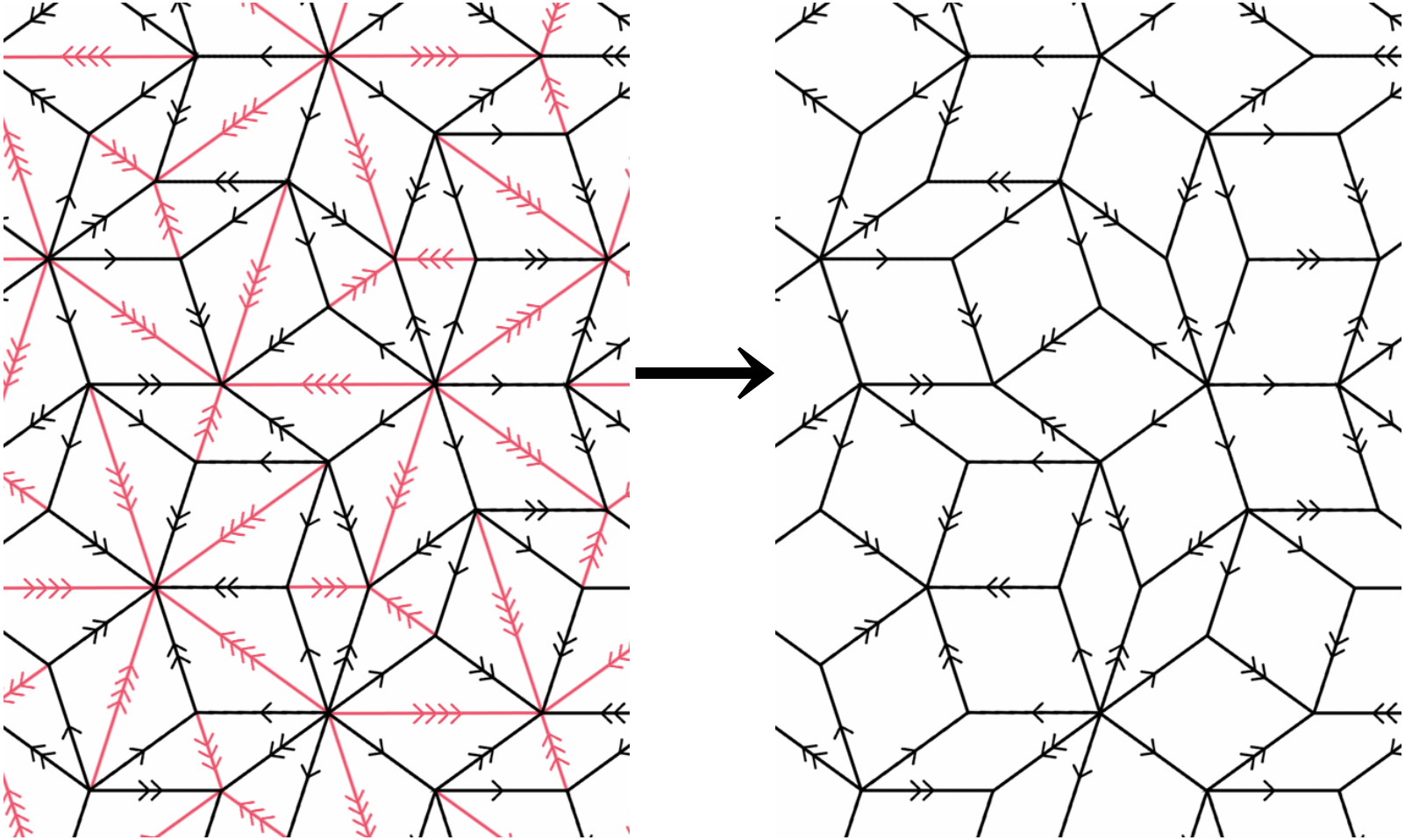
some of 21 expansions

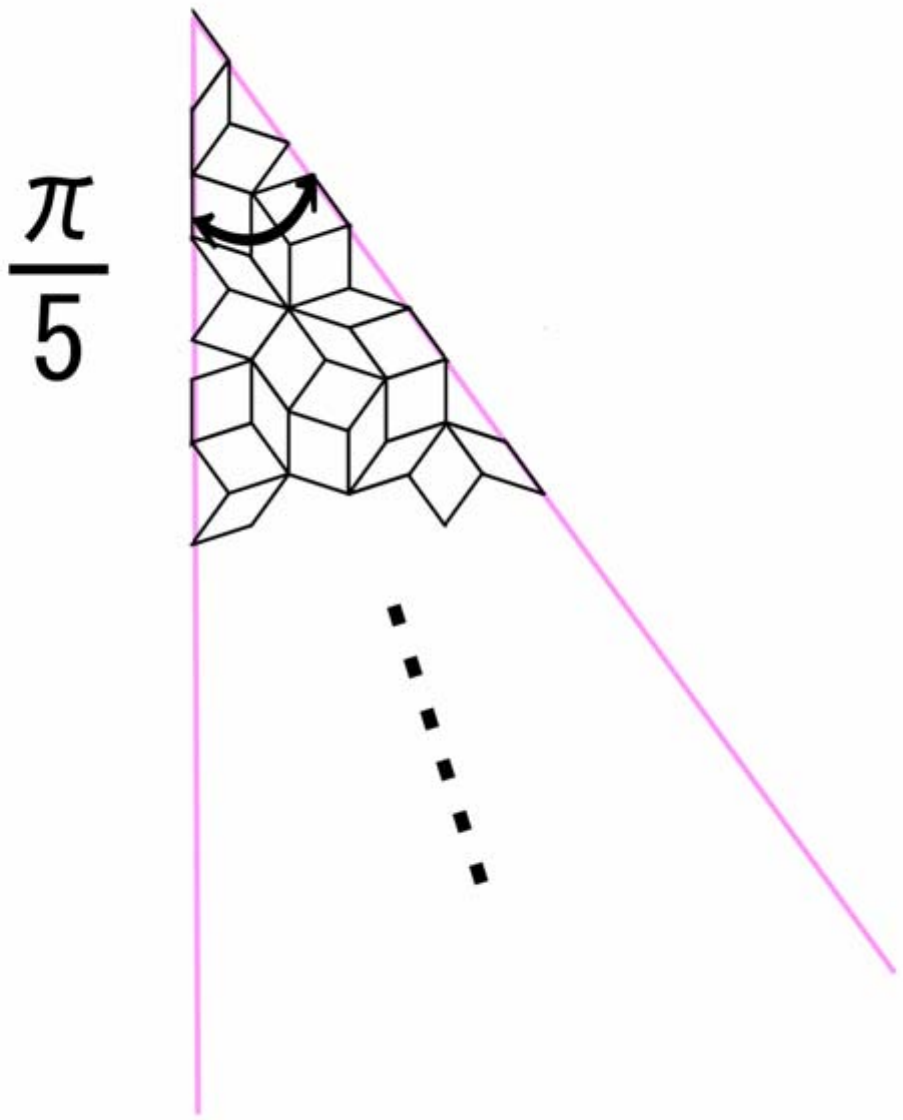


Up-down generation



By erasing edges, we have the Penrose tiling.

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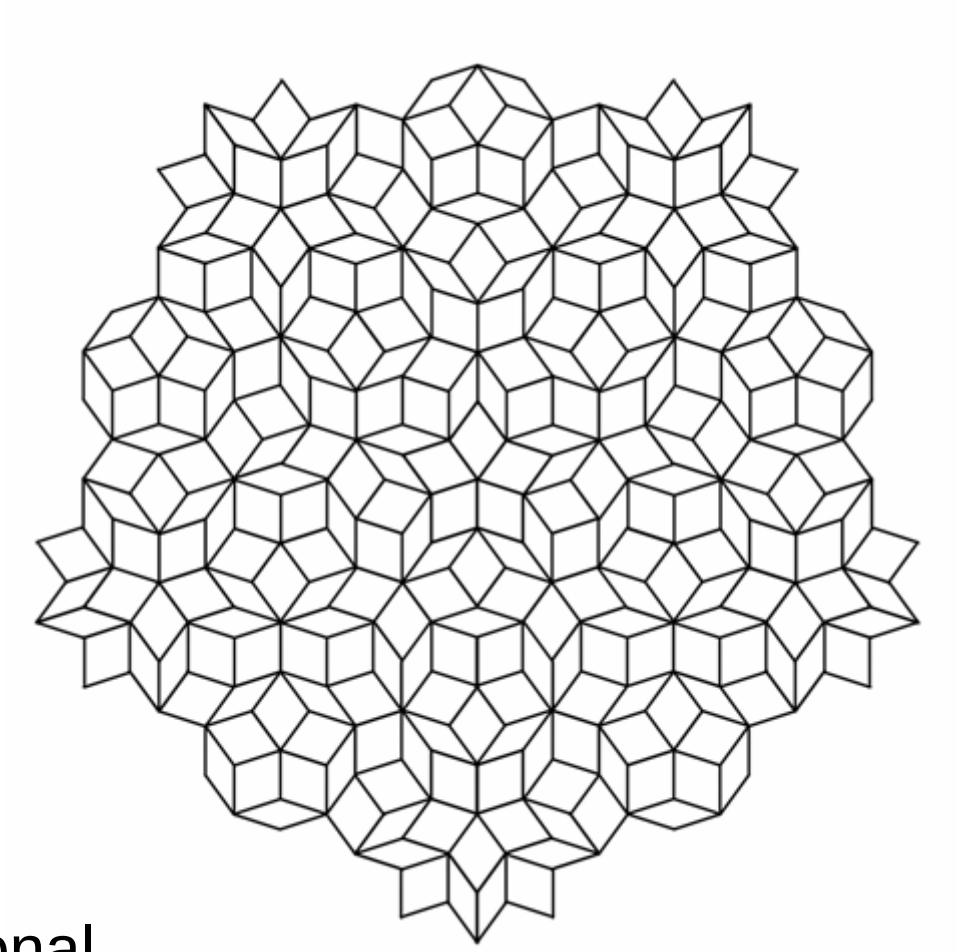
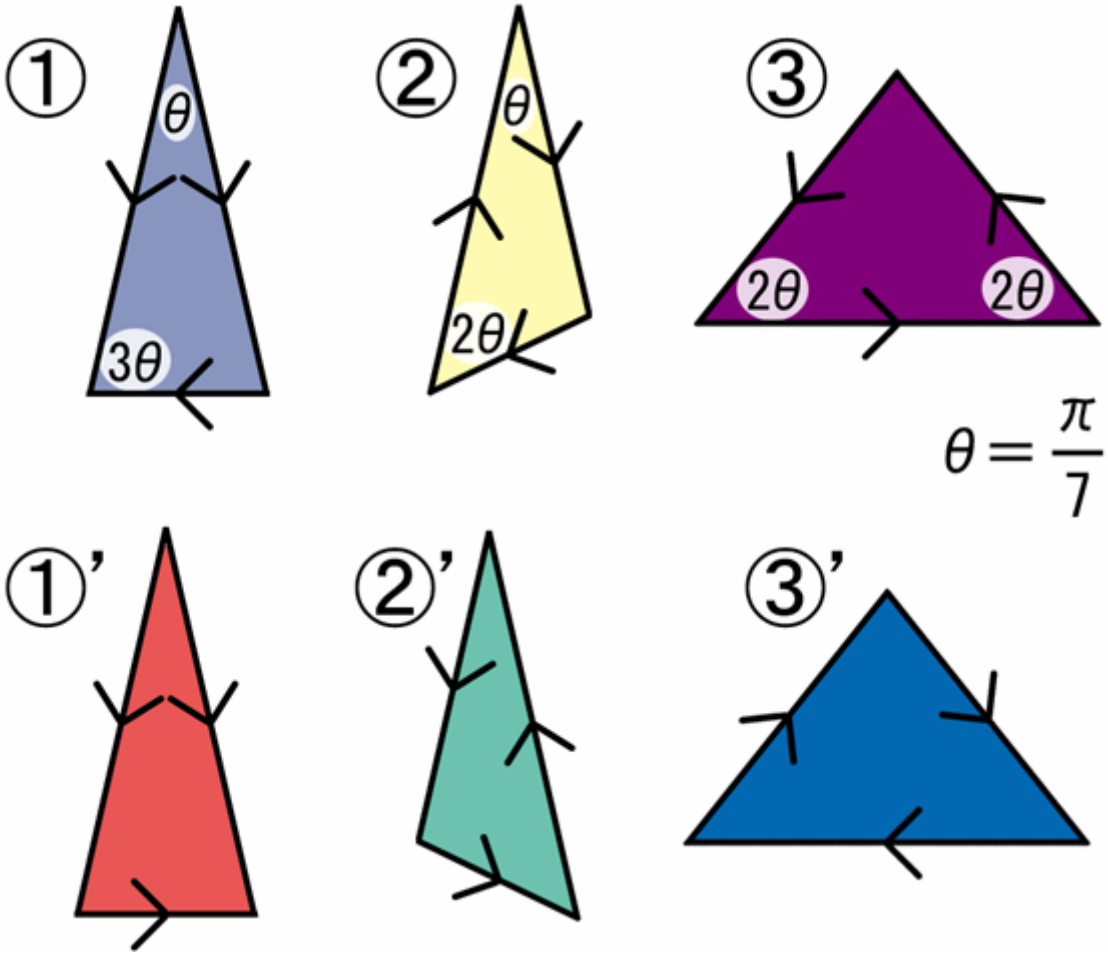
Penrose tiling

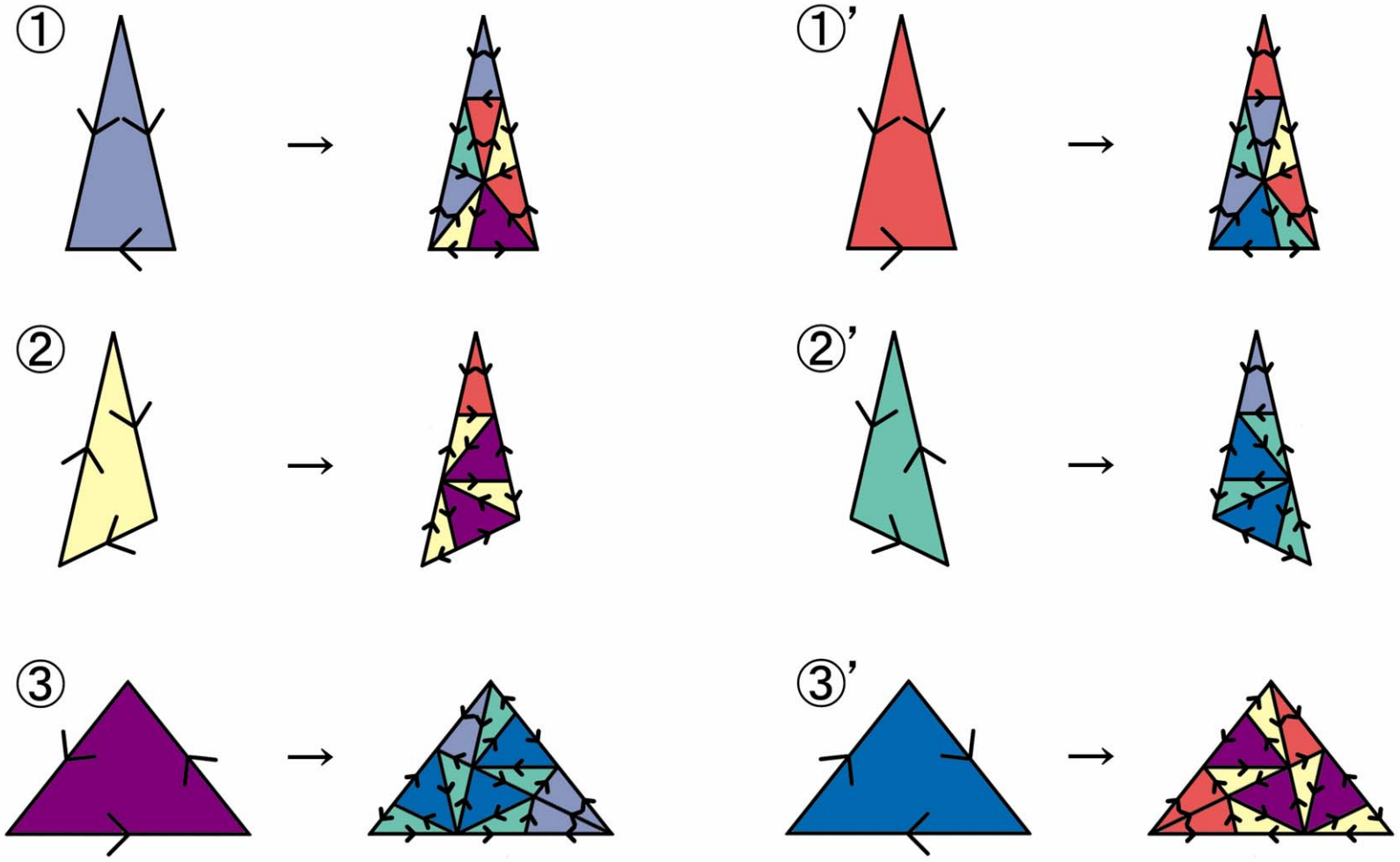


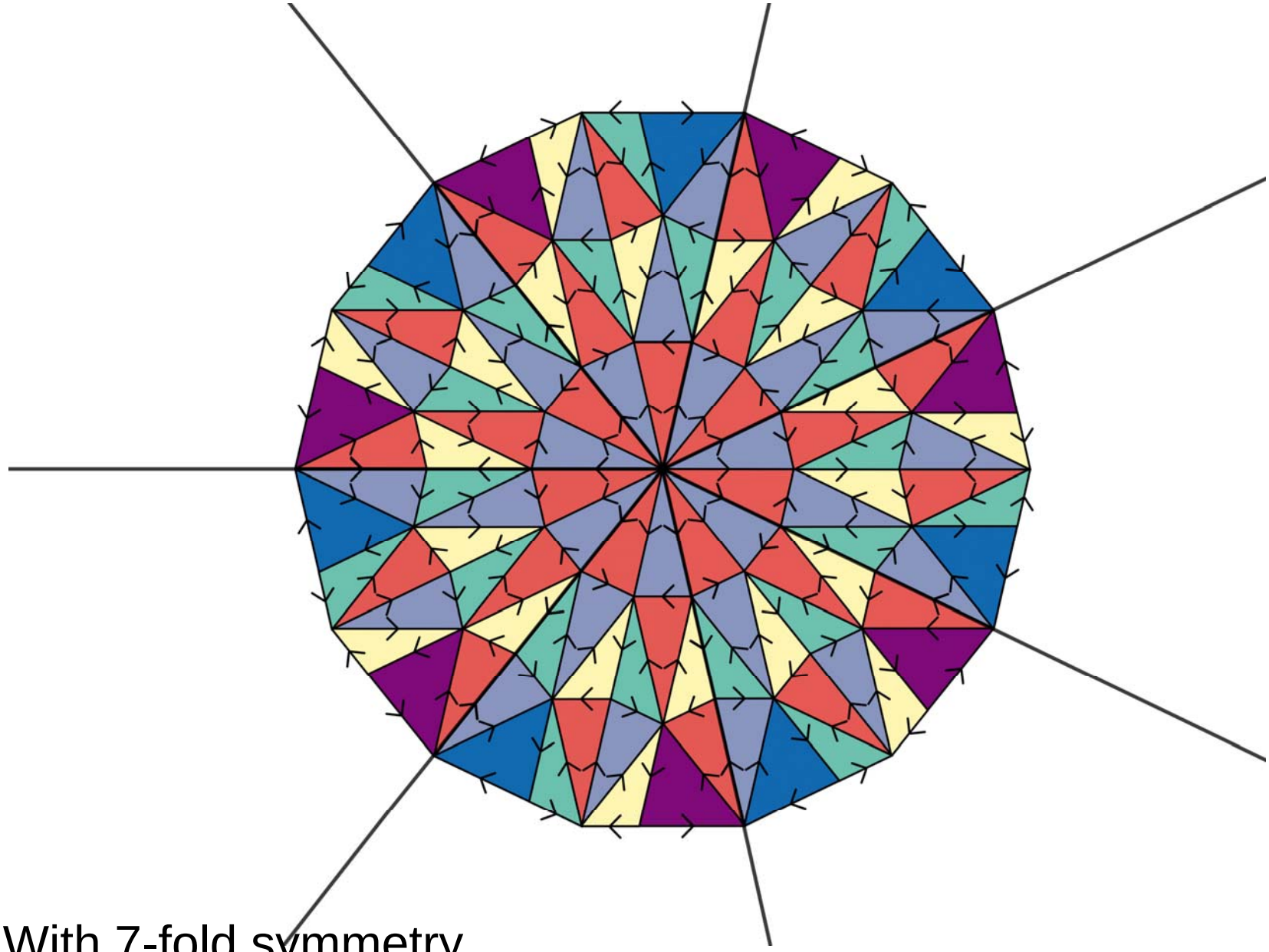
5-fold rotational symmmetry

Prototiles of Danzer tilings



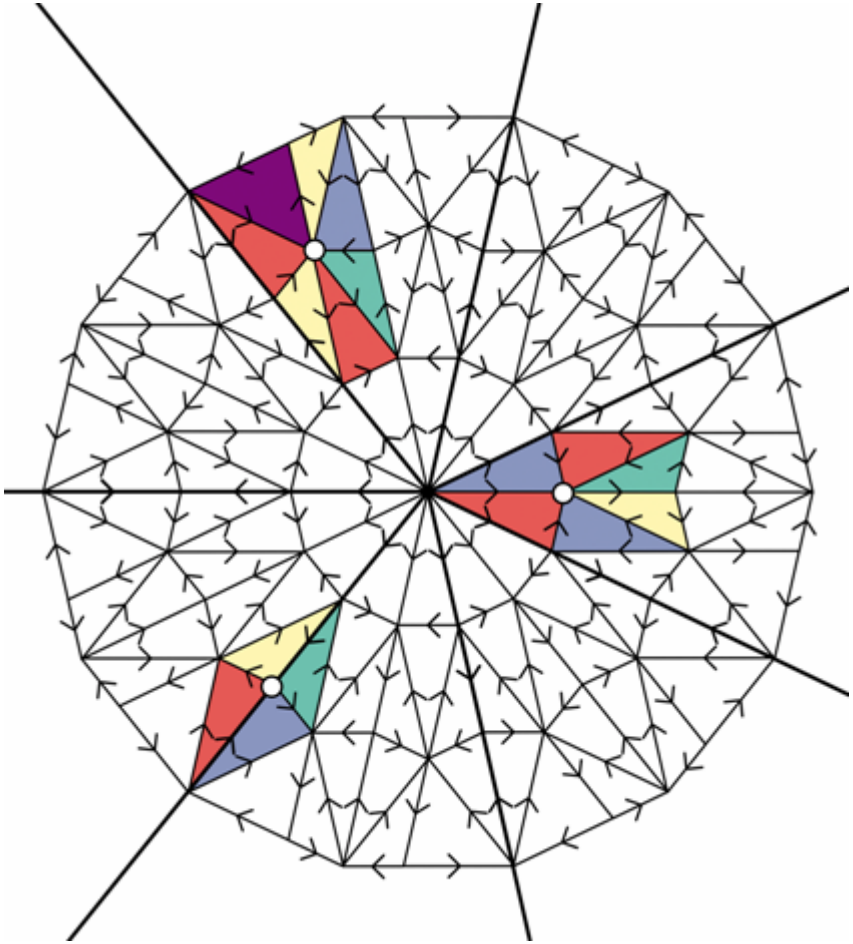
The substitution rule





With 7-fold symmetry

Vertex atlas

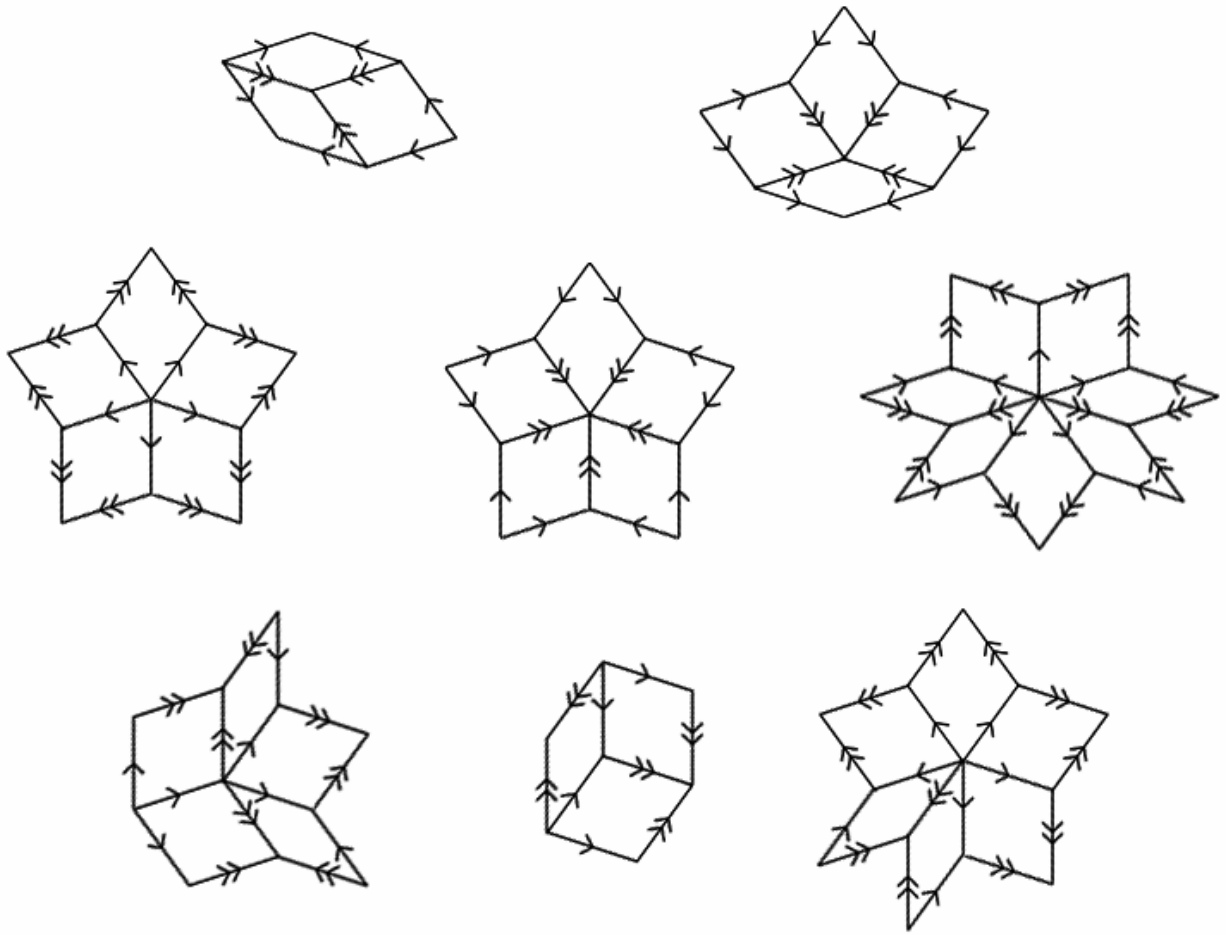


- Configurations (without gap and overlapping)

  of tiles around a vertex is called the **vertex atlas**

  in a tiling.

8 kinds of vertex atlases of Penrose tiling

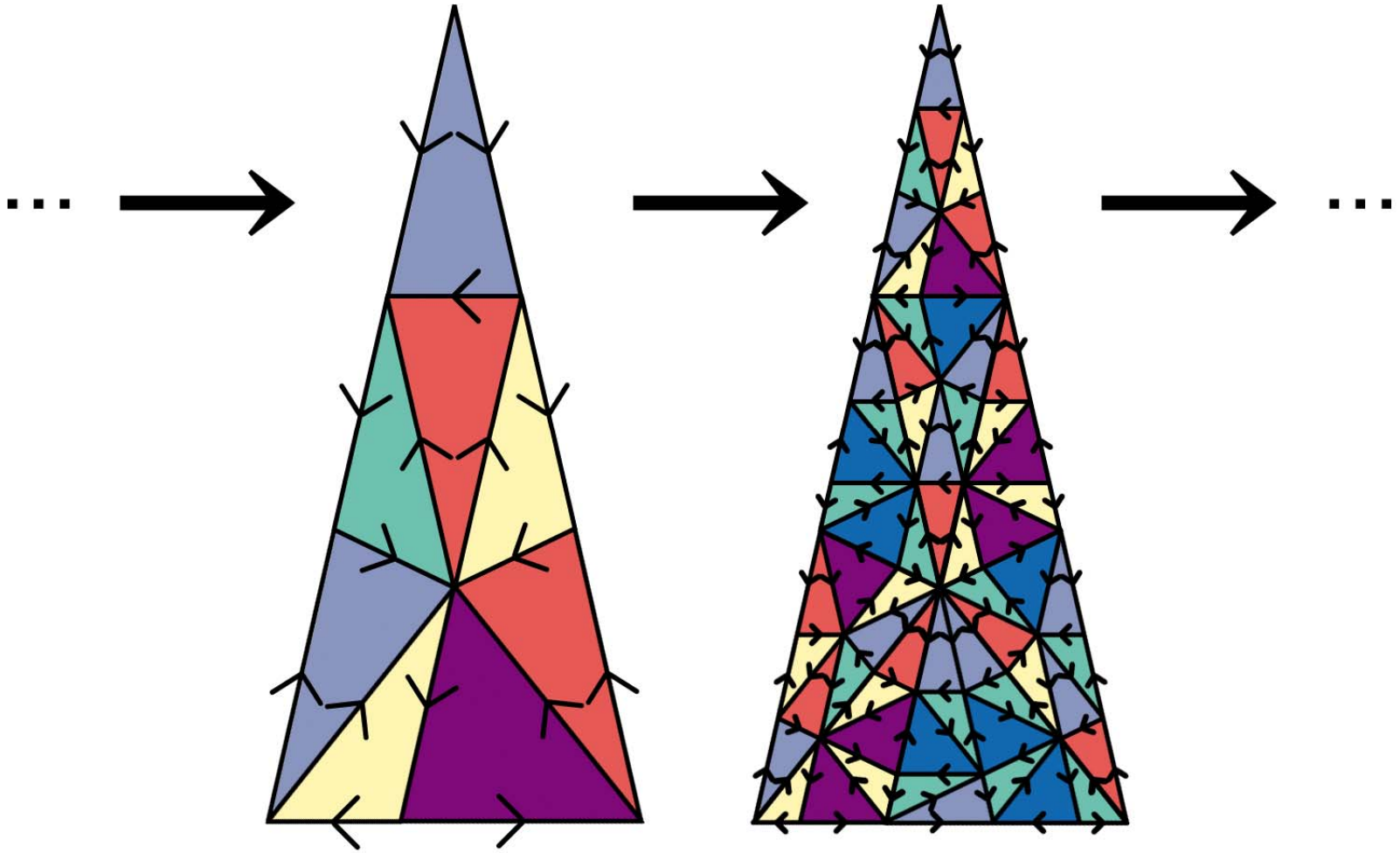


【2】Motivation

Motivation

- Prof. Danzer said as a remark in the appendix of Danzer's paper that

"29 kinds vertex atlases
appear in the Danzer tiling,
and that these vertex atlases
may serve as a matching rule."



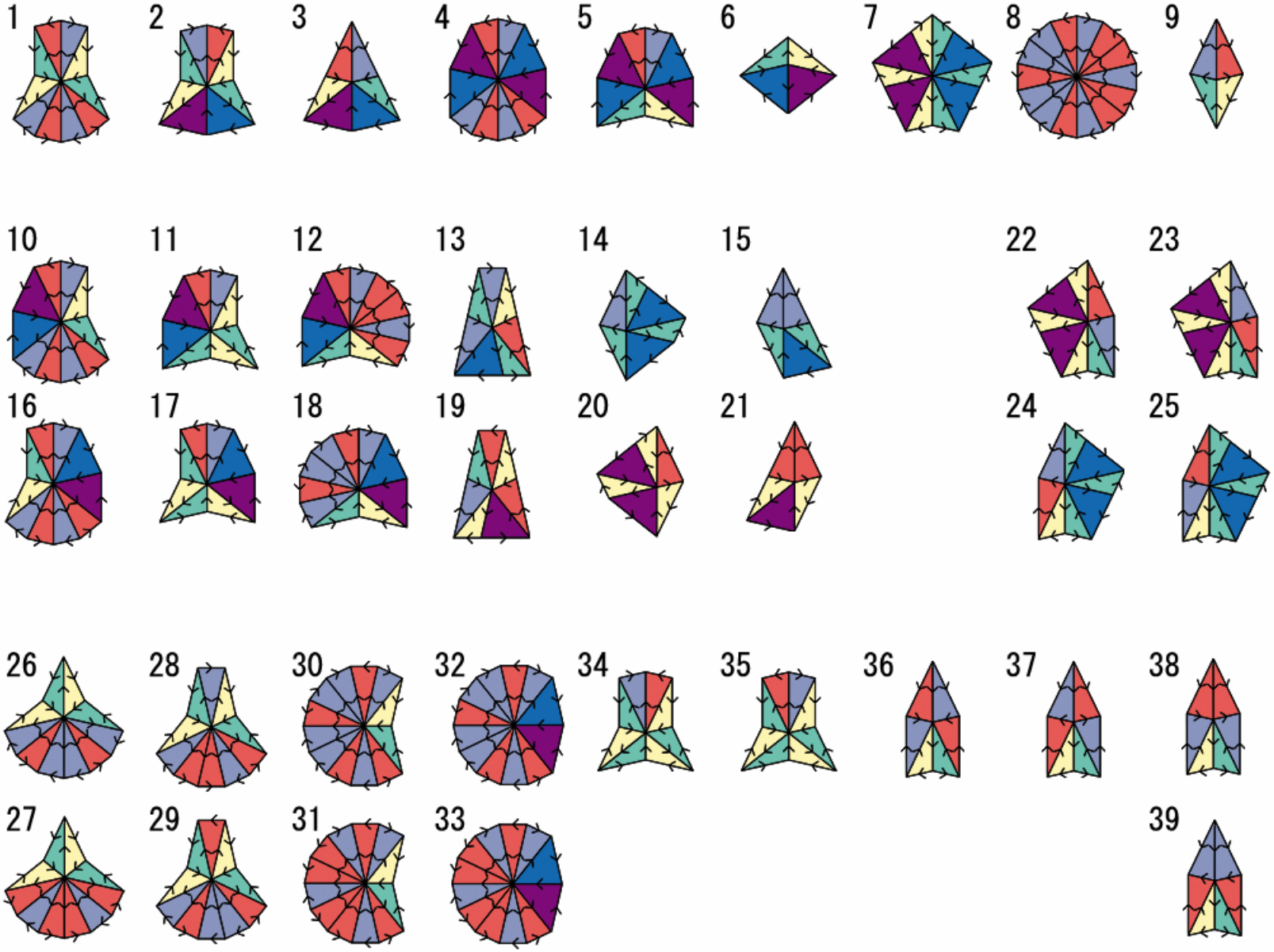
How many vertex atlases ?

- 39 vertex atlases with arrows.

(our result)

- 29 vertex atlases without arrows.

(remark of Prof. Danzer)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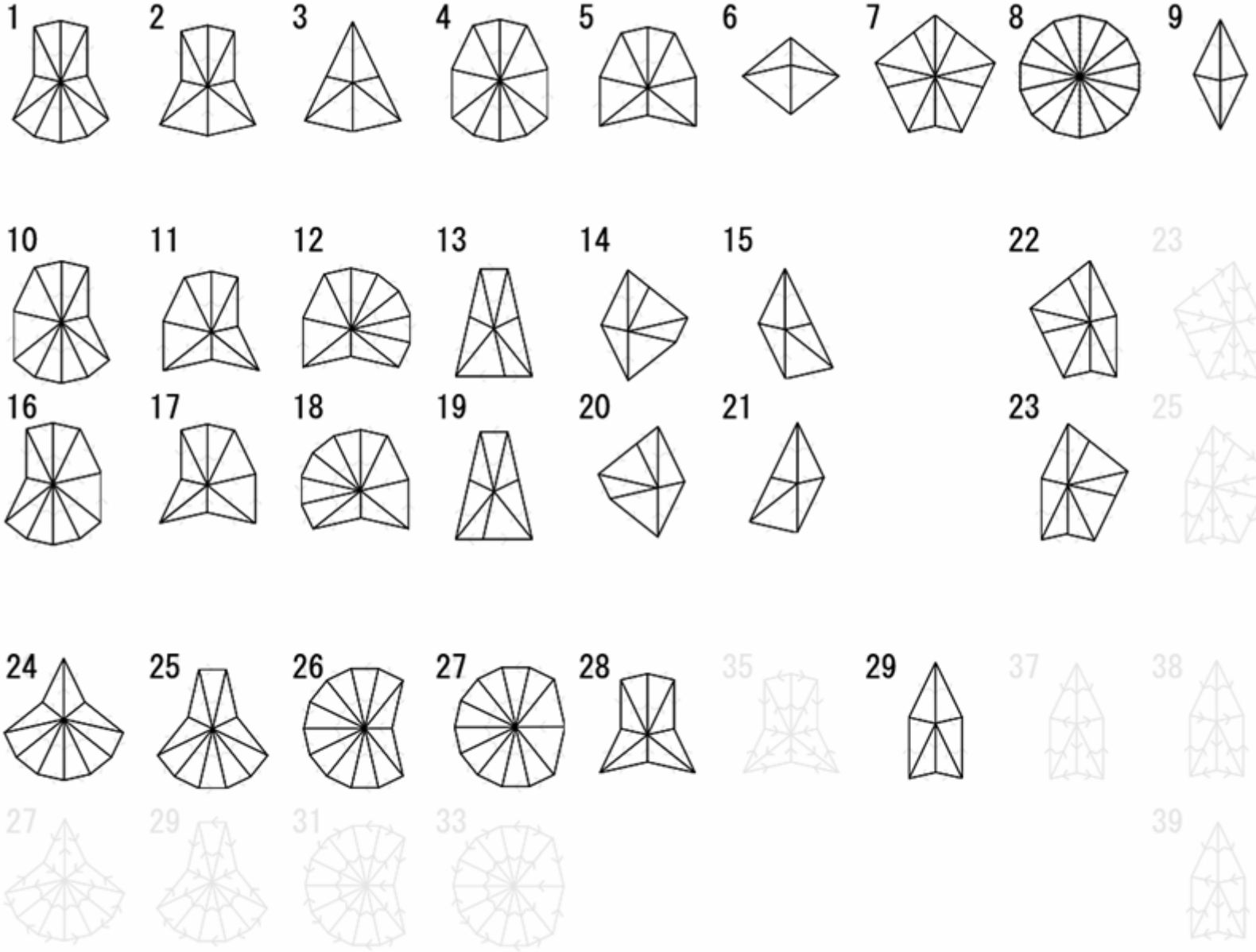
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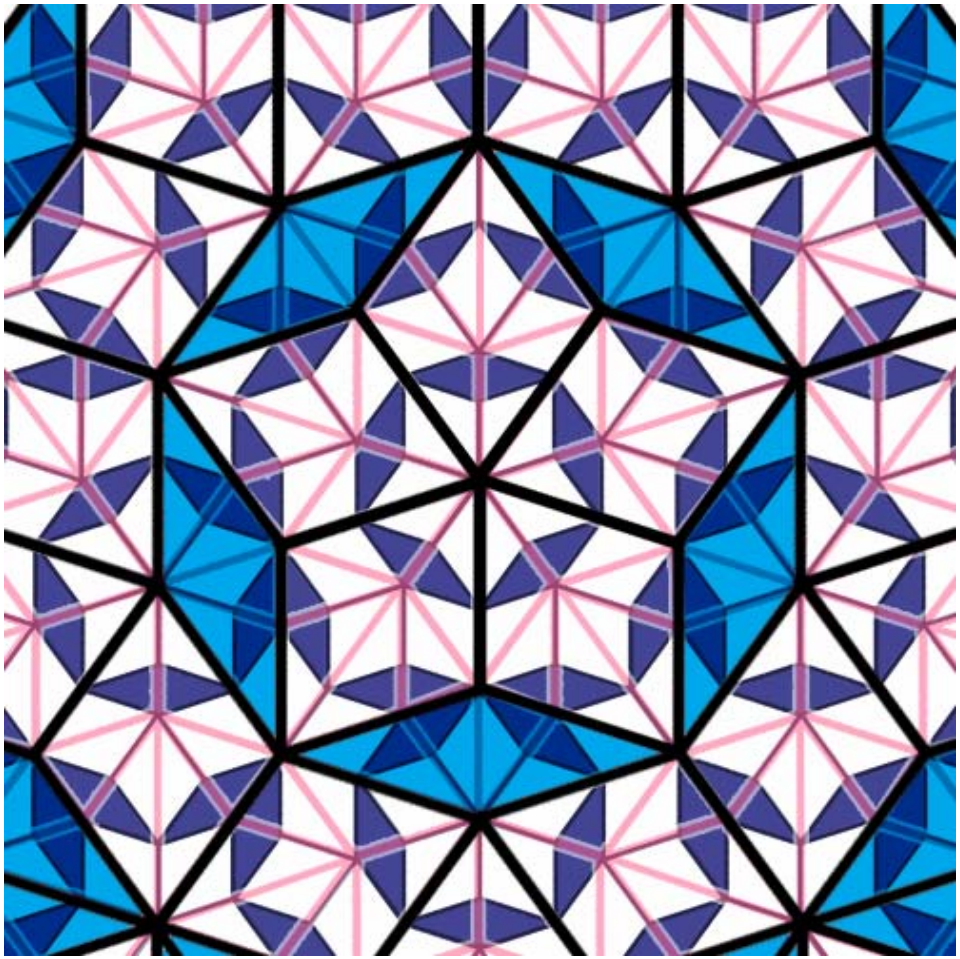
Up-down generation

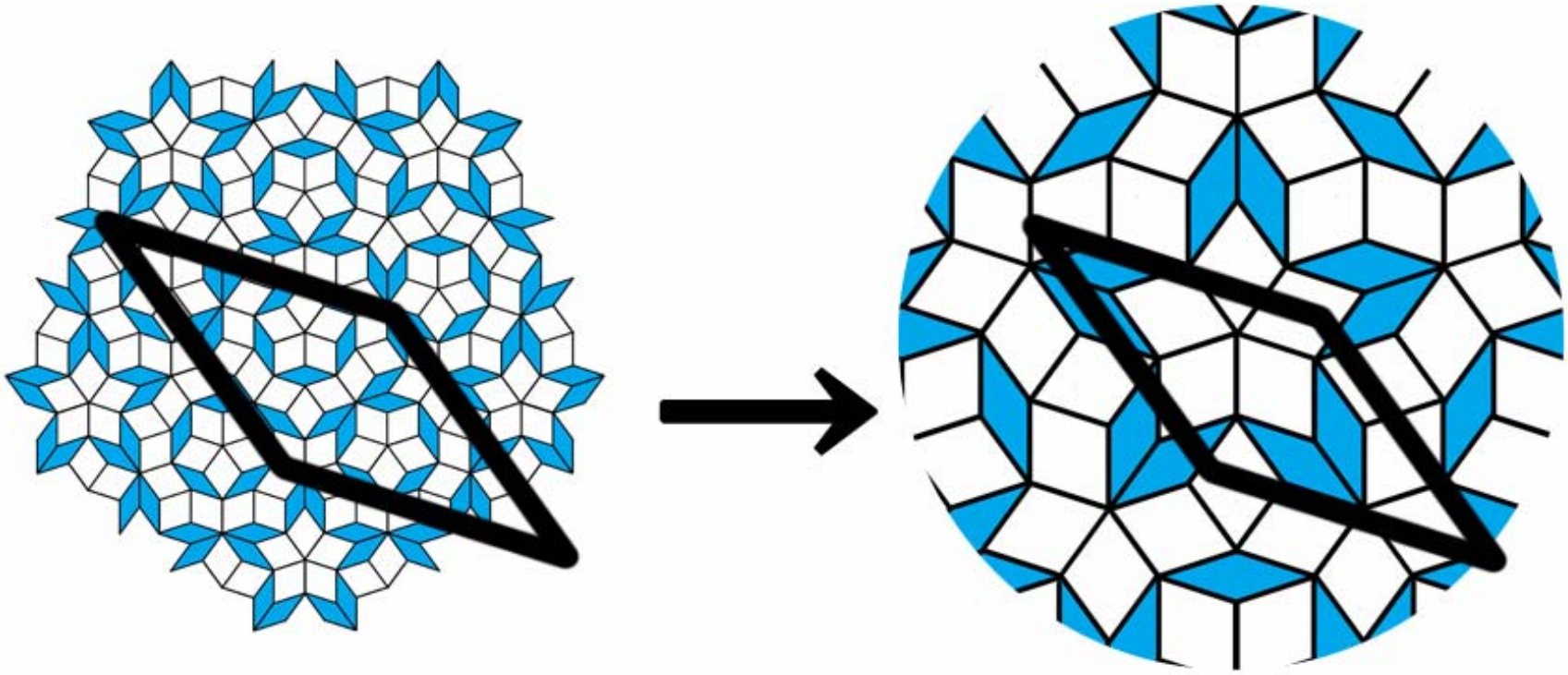
- Penrose tiling and Danzer tiling with rotational symmetry cannot be constructed only by the up-down generation procedure.

- It is necessary to extend the tilings to the whole plane by using reflection and rotation.

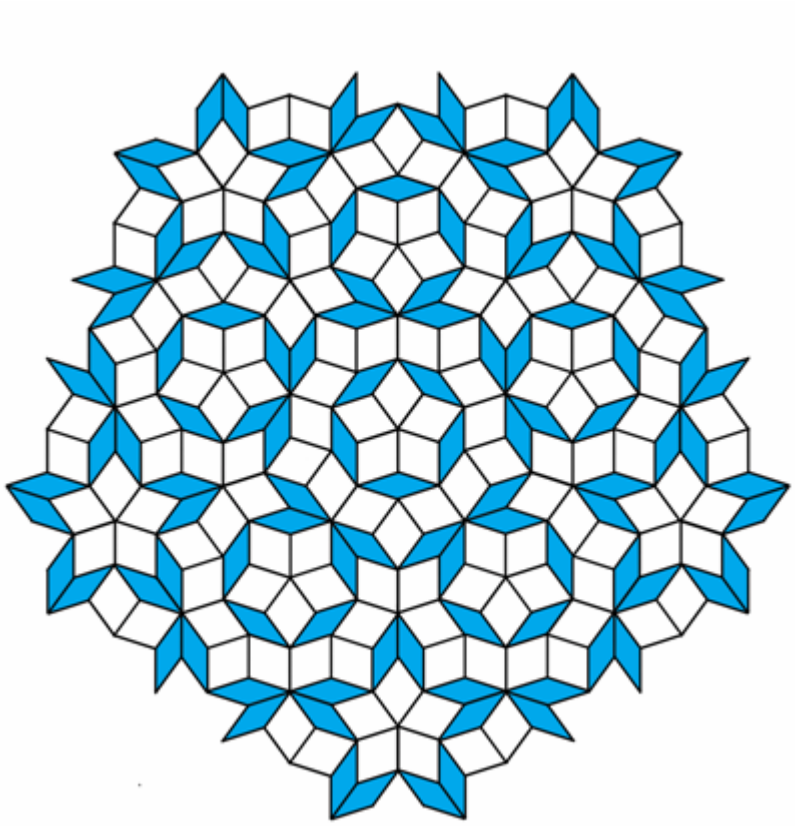
Super tile

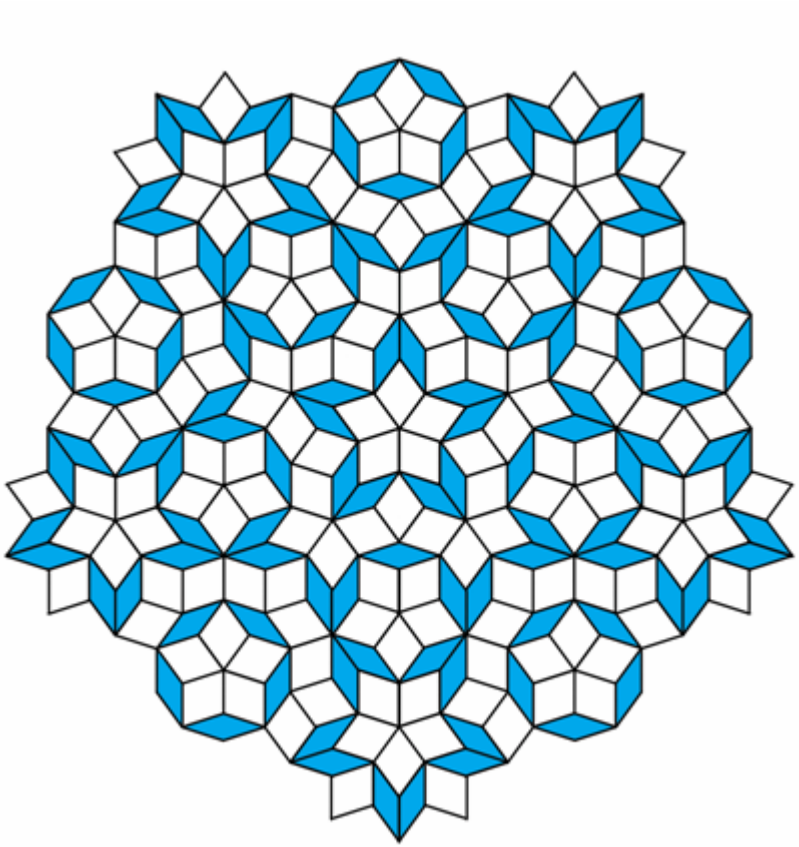


Scale-up procedure

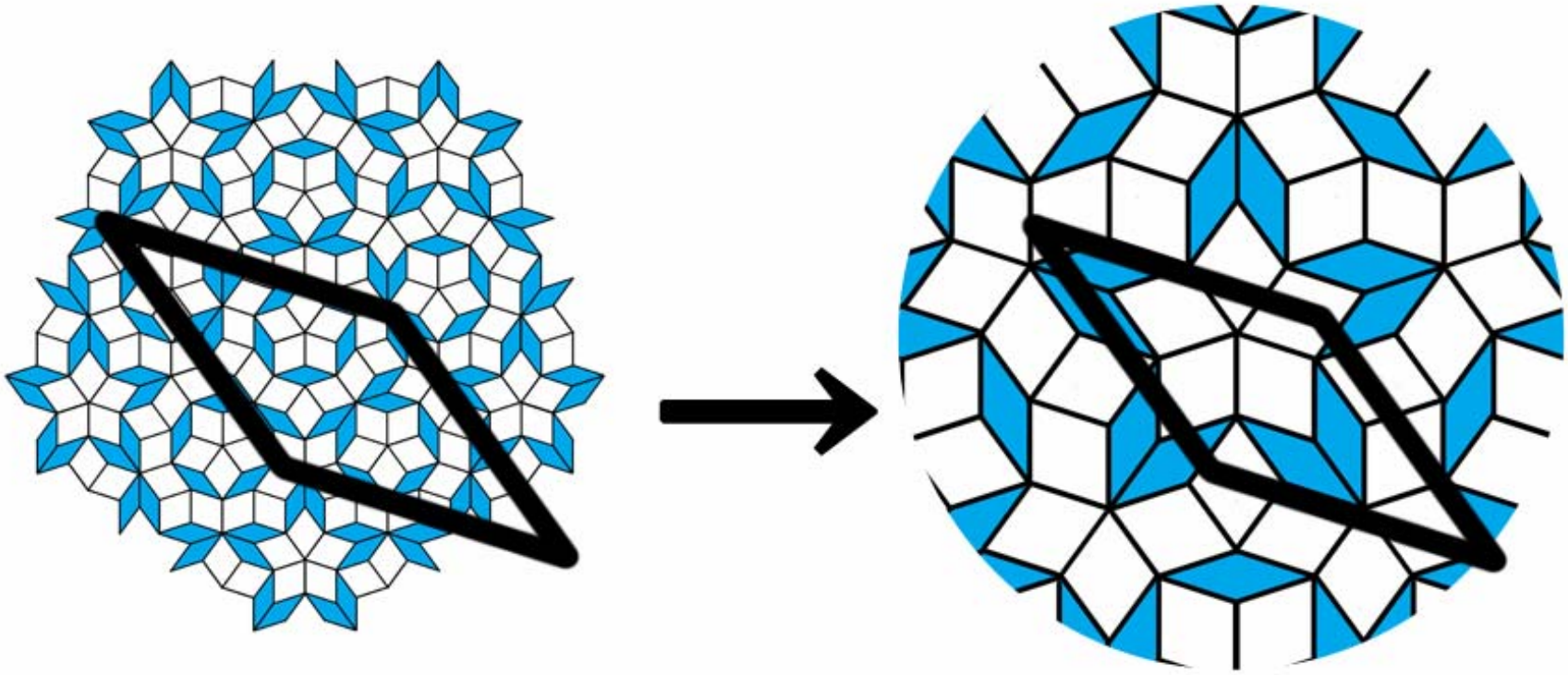


All of Penrose tilings with 5-fold symmetry

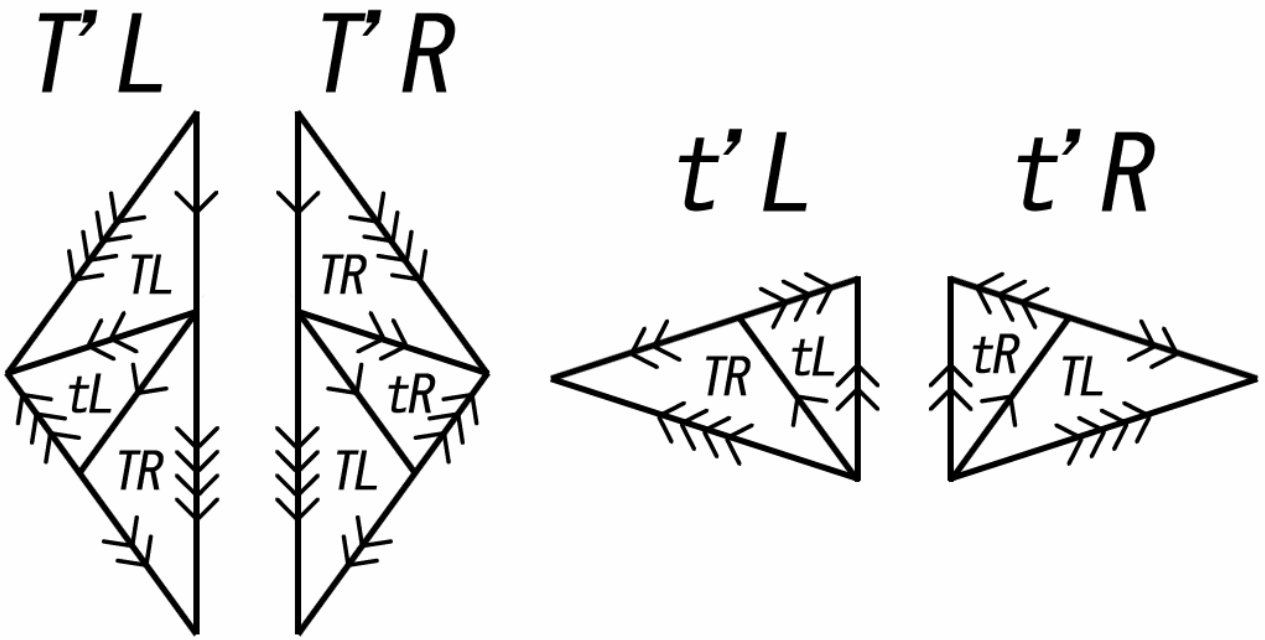




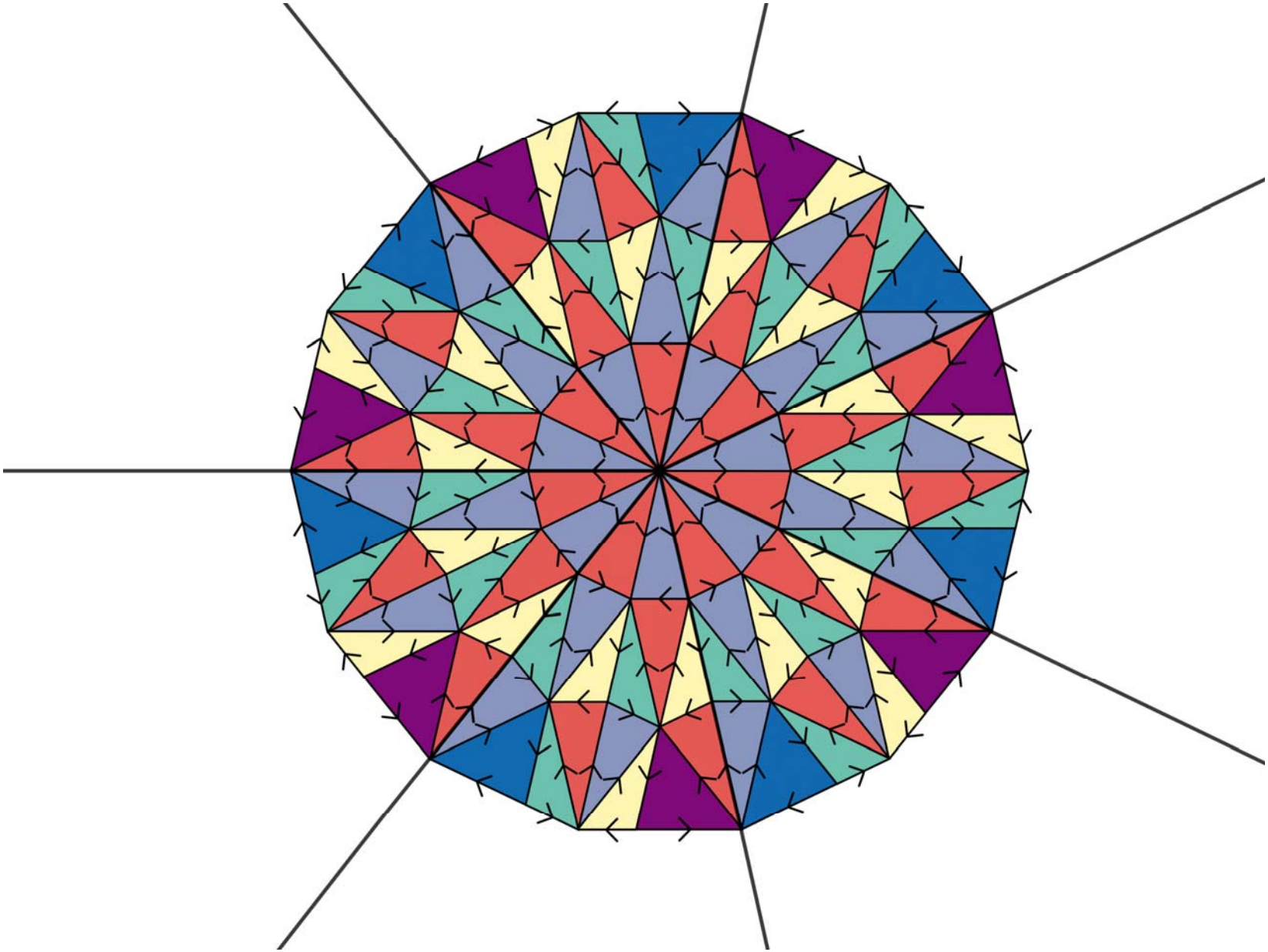
Scale-up procedure

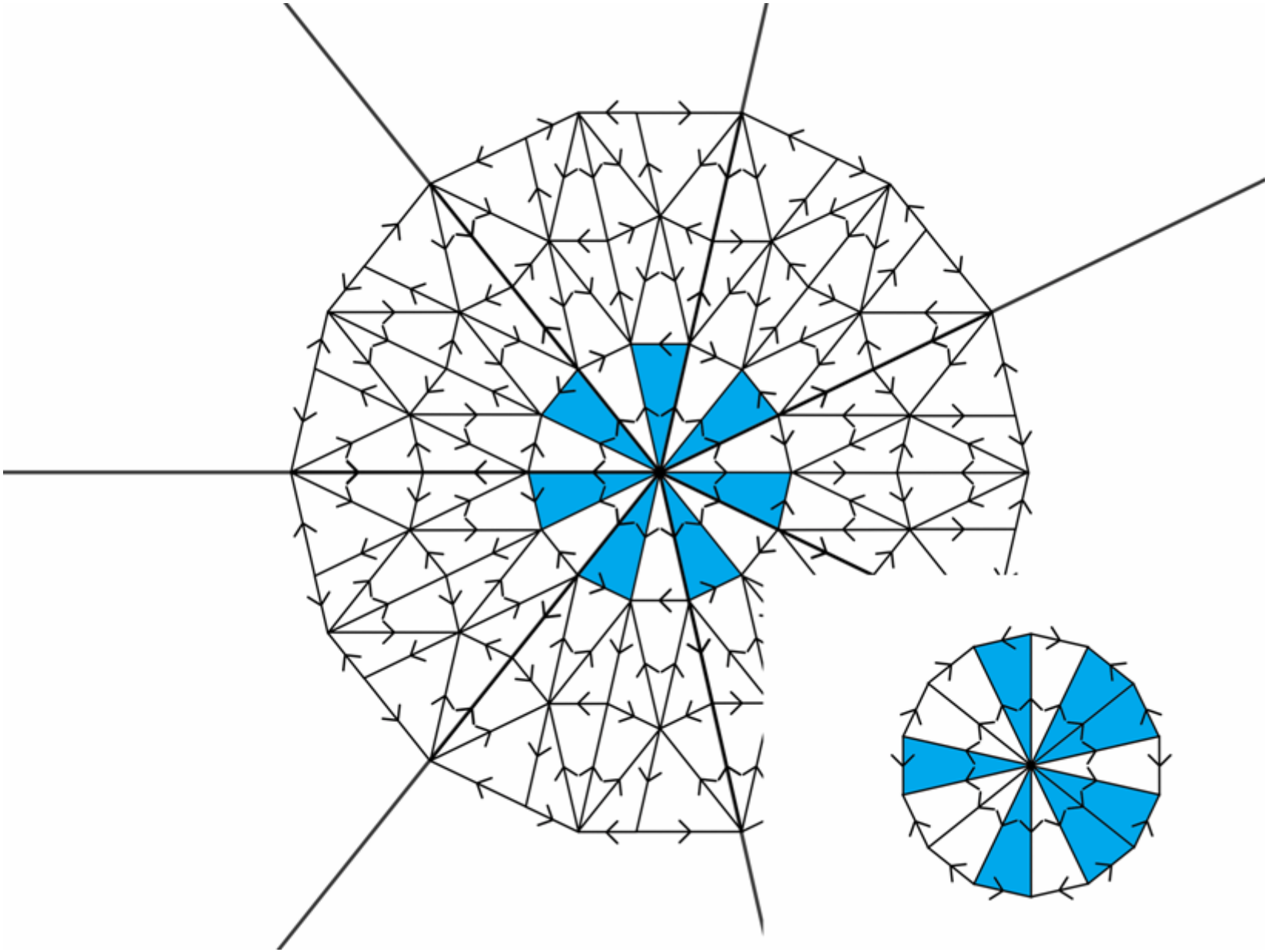


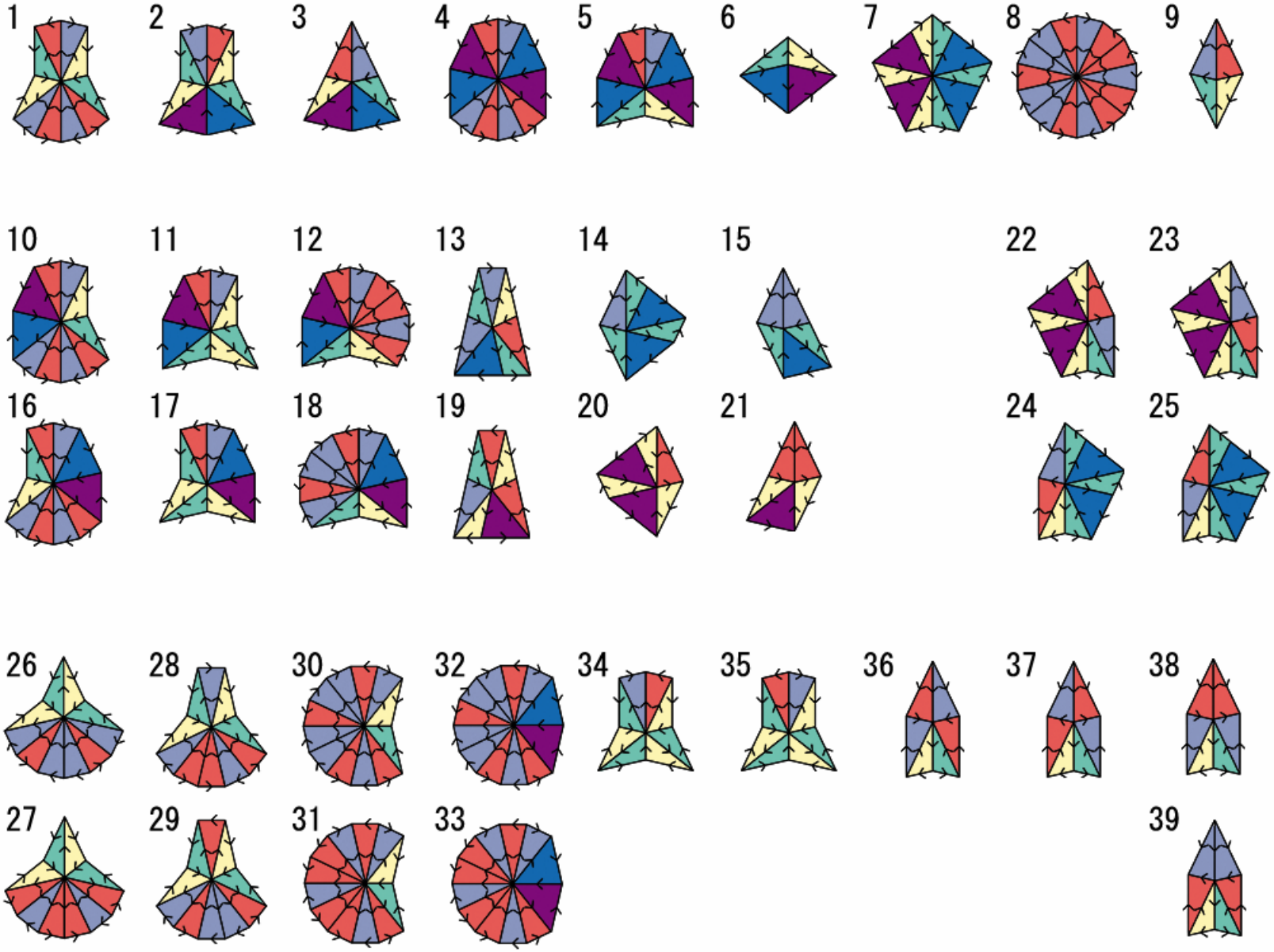
The substitution rule



The center of 5-fold symmetry can not be found.



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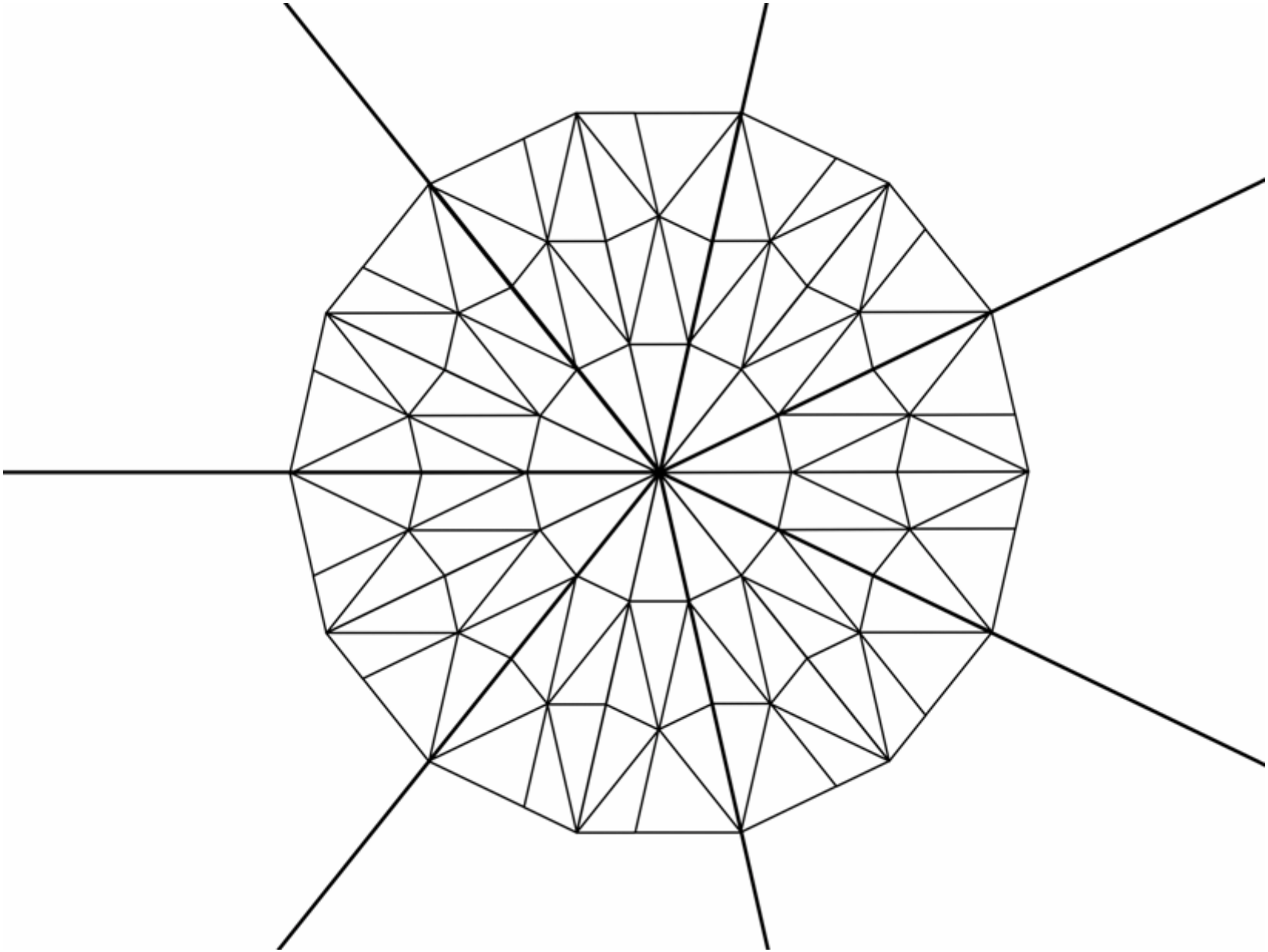
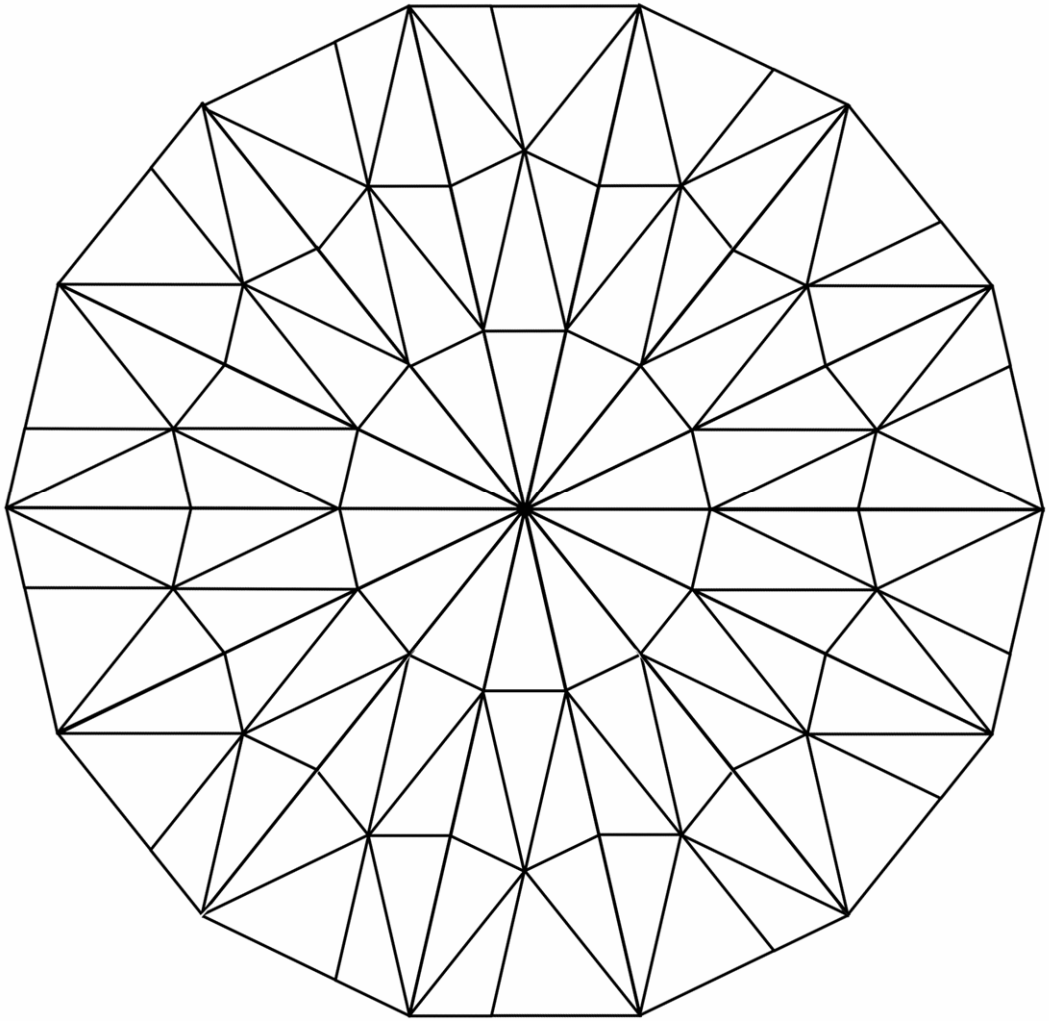
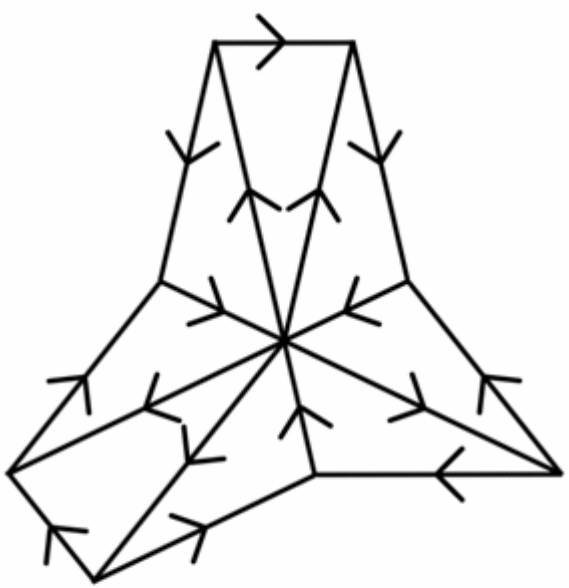
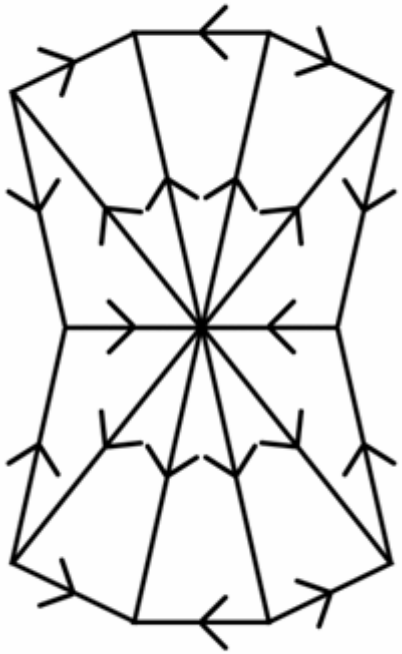




Vertex atlas



(i)

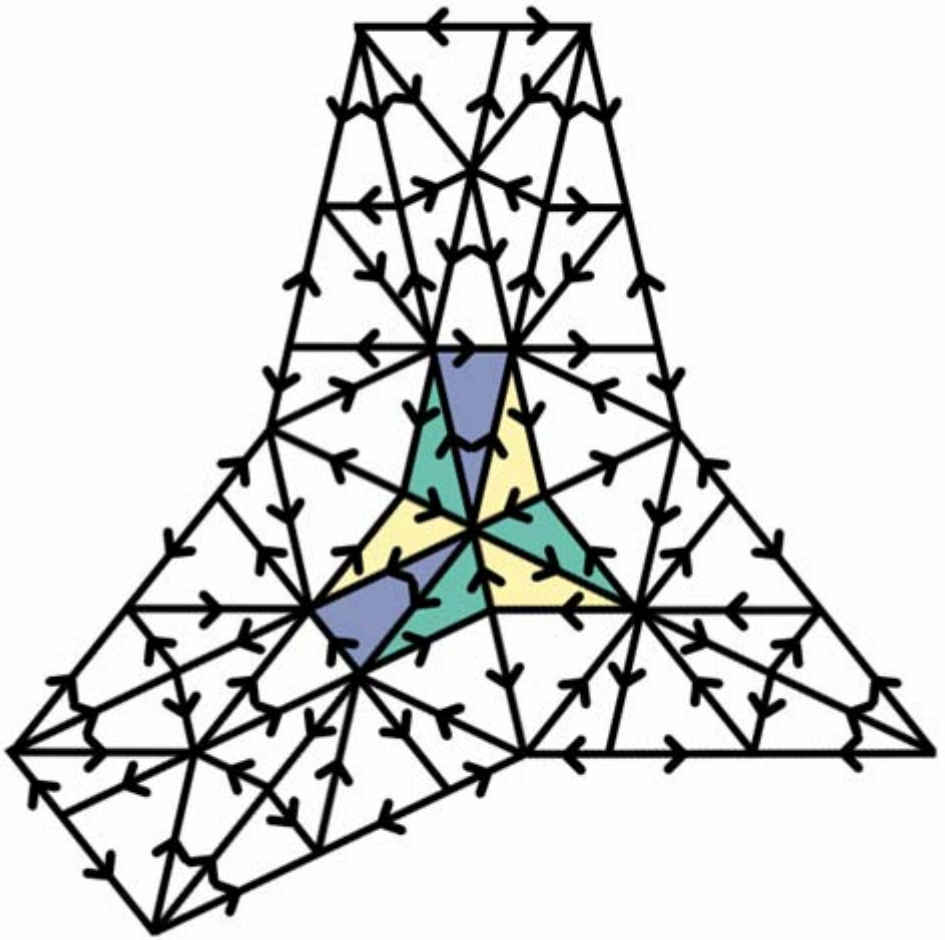


(ii)

The properties of (i) , (ii)

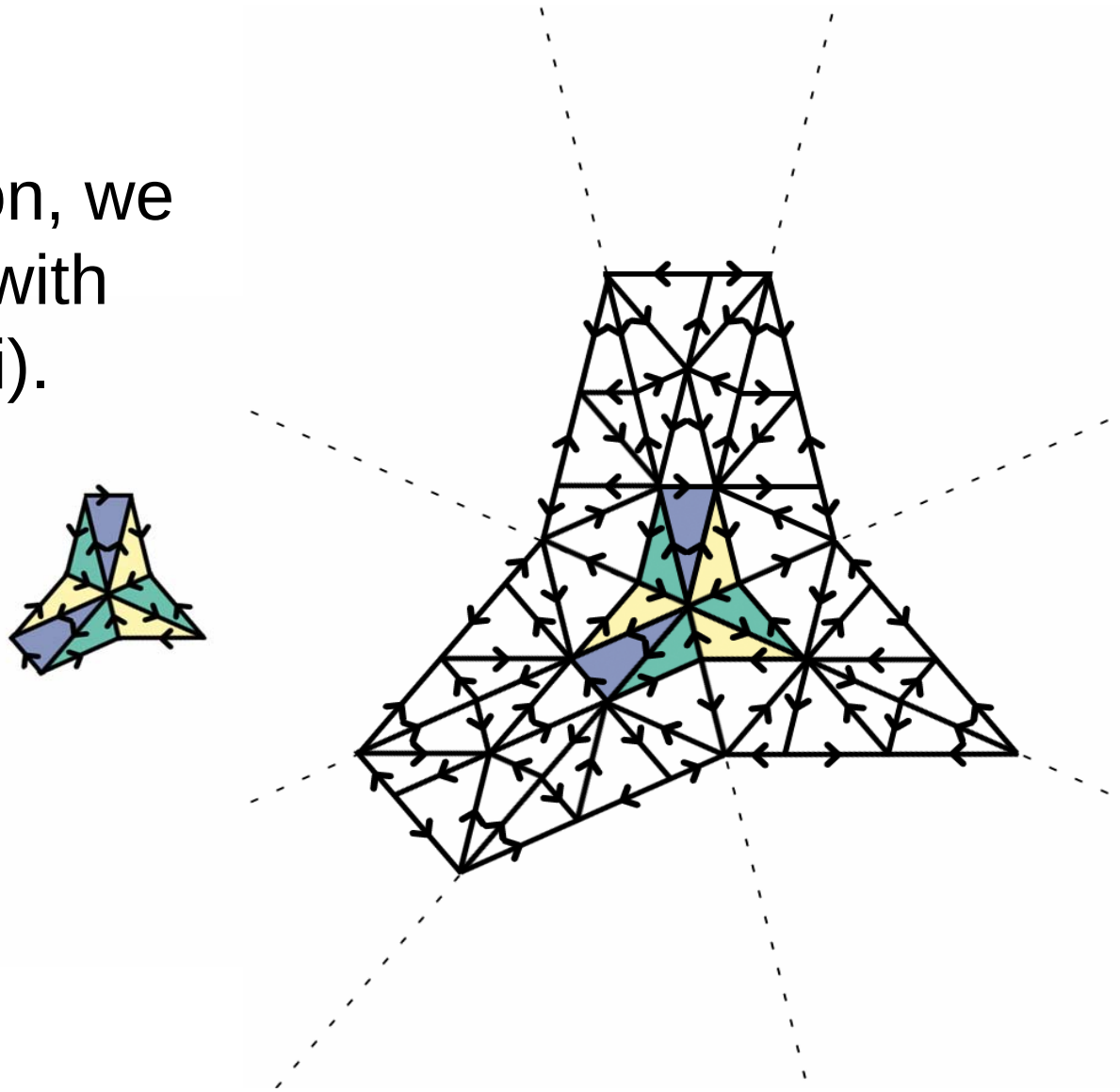
- These do not exist in the list of 39 vertex atlases.
- By substitution rule, the same type appears at the center when these subdivide.
- Therefore, the plane can be covered by 39 kinds of vertex atlas and substitution.
- （There is no one of (i) and (ii) in the Penrose tiling.）

(i)



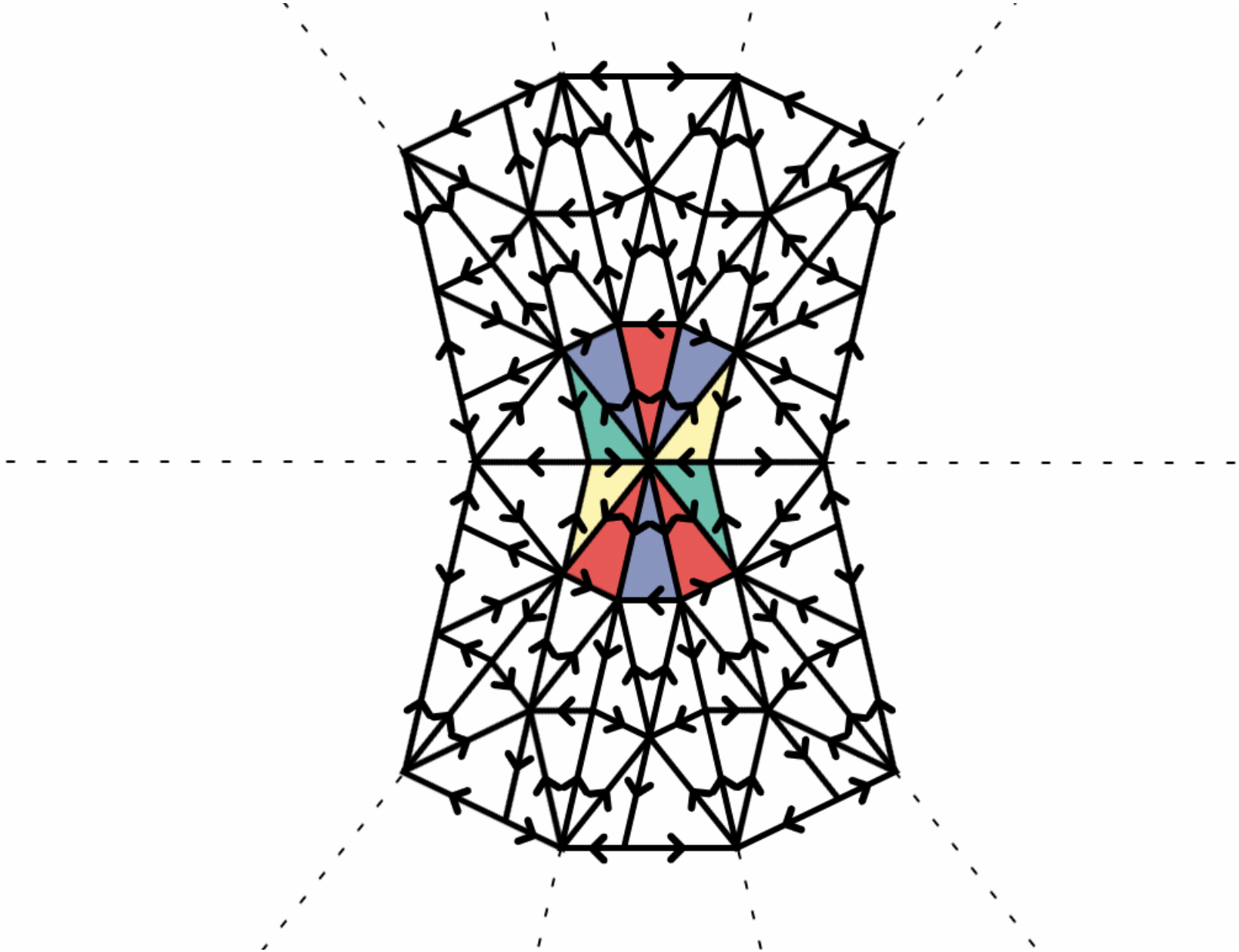
(I)

- By substitution, we have a tiling with vertex atlas (i).



- By trial and error, we can show that configuration (I) is uniquely obtained from (i).

- If a tiling have just 39 vertex atlases and the vertex atlas (i), then vertex atlas (i) appear only in one place.

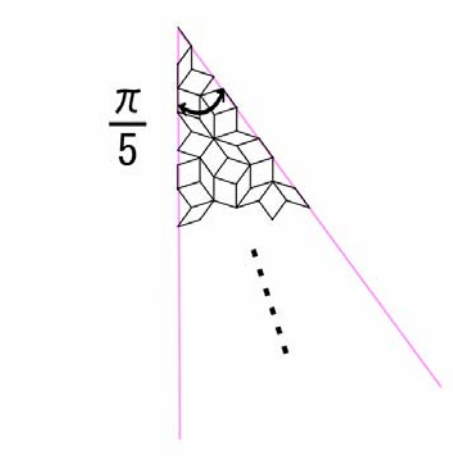


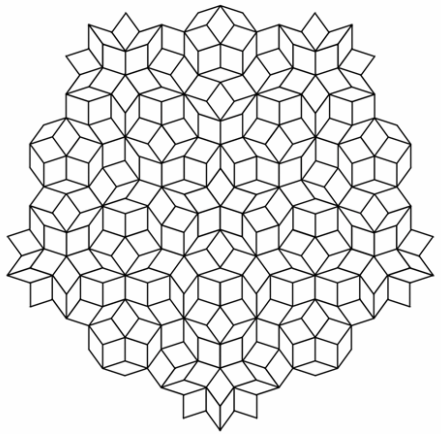
【3】Our observations

Our questions

Question 1

- For which *n*, can tilings with *n*-fold symmetry be constructed only by the up-down generation procedure ?

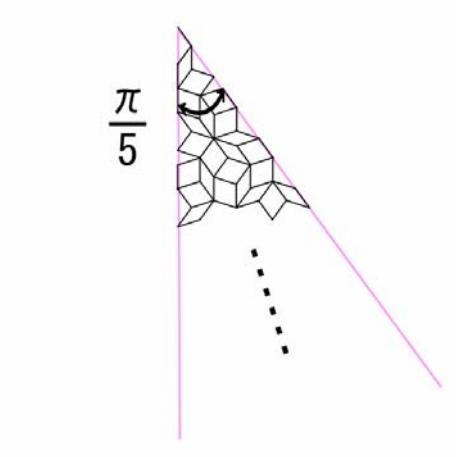


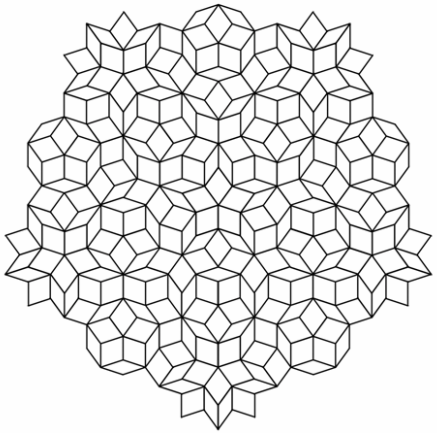


Our questions

Question 2

- How can tilings be constructed by attaching unbounded configuration?





Primitive substitution

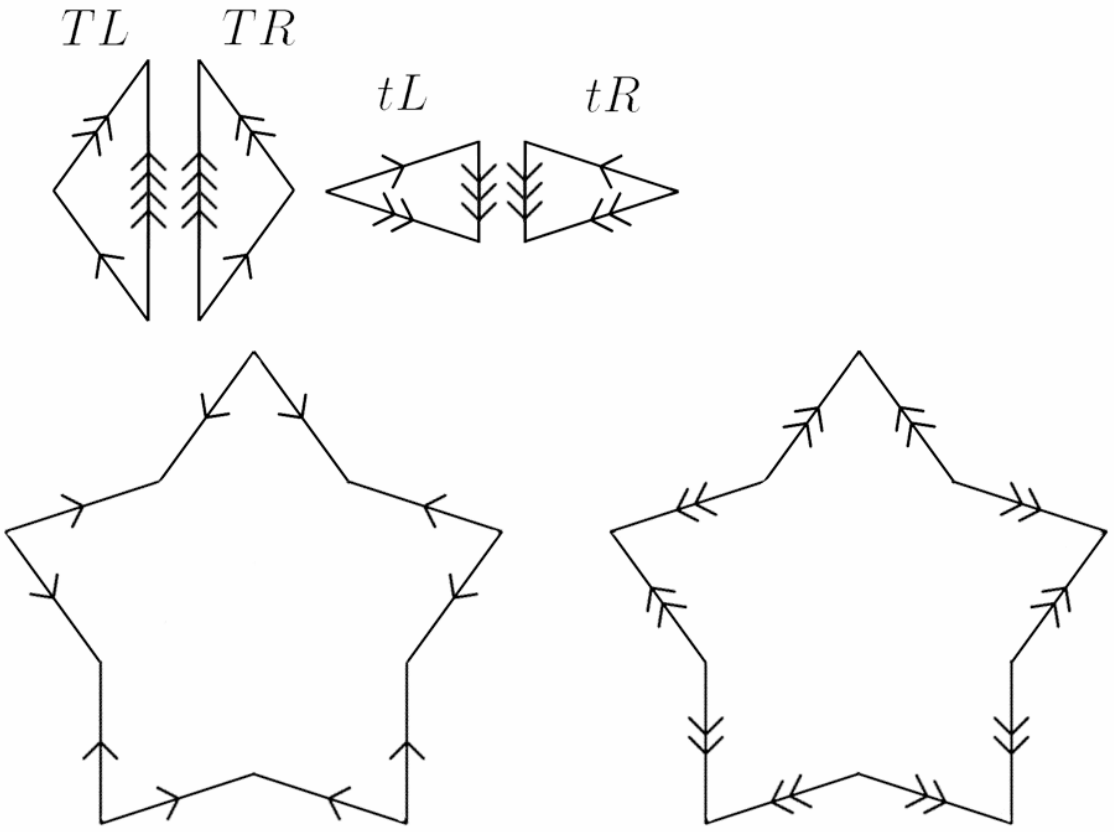
- A substitution is primitive if the substitution rule is a linear map that can be represented by a **primitive** matrix.

Primitive matrix

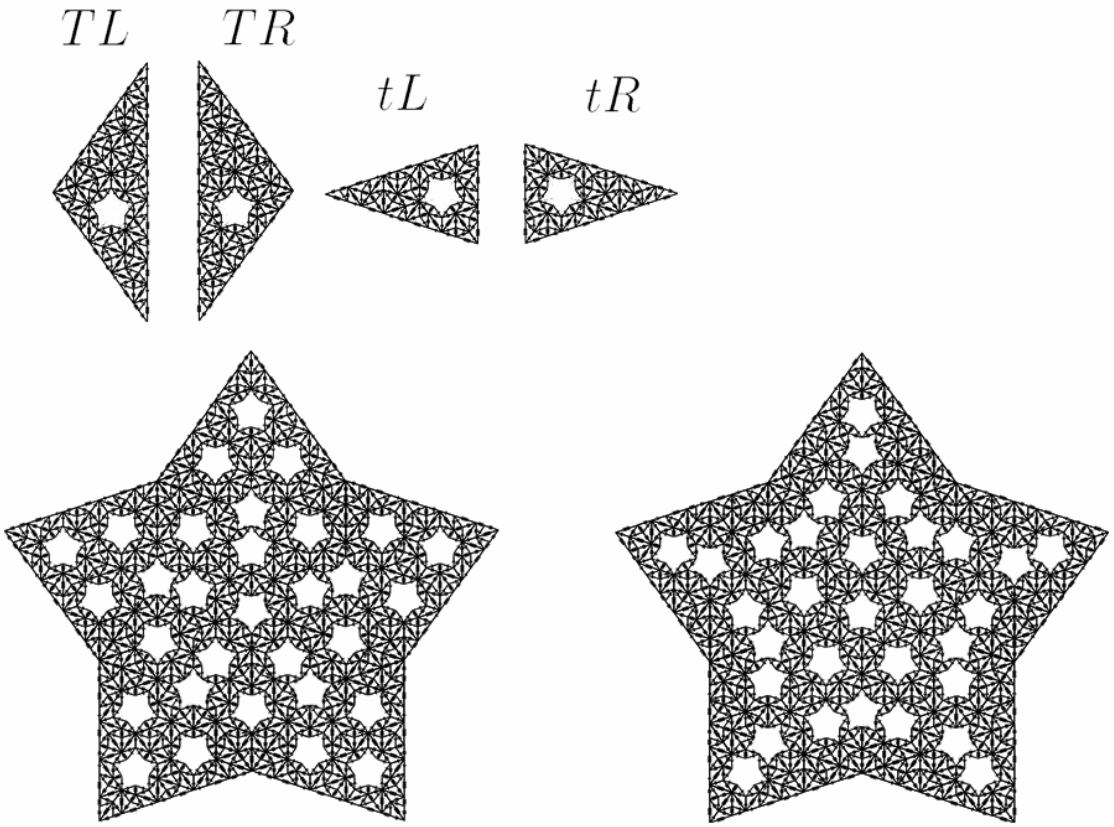
- An $n \times n$ matrix A is said to be **primitive** if its entries are nonnegative integers and if there exists a positive integer k such that all the entries of A^k are positive.

- If there is a substitution with *n*-fold tiling, we can make a substitution that can generate an *n*-fold tiling from up-down generation.

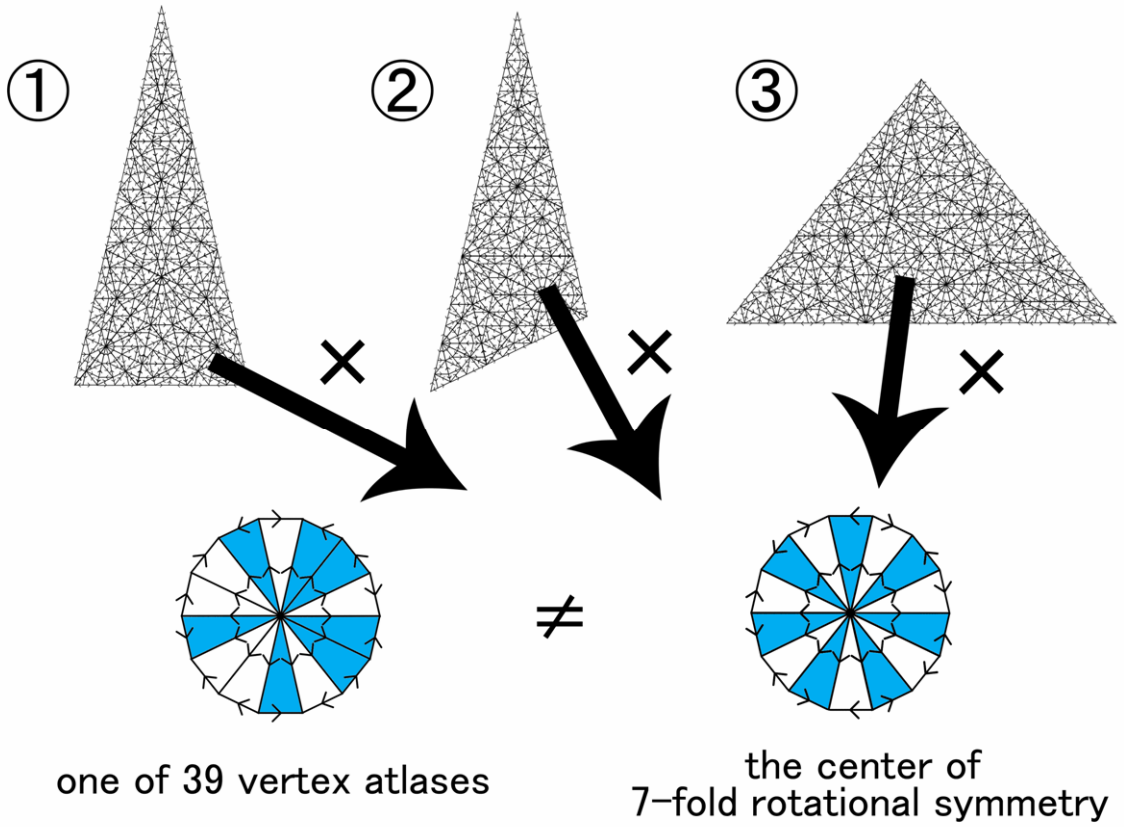
Prototiles



Substitution



This is not primitive.



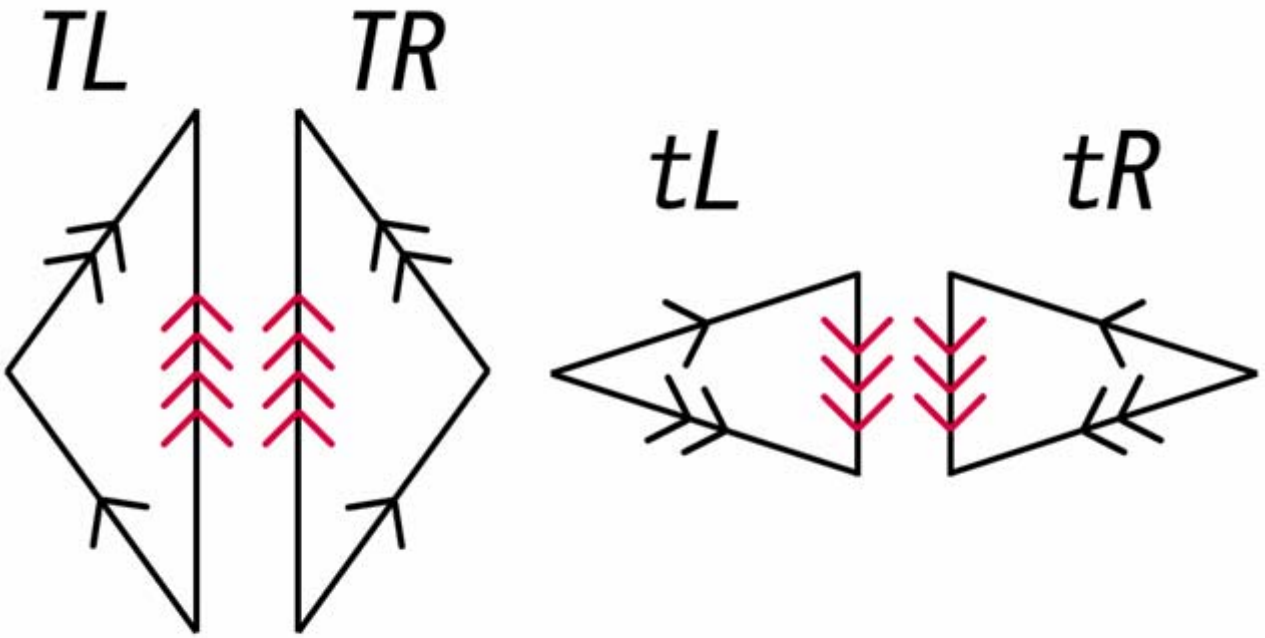
New question 1

- For which *n*, can tilings with *n*-fold symmetry be constructed only by the up-down generation procedure of some primitive substitution?

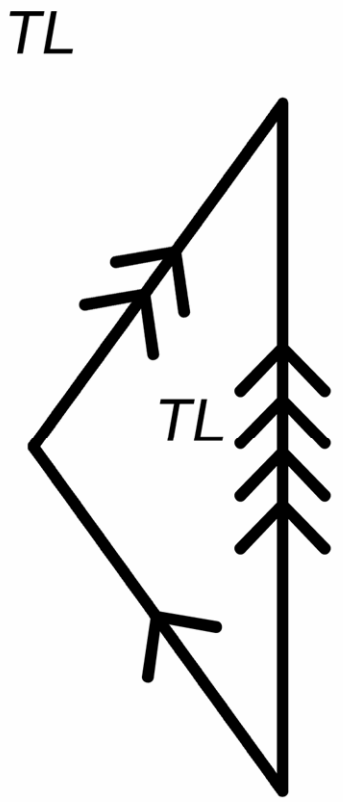
Forcing the border

- A substitution is said to **force the border** if there exists a positive integer *n* such that any two level-*n* supertiles of the same type have the same pattern of neighboring tiles.

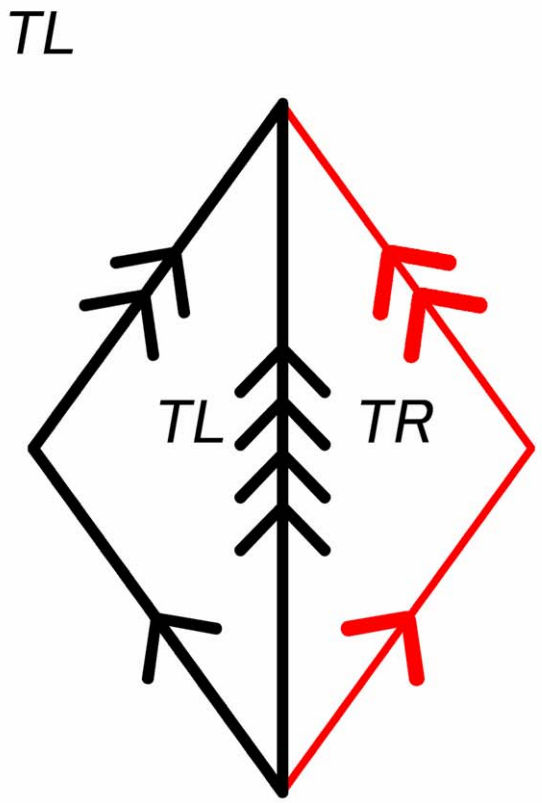
Divided Penrose tiles



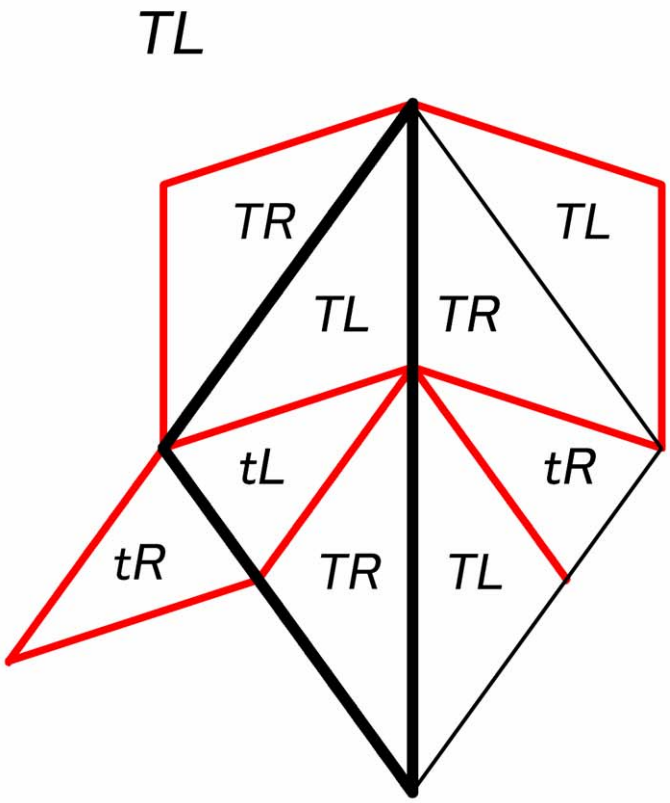
TL : level 0



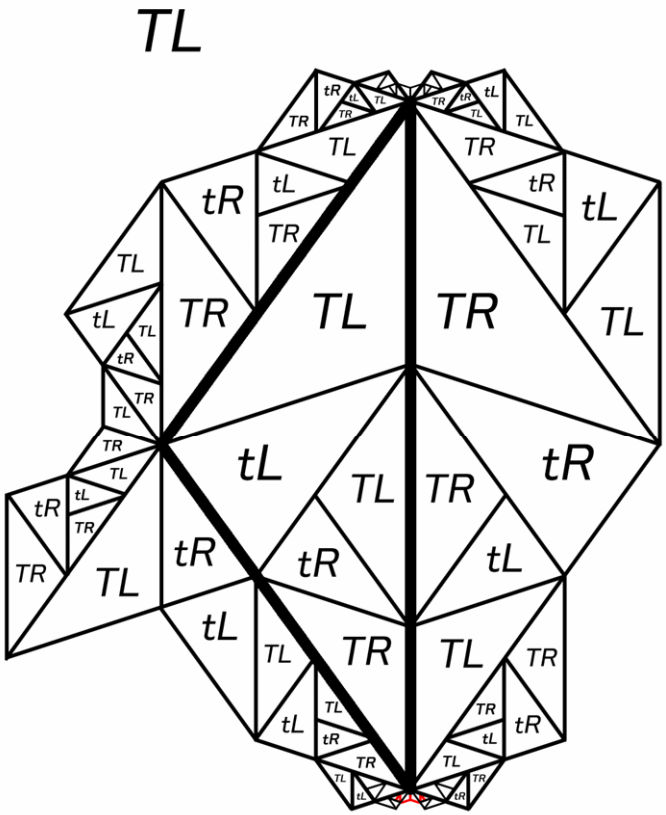
TL : level 0

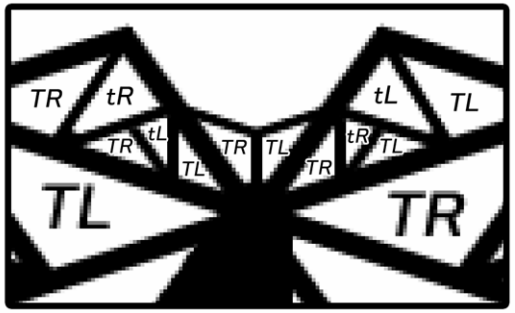


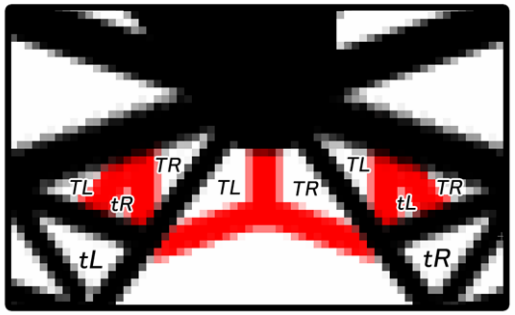
TL : level 1



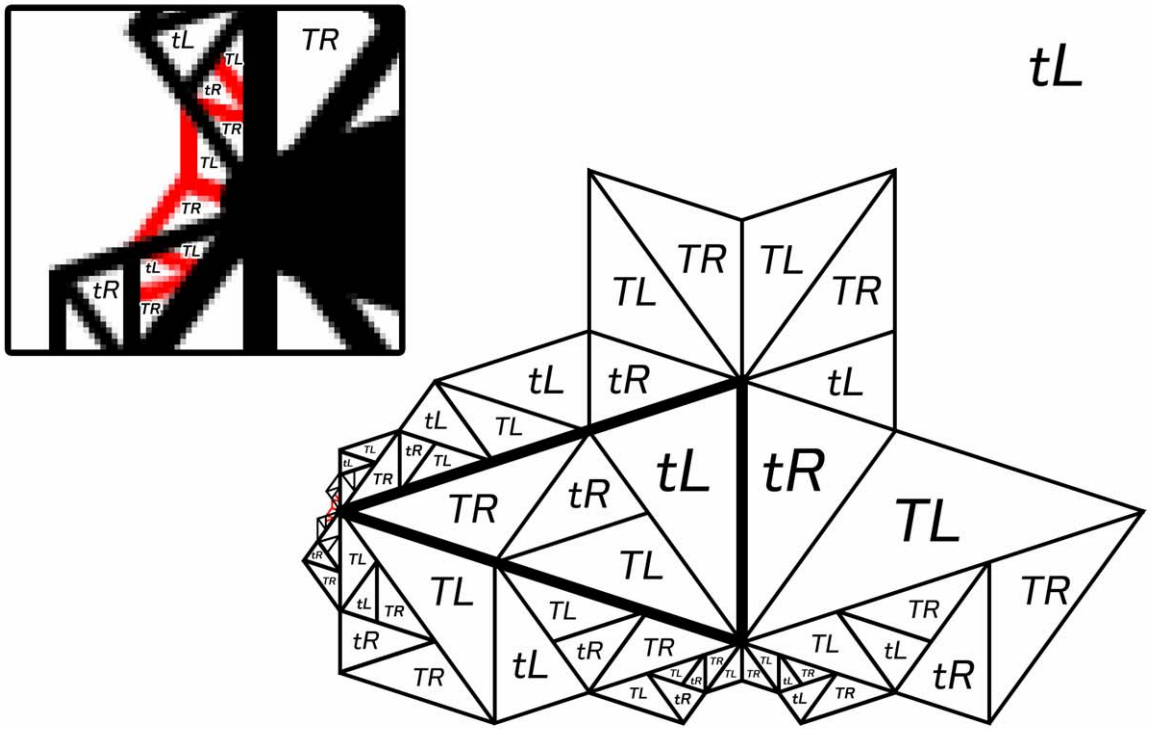
TL : level 8



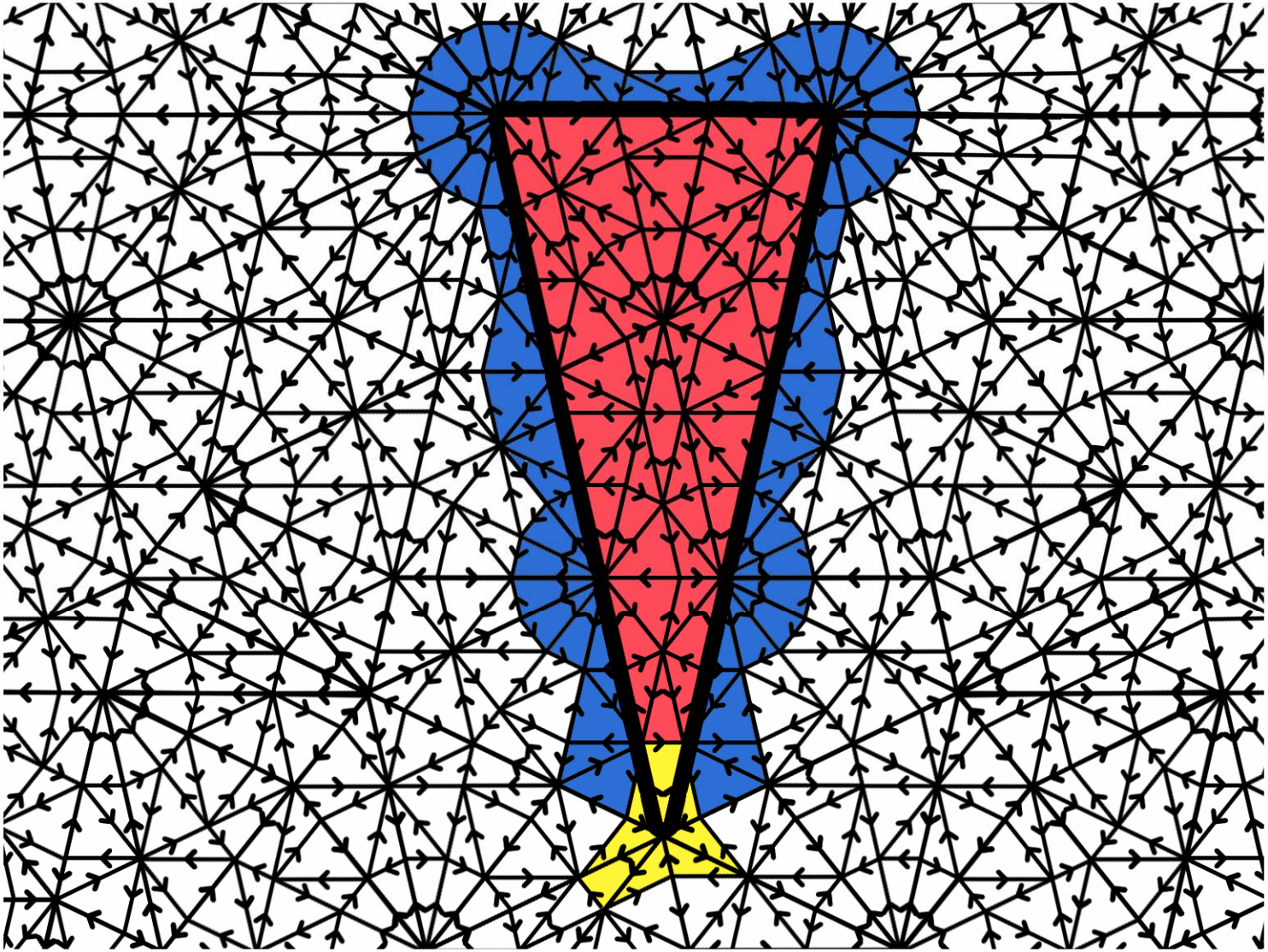




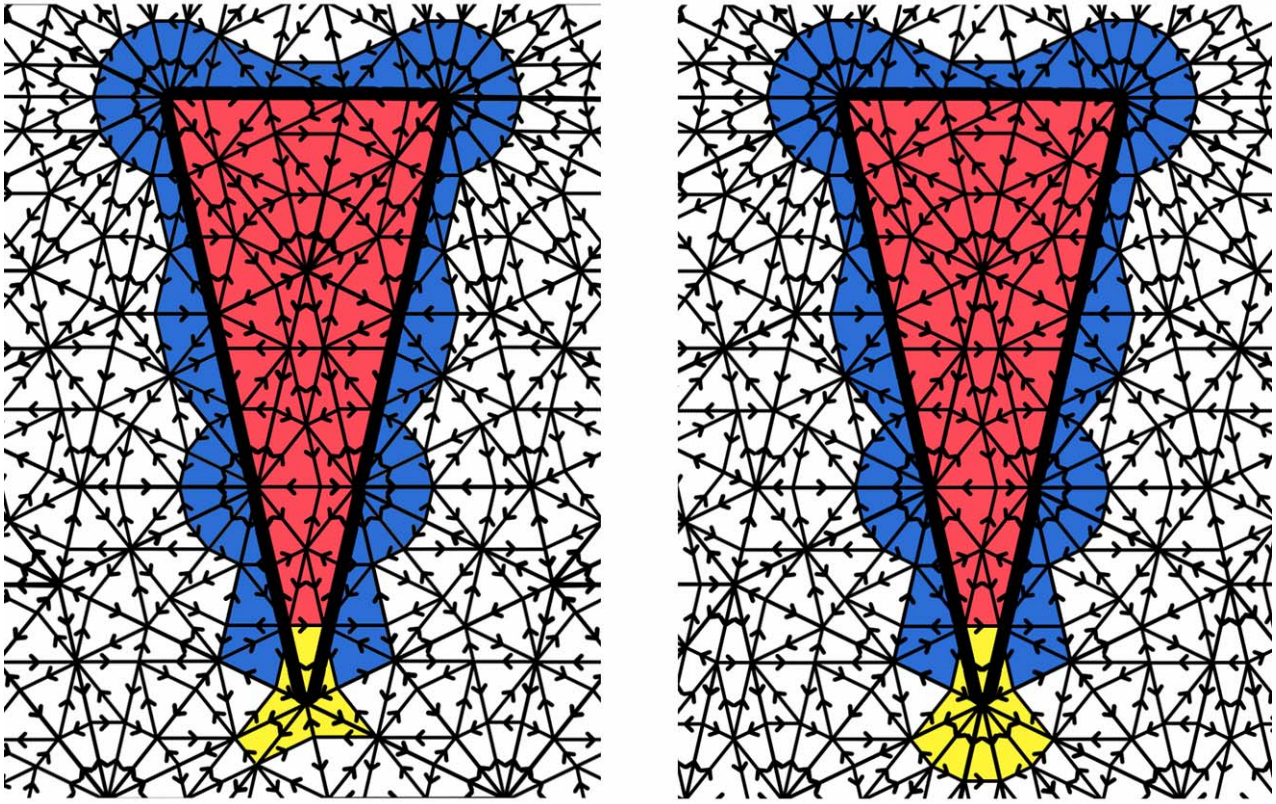
tL : level 8



red : the supertile
blue : the neighboring tiles
yellow : vertex atlas (i)



Two supertiles of the same type



- The way of attaching of Penrose tilings is unique.

- However, that of Danzer tiling is not unique.

New question 2

- Nonexistence of singular vertex atlases .

  $\overset{\text{iff ?}}{\Leftrightarrow}$ Forcing the boder.

- In particular, is the attachment of unbouded configurations unique, if singular vertex atlases do not exist?