Substitutions from Rauzy Induction on 4-interval exchange transformations and Quasi-periodic tilings

Pierre ARNOUX[*] Maki FURUKADO[†] Shunji ITO[‡]

Jun. 2010

This is a survey that focuses on the 2-dimensional quasi-periodic tilings by using the non-Pisot hyperbolic substitution generated by the Rauzy induction on exchanges of four intervals.

Key word: Rauzy induction, 4-interval exchange transformation, Quasi-periodic tiling

1 A Rauzy induction on 4-interval exchange transformations

Let $\mathcal{A}$ be the alphabet given by $\mathcal{A} = \{A, B, C, D\}$ and let us consider seven 2×4 matrices as follows:

$$\mathrm{I} = \begin{bmatrix} A & B & C & D \\ D & C & B & A \end{bmatrix}, \quad \mathrm{II} = \begin{bmatrix} A & C & D & B \\ D & C & B & A \end{bmatrix}, \quad \mathrm{III} = \begin{bmatrix} A & D & B & C \\ D & C & B & A \end{bmatrix},$$

$$\mathrm{IV} = \begin{bmatrix} A & D & B & C \\ D & C & A & B \end{bmatrix}, \quad \mathrm{V} = \begin{bmatrix} A & B & C & D \\ D & B & A & C \end{bmatrix}, \quad \mathrm{VI} = \begin{bmatrix} A & B & C & D \\ D & A & C & B \end{bmatrix},$$

$$\mathrm{VII} = \begin{bmatrix} A & B & D & C \\ D & A & C & B \end{bmatrix}.$$

For each $\mathrm{J} \in \{\mathrm{I}, \mathrm{II}, \cdots, \mathrm{VII}\}$, let us define the two bijections ${}_{\mathrm{J}}\pi_0 : \mathcal{A} \to \{1, 2, 3, 4\}$ and ${}_{\mathrm{J}}\pi_1 : \mathcal{A} \to \{1, 2, 3, 4\}$ by

$$\begin{aligned} {}_{\mathrm{J}}\pi_0 &= \text{the location of } \alpha \in \mathcal{A} \text{ in the first row vector of J}, \\ {}_{\mathrm{J}}\pi_1 &= \text{the location of } \alpha \in \mathcal{A} \text{ in the second row vector of J}. \end{aligned}$$

[*]IML UPR-CNRS 9016
[†]Faculty of Business Administration of Yokohama National University
[‡]Graduate School of Natural Science & Technology of Kanazawa Univerisity

For example, if $\mathrm{J} = \mathrm{II} = \left[\begin{array}{cccc} A & C & D & B \\ D & C & B & A \end{array}\right]$, then we obtain

$$\begin{aligned} \left({}_{\mathrm{J}}\pi_0(A),\ {}_{\mathrm{J}}\pi_0(B),\ {}_{\mathrm{J}}\pi_0(C),\ {}_{\mathrm{J}}\pi_0(D)\right) &= (1,4,2,3), \\ \left({}_{\mathrm{J}}\pi_1(A),\ {}_{\mathrm{J}}\pi_1(B),\ {}_{\mathrm{J}}\pi_1(C),\ {}_{\mathrm{J}}\pi_1(D)\right) &= (4,3,2,1), \\ \left({}_{\mathrm{J}}\pi_0^{-1}(1),\ {}_{\mathrm{J}}\pi_0^{-1}(2),\ {}_{\mathrm{J}}\pi_0^{-1}(3),\ {}_{\mathrm{J}}\pi_0^{-1}(4)\right) &= (A,C,D,B), \\ \left({}_{\mathrm{J}}\pi_1^{-1}(1),\ {}_{\mathrm{J}}\pi_1^{-1}(2),\ {}_{\mathrm{J}}\pi_1^{-1}(3),\ {}_{\mathrm{J}}\pi_1^{-1}(4)\right) &= (D,C,B,A). \end{aligned}$$

For each J, let us consider the 4-*interval exchange transformation* R_{J}, $\mathrm{J} \in \{\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VII}\}$ with the subintervals $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ of $[0,1)$ as follows [Y] (see Figure 1).

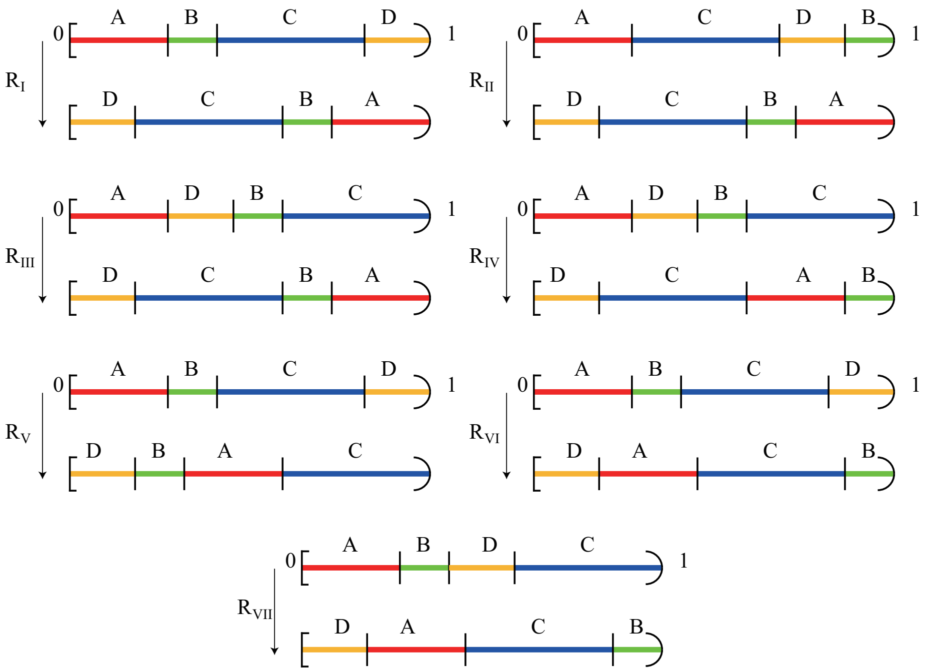


Figure 1: The 4-interval exchange transformations R_{J}.

Let $(\lambda_\alpha)_{\alpha\in\mathcal{A}}$ be the *length data* of the intervals I_α satisfying $\sum_{\alpha\in\mathcal{A}} \lambda_\alpha = 1$. Then, the transformation $R_J : [0,1) \to [0,1)$ is explicitly given by

$$R_{\mathrm{J}}(x) := x - \sum_{\substack{\beta :\\ {}_{\mathrm{J}}\pi_0(\beta) <_{\mathrm{J}} \pi_0(\alpha)}} \lambda_{{}_{\mathrm{J}}\pi_0(\beta)} + \sum_{\substack{\beta :\\ {}_{\mathrm{J}}\pi_1(\beta) <_{\mathrm{J}} \pi_1(\alpha)}} \lambda_{{}_{\mathrm{J}}\pi_1(\beta)} \quad \text{if } x \in I_\alpha.$$

For example, if $\mathrm{J} = \mathrm{I}$,

$$R_{\mathrm{I}}(x) := \begin{cases} x + \lambda_D + \lambda_C + \lambda_B & \text{if} \quad x \in I_A \\ x - \lambda_A + \lambda_D + \lambda_C & \text{if} \quad x \in I_B \\ x - (\lambda_A + \lambda_B) + \lambda_D & \text{if} \quad x \in I_C \\ x - (\lambda_A + \lambda_B + \lambda_C) & \text{if} \quad x \in I_D \end{cases}$$

(see Figure 2).

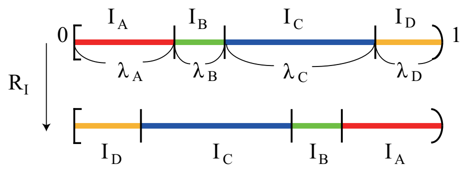


Figure 2: The 4-interval exchange transformation R_I given by the length data $(\lambda_\alpha)_{\alpha\in\mathcal{A}}$.

For each $\mathrm{J} \in \{\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VII}\}$, let us consider the *induced transformation* $(R_\mathrm{J})_{[0,\lambda^*_\varepsilon(\mathrm{J}))}$ of R_J where ε is given by

$$\varepsilon := \begin{cases} 0 & \text{if} \quad \lambda_{{}_\mathrm{J}\pi_0^{-1}(4)} > \lambda_{{}_\mathrm{J}\pi_1^{-1}(4)} \\ 1 & \text{if} \quad \lambda_{{}_\mathrm{J}\pi_0^{-1}(4)} < \lambda_{{}_\mathrm{J}\pi_1^{-1}(4)} \end{cases}$$

and $\lambda^*_\varepsilon(\mathrm{J})$ is given by

$$\lambda^*_\varepsilon(\mathrm{J}) := 1 - \min\left\{\lambda_{{}_\mathrm{J}\pi_0^{-1}(4)}, \lambda_{{}_\mathrm{J}\pi_1^{-1}(4)}\right\}.$$

Then, for J and ε, there exists J' such that the induced transformation $(R_\mathrm{J})_{[0,\lambda^*_\varepsilon(\mathrm{J}))}$ is *isomorphic* to $R_{\mathrm{J}'}$ by the isomorphism $\varphi_{\binom{\mathrm{J}}{\varepsilon}}(x) = \frac{x}{\lambda^*_\varepsilon(\mathrm{J})}$ from $[0, \lambda^*_\varepsilon(\mathrm{J}))$ to $[0, 1)$. For example, if $\mathrm{J} = \mathrm{I}$, the induced transformations $(R_\mathrm{I})_{[0,\lambda^*_\varepsilon(\mathrm{J}))}$, $\varepsilon \in \{0, 1\}$ are following (see Figure 3):

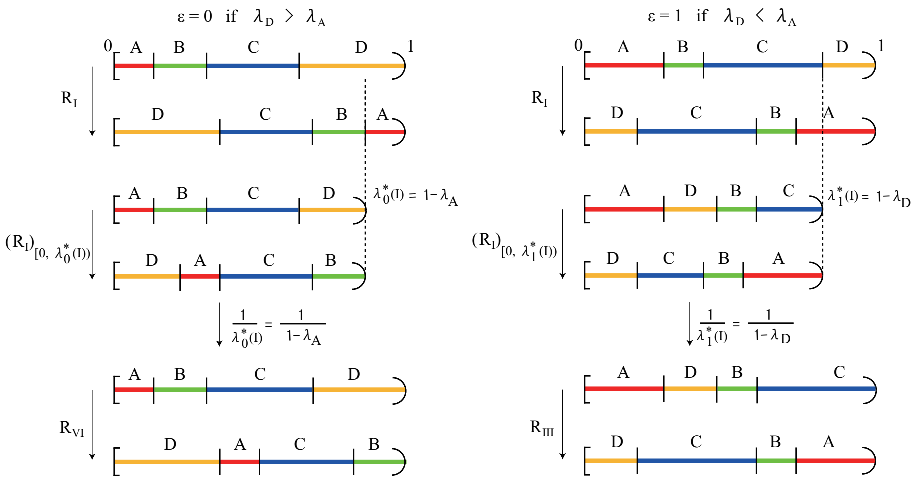


Figure 3: The induced transformations $(R_\mathrm{I})_{[0,\lambda^*_\varepsilon(\mathrm{J}))}$ of R_I, $\varepsilon = 0, 1$ and the renormalized transformations R_{VI} and R_{III} of $(R_\mathrm{I})_{[0,\lambda^*_\varepsilon(\mathrm{J}))}$, $\varepsilon = 0, 1$.

The other cases of $\mathrm{J} = \mathrm{II}, \mathrm{III}, \ldots, \mathrm{VII}$ are defined analogoulsy.

By the length $\lambda_{_{\mathrm{J}}\pi_0^{-1}(4)}$ and $\lambda_{_{\mathrm{J}}\pi_1^{-1}(4)}$ of the subintervals $I_{_{\mathrm{J}}\pi_0^{-1}(4)}$ and $I_{_{\mathrm{J}}\pi_1^{-1}(4)}$ respectively, we have a part of the *directed graph* with the vertices $\{\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VII}\}$ and the labels $\varepsilon \in \{0, 1\}$. For example, if $\mathrm{J} = \mathrm{I}$, see Figure 4.

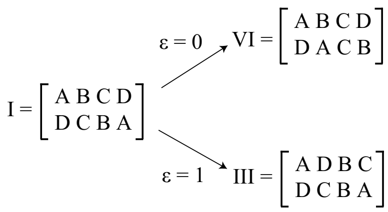


Figure 4: The directed graph that starting vertex is I.

The other cases are defined analogously.

Then we have the following *Rauzy induction diagram* from the 4-interval exchange transformations (see [Y]).

Proposition 1.1 (The Rauzy induction diagram)**.** *We have the following Rauzy induction diagram (see Figure 5):*

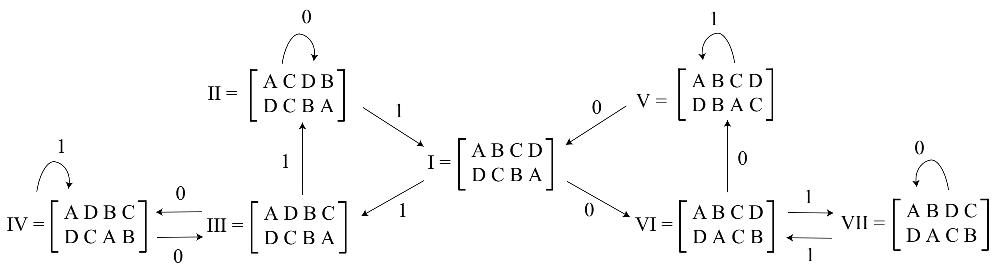


Figure 5: The Rauzy induction diagram (RID).

Using the Rauzy induction diagram (RID), we obtain the *RID-admissible path* $\left(\binom{J_1}{\varepsilon_1}\binom{J_2}{\varepsilon_2}\cdots\binom{J_i}{\varepsilon_i}\cdots\right)$ of $\binom{J_i}{\varepsilon_i} \in \{\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VII}\} \times \{0, 1\}$.

Now let us introduce the family of the substitutions $\sigma_{\binom{\mathrm{J}}{\varepsilon}}$ on $\mathcal{A}^*$ related to the induced transformation $(R_{\mathrm{J}})_{[0, \lambda^*_\varepsilon(\mathrm{J}))}$ as follows:

$$\sigma_{\binom{\mathrm{I}}{0}} : \begin{array}{lcl} A & \to & AD \\ B & \to & B \\ C & \to & C \\ D & \to & D \end{array} \quad \sigma_{\binom{\mathrm{I}}{1}} : \begin{array}{lcl} A & \to & A \\ B & \to & B \\ C & \to & C \\ D & \to & AD \end{array} \quad \sigma_{\binom{\mathrm{II}}{0}} : \begin{array}{lcl} A & \to & AB \\ B & \to & B \\ C & \to & C \\ D & \to & D \end{array} \quad \sigma_{\binom{\mathrm{II}}{1}} : \begin{array}{lcl} A & \to & A \\ B & \to & AB \\ C & \to & C \\ D & \to & D \end{array}$$

$$\sigma_{\binom{\text{III}}{0}}:\begin{array}{lcl}A&\to&AC\\B&\to&B\\C&\to&C\\D&\to&D\end{array}\quad \sigma_{\binom{\text{III}}{1}}:\begin{array}{lcl}A&\to&A\\B&\to&B\\C&\to&AC\\D&\to&D\end{array}\quad \sigma_{\binom{\text{IV}}{0}}:\begin{array}{lcl}A&\to&A\\B&\to&BC\\C&\to&C\\D&\to&D\end{array}\quad \sigma_{\binom{\text{IV}}{1}}:\begin{array}{lcl}A&\to&A\\B&\to&B\\C&\to&BC\\D&\to&D\end{array}$$

$$\sigma_{\binom{\text{V}}{0}}:\begin{array}{lcl}A&\to&A\\B&\to&B\\C&\to&CD\\D&\to&D\end{array}\quad \sigma_{\binom{\text{V}}{1}}:\begin{array}{lcl}A&\to&A\\B&\to&B\\C&\to&C\\D&\to&CD\end{array}\quad \sigma_{\binom{\text{VI}}{0}}:\begin{array}{lcl}A&\to&A\\B&\to&BD\\C&\to&C\\D&\to&D\end{array}\quad \sigma_{\binom{\text{VI}}{1}}:\begin{array}{lcl}A&\to&A\\B&\to&B\\C&\to&C\\D&\to&BD\end{array}$$

$$\sigma_{\binom{\text{VII}}{0}}:\begin{array}{lcl}A&\to&A\\B&\to&BC\\C&\to&C\\D&\to&D\end{array}\quad \sigma_{\binom{\text{VII}}{1}}:\begin{array}{lcl}A&\to&A\\B&\to&B\\C&\to&BC\\D&\to&D\end{array}.$$

We write the *incidence* matrices of the above substitutions $\sigma_{\binom{J_i}{\varepsilon_i}}$ as M_i.

Then, we have the following RID with the substitutions.

Proposition 1.2 (The RID with the substitutions)**.** *We have the following RID with the substituions (see Figure 6):*

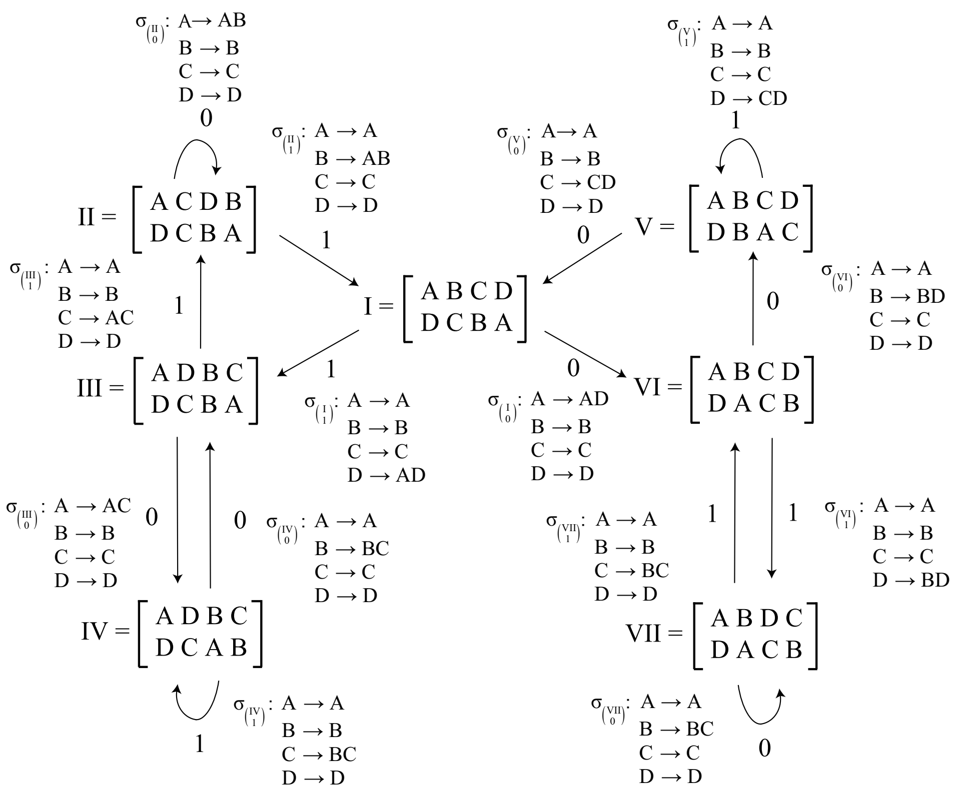


Figure 6: The RID with the substitutions.

For any *RID-admissible periodic path* $\overline{\left(\binom{\mathrm{J}_0}{\varepsilon_0}\binom{\mathrm{J}_1}{\varepsilon_1}\cdots\binom{\mathrm{J}_i}{\varepsilon_i}\cdots\binom{\mathrm{J}_{k-1}}{\varepsilon_{k-1}}\right)}$ with period k, we have the substitution σ_i as follows:

$$\sigma_i = \sigma_{\binom{\mathrm{J}_i}{\varepsilon_i}} \circ \sigma_{\binom{\mathrm{J}_{i+1}}{\varepsilon_{i+1}}} \circ \cdots \circ \sigma_{\binom{\mathrm{J}_{k-1}}{\varepsilon_{k-1}}} \circ \sigma_{\binom{\mathrm{J}_0}{\varepsilon_0}} \circ \cdots \circ \sigma_{\binom{\mathrm{J}_{i-1}}{\varepsilon_{i-1}}}$$

on $\mathcal{A}^*$. In this survey, we only consider the following RID-admissible periodic path:

$$\overline{\left(\begin{pmatrix}\mathrm{J}_0\\ \varepsilon_0\end{pmatrix}\begin{pmatrix}\mathrm{J}_1\\ \varepsilon_1\end{pmatrix}\cdots\begin{pmatrix}\mathrm{J}_i\\ \varepsilon_i\end{pmatrix}\cdots\begin{pmatrix}\mathrm{J}_7\\ \varepsilon_7\end{pmatrix}\right)} = \overline{\left(\begin{pmatrix}\mathrm{II}\\ 0\end{pmatrix}\begin{pmatrix}\mathrm{II}\\ 1\end{pmatrix}\begin{pmatrix}\mathrm{I}\\ 0\end{pmatrix}\begin{pmatrix}\mathrm{VI}\\ 0\end{pmatrix}\begin{pmatrix}\mathrm{V}\\ 1\end{pmatrix}\begin{pmatrix}\mathrm{V}\\ 0\end{pmatrix}\begin{pmatrix}\mathrm{I}\\ 1\end{pmatrix}\begin{pmatrix}\mathrm{III}\\ 1\end{pmatrix}\right)}$$

with period 8.

The substitution σ will be sometimes written by

$$\sigma(\alpha) = W_1^{(\alpha)} W_2^{(\alpha)} \cdots W_{l_\alpha}^{(\alpha)} = P_k^{(\alpha)} W_k^{(\alpha)} S_k^{(\alpha)}$$

where $P_k^{(\alpha)}$ (resp. $S_k^{(\alpha)}$) is the prefix (resp. suffix) of the letter $W_k^{(\alpha)}$.

2 On an example

Let us consider the following substitution σ as an example:

$$\sigma = \sigma_{\binom{\mathrm{II}}{0}} \circ \sigma_{\binom{\mathrm{II}}{1}} \circ \sigma_{\binom{\mathrm{I}}{0}} \circ \sigma_{\binom{\mathrm{VI}}{0}} \circ \sigma_{\binom{\mathrm{V}}{1}} \circ \sigma_{\binom{\mathrm{V}}{0}} \circ \sigma_{\binom{\mathrm{I}}{1}} \circ \sigma_{\binom{\mathrm{III}}{1}}$$

generated by a RID-admissible periodic path with period 8 (see Fig. 7).
The substitution σ is explicitly given by

$$\sigma: \begin{array}{lcl} A & \to & ABD \\ B & \to & ABBD \\ C & \to & ABDCCD \\ D & \to & ABDCD \end{array},$$

and its incidence matrix M_σ and its characteristic polynomial $\Phi_\sigma(x)$ are given by

$$M_\sigma = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}, \quad \Phi_\sigma(x) = x^4 - 7x^3 + 13x^2 - 7x + 1$$

respectively. Then, we see that the root of $\Phi_\sigma(x)$ is distributed by Figure 8.
Therefore we have the Perron-Frobenius eigenvector $\boldsymbol{v}_1$ satisfying

$$\boldsymbol{v}_1 = {}^t[\lambda_A, \lambda_B, \lambda_C, \lambda_D], \ \lambda_\alpha > 0, \ \text{and} \ \sum_{\alpha\in\mathcal{A}} \lambda_\alpha = 1$$

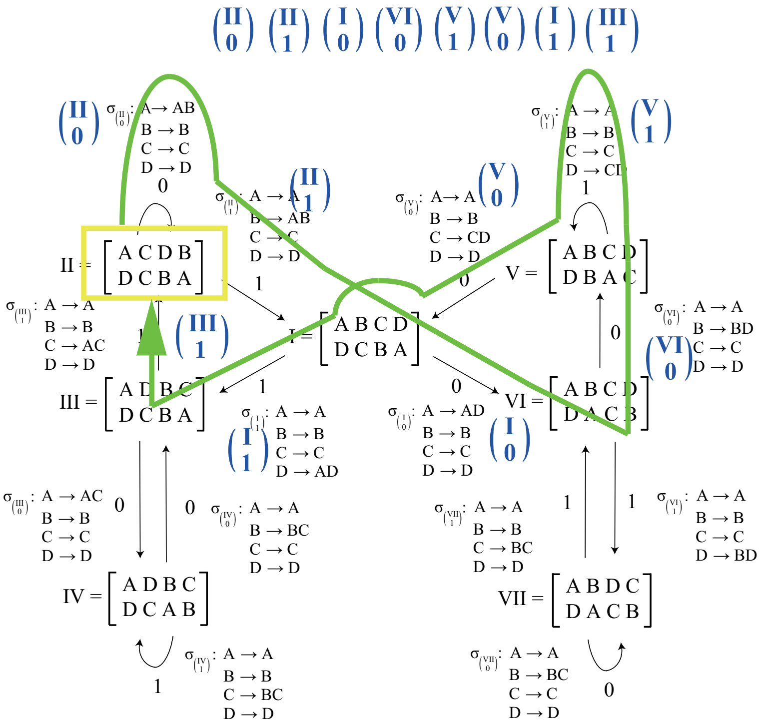


Figure 7: An example of a RID-admissible periodic path with period 8.

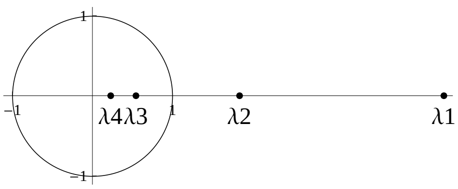


Figure 8: The distribution of the roots of $\Phi_{\sigma}(x)$.

where ${}^{t}M$ means the transpose of the matrix M.

Starting from σ, we obtian the following 4-interval exchange transformation. Let us define the partition $\{I_\alpha \mid \alpha \in \mathcal{A}\}$ of $[0,1)$ by

$$I_A = [0, \lambda_A), \ I_B = [\lambda_A, \lambda_A + \lambda_C), \ I_C = [\lambda_A + \lambda_C, \lambda_A + \lambda_C + \lambda_D), \\ I_D = [\lambda_A + \lambda_C + \lambda_D, 1)$$

(see Figure 9).

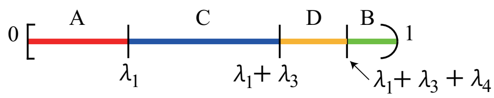


Figure 9: The partition of $[0,1)$.

From the definition, R_{II} is explicitly given by

$$R_{\text{II}}(x) = \begin{cases} x + \lambda_D + \lambda_C + \lambda_B & \text{if} \quad x \in I_A \\ x - \lambda_A + \lambda_D & \text{if} \quad x \in I_C \\ x - (\lambda_A + \lambda_C) & \text{if} \quad x \in I_D \\ x - \lambda_A & \text{if} \quad x \in I_B \end{cases}.$$

Then, $R_{\text{II}}(x)$ by $\lambda_B > \lambda_A$ and the induced transformation $(R_{\text{II}})_{[0,\lambda_0^*(\text{II}))}$ of R_{II} is isomorphic to R_{II} by the isomorphism $\varphi_{\binom{\text{II}}{0}}(x) = \frac{x}{\lambda_0^*(\text{II})}$ from $[0, \lambda_0^*(\text{II}))$ to $[0,1)$ (see Figure 10).

Let W be the fixed point of σ, that is,

$$W = s_1 s_2 \ldots s_k \ldots = \lim_{n\to\infty} \sigma^n(A).$$

Let

$$\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4) := \mathcal{L}(\boldsymbol{v}_1) \oplus \mathcal{L}(\boldsymbol{v}_2) \oplus \mathcal{L}(\boldsymbol{v}_3) \oplus \mathcal{L}(\boldsymbol{v}_4)$$

and let us define the projection π_i and π_{ij} by

$$\begin{aligned} \pi_i : \ \mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4) &\to \mathcal{L}(\boldsymbol{v}_i) \\ \pi_{ij} : \ \mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4) &\to \mathcal{L}(\boldsymbol{v}_i, \boldsymbol{v}_j) \end{aligned}$$

where $\boldsymbol{v}_i$, $i = 1,2,3,4$ are the eigenvectors associated to the eigenvalues λ_i, $i = 1,2,3,4$ of M_σ satisfying $\lambda_1 > \lambda_2 > 1 > \lambda_3 > \lambda_4 > 0$ respectively.

Moreover, let us define the homomorphism $f : \mathcal{A}^* \to \mathbb{Z}^4$ by

$$f(A) := \boldsymbol{e}_1, \ f(B) := \boldsymbol{e}_2, \ f(C) := \boldsymbol{e}_3, \ f(D) := \boldsymbol{e}_4, \ f(\emptyset) := \emptyset \\ f(W_1 W_2 \ldots W_k) := f(W_1) + f(W_2) + \cdots + f(W_k).$$

On the above notation, we have firstly the following proposition.

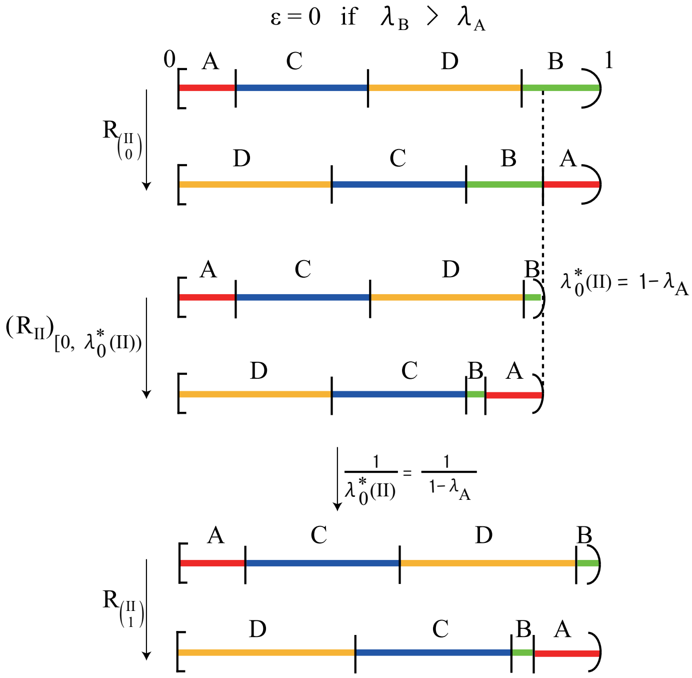


Figure 10: The induced transformation $(R_{\mathrm{II}})_{[0,\lambda_0^*(\mathrm{II}))}$ of R_{II}.

Proposition 2.1. *Let us define the set X_α, X'_α, X as follows:*

$$\begin{aligned} X_\alpha &:= \text{the closure of } \pi_4\left\{f\left(s_1 s_2 \dots s_{k-1}\right) \mid s_k=\alpha, k=1,2,\dots\right\}, \alpha\in\mathcal{A} \\ X'_\alpha &:= \text{the closure of } \pi_4\left\{f\left(s_1 s_2 \dots s_{k}\right) \mid s_k=\alpha, k=1,2,\dots\right\}, \alpha\in\mathcal{A} \\ X &:= \text{the closure of } \pi_4\left\{f\left(s_1 s_2 \dots s_{k-1}\right) \mid k=1,2,\dots\right\}. \end{aligned}$$

Then, we have the following properties:

(1) X_α is the interval of the line $\mathcal{L}(\boldsymbol{v}_4)$;

(2) $X=\bigcup_{\alpha\in\mathcal{A}} X_\alpha = \bigcup_{\alpha\in\mathcal{A}} X'_\alpha$;

(3) $X_\alpha \cap X_\beta$ $(\alpha\neq\beta)$, $\alpha,\beta\in\mathcal{A}$ are not overlapped;

(4) $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ satisfies the set equation:

$$\lambda_1 X_\alpha \left(=M_\sigma^{-1}X_\alpha\right) = \bigcup_{\beta\in\mathcal{A}} \bigcup_{W_k^{(\beta)}=\alpha} \left(\pi_4 f\left(P_k^{(\beta)}\right)+X_\beta\right);$$

(5) The interval exchange transformation $D: X\to X$ such that $D(X_\alpha)=X'_\alpha$ is isomorphic to $R_{\binom{\mathrm{II}}{0}}$ where $D: X\to X$ such that $D(X_\alpha)=X'_\alpha$ is isomorphic to $R_{\mathrm{II}} = \begin{bmatrix} A & C & D & B \\ D & C & B & A \end{bmatrix}$ (see Figure 11).

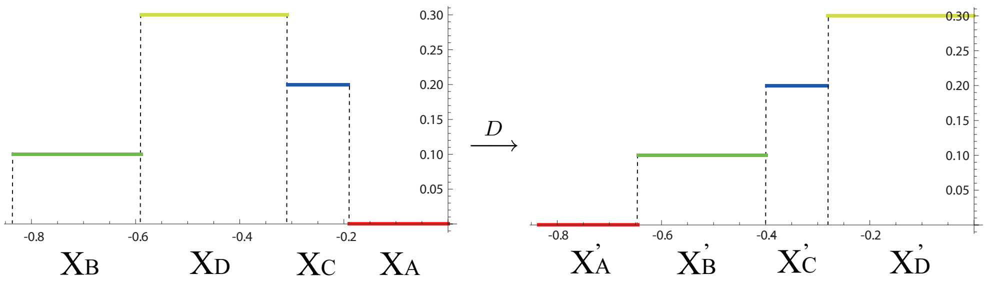


Figure 11: X_α and X'_α.

Moreover, we have the following theorem.

Theorem 2.2. *(cf. [F-I-Rao]) Let us define*

$$\begin{aligned}
\widehat{X}_\alpha &:= \text{the closure of } \pi_{34}\left\{f\left(s_1 s_2 \dots s_{k-1}\right) \mid s_k = \alpha, k = 1, 2, \dots\right\} \\
\widehat{X}'_\alpha &:= \text{the closure of } \pi_{34}\left\{f\left(s_1 s_2 \dots s_k\right) \mid s_k = \alpha, k = 1, 2, \dots\right\} \\
\widehat{X} &:= \text{the closure of } \pi_{34}\left\{f\left(s_1 s_2 \dots s_{k-1}\right) \mid k = 1, 2, \dots\right\}.
\end{aligned}$$

Then,

(1) $\widehat{X} = \bigcup_{\alpha \in \mathcal{A}} \widehat{X}_\alpha = \bigcup_{\alpha \in \mathcal{A}} \widehat{X}_{\alpha'}$ *(non-overlapping);*

(2) $\left\{\widehat{X}_\alpha\right\}_{\alpha \in \mathcal{A}}$ *satisfies the set set equation:*

$$M_\sigma^{-1} \widehat{X}_\alpha = \bigcup_{\beta \in \mathcal{A}} \bigcup_{W_k^{(\beta)} = \alpha} \left(\pi_{34}\left(f\left(P_k^{(\beta)}\right) + \widehat{X}_\beta\right)\right);$$

(3) The above set equation satisfies open set condition, that is, there exist a family of open set U_α*,* $\alpha \in \mathcal{A}$ *such that*

$$M_\sigma^{-1} U_\alpha \supset \bigcup_{\beta \in \mathcal{A}} \bigcup_{W_k^{(\beta)} = \alpha} \left(\pi_{34}\left(f\left(P_k^{(\beta)}\right) + U_\beta\right)\right)$$

where the right-hand side is non-overlapping union;

(4) The domain exchange transformation $\widehat{D} : \widehat{X} \to \widehat{X}$ *satisfying* $\widehat{D}\left(\widehat{X}_\alpha\right) = \widehat{X}_{\alpha'}$ *is well-defined (see Figure 12).*

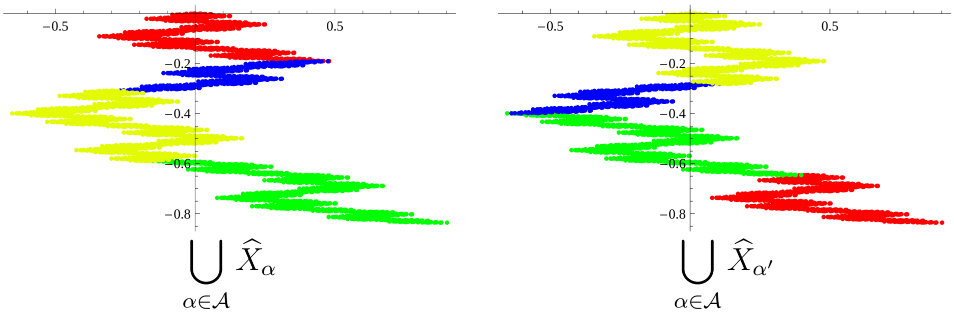


Figure 12: $\bigcup_{\alpha\in\mathcal{A}} \widehat{X}_\alpha$ and $\bigcup_{\alpha\in\mathcal{A}} \widehat{X}_{\alpha'}$.

3 Quasi-periodic tiling

Starting from the hyperbolic and non-Pisot substitutions (automorphism) σ of degree 4, the generating method of quasi-periodic tiling on $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ and $\mathcal{L}(\boldsymbol{v}_3, \boldsymbol{v}_4)$ were discussed in [A-F-H-I], [F-I-Rob], [H-F-I]. In this section, we show the existence of the quasi-periodic polygonal/self-affine tilings generated by substitution σ analogously.

Let us observe the figure of $\{\pi_{34}\boldsymbol{e}_i\}_{i=1,2,3,4}$ (see Figure 13). Using the projected basis $\{\pi_{34}\boldsymbol{e}_i\}_{i=1,2,3,4}$, we consider the *proto tiles* of parallelograms on $\mathcal{L}(\boldsymbol{v}_3, \boldsymbol{v}_4)$ (see Figure 14).

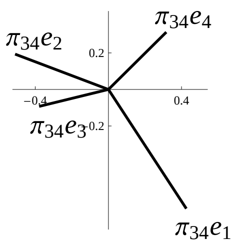


Figure 13: $\{\pi_{34}\boldsymbol{e}_i\}_{i=1,2,3,4}$.

Using the automorphism $\theta := \overleftarrow{\sigma}^{-1}$ (see [E]):

$$\begin{array}{rcl} \theta(A) & = & AD^{-1}CD^{-1}AB^{-1}A \\ \theta(B) & = & AD^{-1}CD^{-1}BA^{-1}DC^{-1}DA^{-1} \\ \theta(C) & = & AD^{-1}CD^{-1} \\ \theta(D) & = & DC^{-1}DA^{-1} \end{array},$$

we try to consider the 2-dim extension of the automorphisms θ as follows:

$$\begin{array}{rcl} E_2(\theta)(\mathbf{0}, \alpha\wedge\beta) & := & (\mathbf{0}, \theta(\alpha)\wedge\theta(\beta)) \\ & = & \displaystyle\sum_{\substack{1\leq i\leq l_\alpha \\ 1\leq j\leq l_\beta}} \left(f\left(P_i^{(\alpha)}\right) + f\left(P_j^{(\beta)}\right), W_i^{(\alpha)} \wedge W_j^{(\beta)} \right) \end{array}$$

(see Figure 15). Attention that we find the negative oriented parallelograms in $E_2(\theta)(\mathbf{0}, \alpha\wedge\beta)$ which is characterized as the strong colored parallelograms in Figure 15.

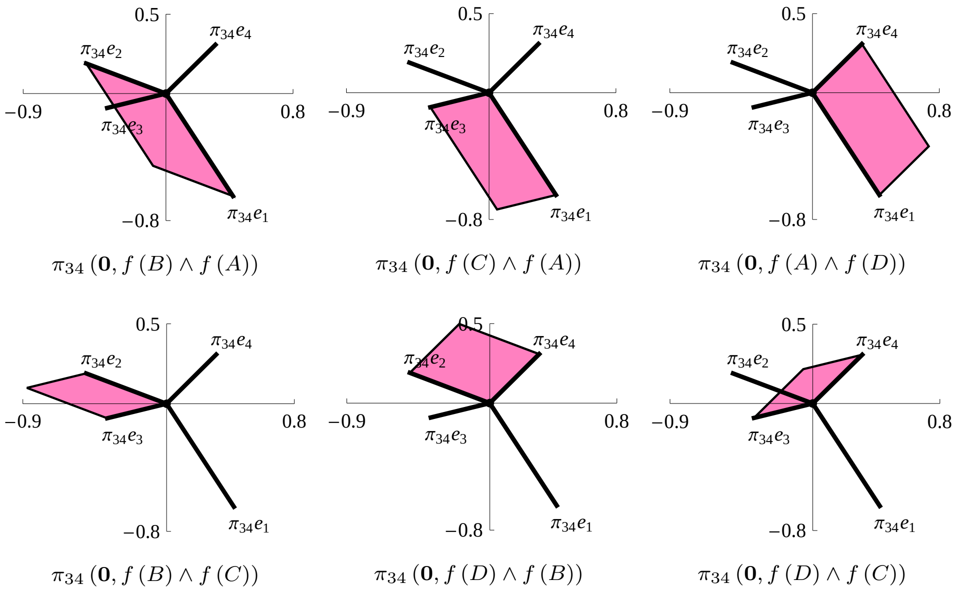


Figure 14: The proto tiles on $\mathcal{L}(\boldsymbol{v}_3, \boldsymbol{v}_4)$ generated by $f(A) = \boldsymbol{e}_1$, $f(B) = \boldsymbol{e}_2$, $f(C) = \boldsymbol{e}_3$, $f(D) = \boldsymbol{e}_4$.

On the example, we know that A^* is positive, that is,

$$A^* = \begin{matrix} A \wedge B \\ C \wedge A \\ D \wedge A \\ B \wedge C \\ B \wedge D \\ D \wedge C \end{matrix} \left[\quad m^*_{i \wedge j, k \wedge l} \quad \right] = \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 3 & 2 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \geq O$$

where $m^*_{i \wedge j, k \wedge l} = \det \begin{bmatrix} m_{ik} & m_{il} \\ m_{jk} & m_{jl} \end{bmatrix}$ for $M_\sigma^{-1} = [m_{ij}]_{1 \leq i,j \leq 4}$.

From this fact, we try to find the *tiling substitution* $\widehat{E_2}(\theta)$ by the *retiling method* in [F-I-Rob] (see Figure 16).

Theorem 3.1. *Let $\mathcal{U}_c$ be a patch generated by the following proto tiles:*

$$\begin{aligned} \mathcal{U}_c &= (\boldsymbol{z}, f(B) \wedge f(A)) + (\boldsymbol{z} + \boldsymbol{e}_4 - \boldsymbol{e}_3, f(C) \wedge f(A)) + (\boldsymbol{z}, f(A) \wedge f(D)) \\ &\quad + (\boldsymbol{z} + \boldsymbol{e}_4 - \boldsymbol{e}_3, f(B) \wedge f(C)) + (\boldsymbol{z}, f(D) \wedge f(B)) \\ &\quad + (\boldsymbol{z} + \boldsymbol{e}_1 - \boldsymbol{e}_3, f(D) \wedge f(C)) \end{aligned}$$

where $\boldsymbol{z} = \left(-\frac{4}{19}, -\frac{5}{19}, -\frac{1}{19}, 0\right)$. Then, we see that $\mathcal{U}_c$ is the seed, that is, $\widehat{E_2}(\theta)^3(\mathcal{U}_c) \succ \mathcal{U}_c$ (see Figure 17).

Moreover,

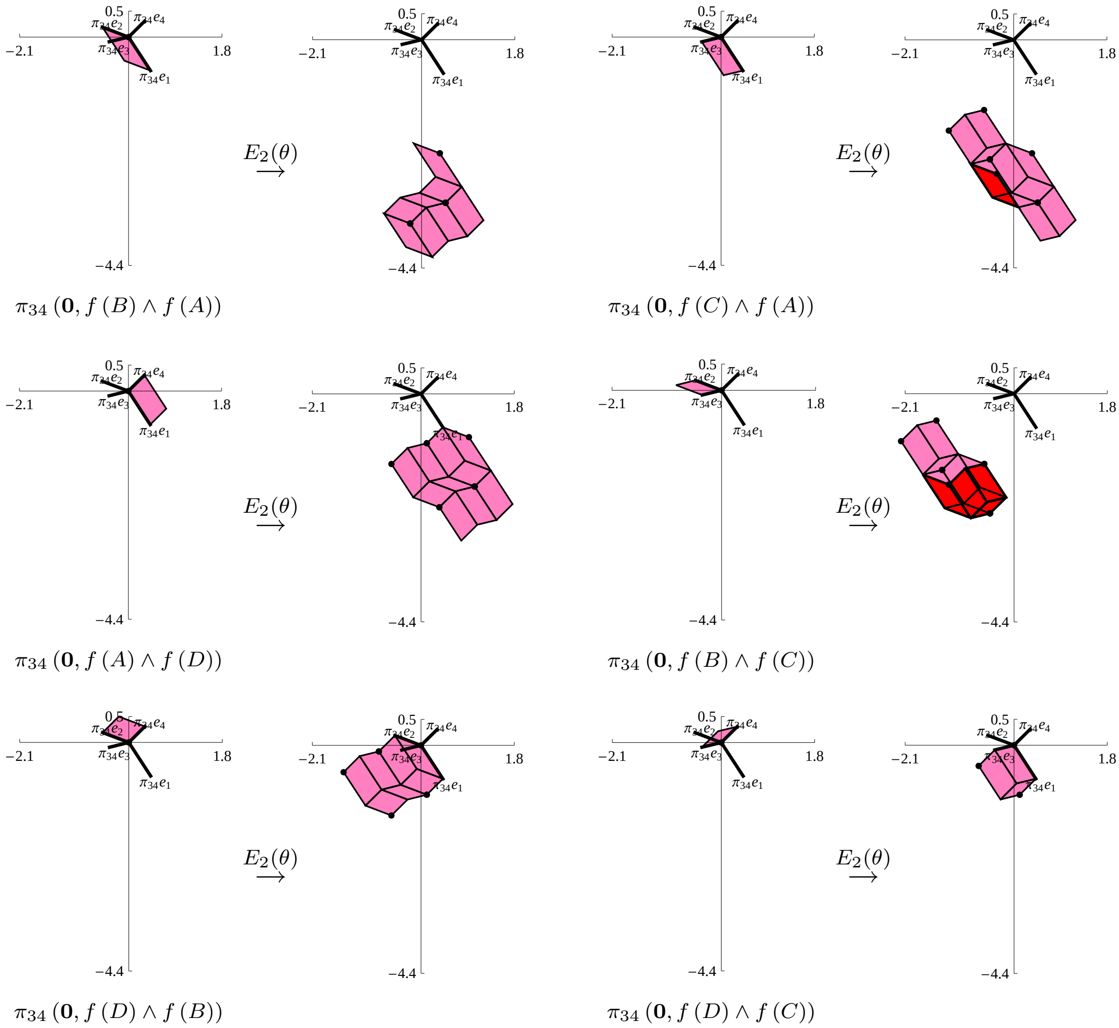
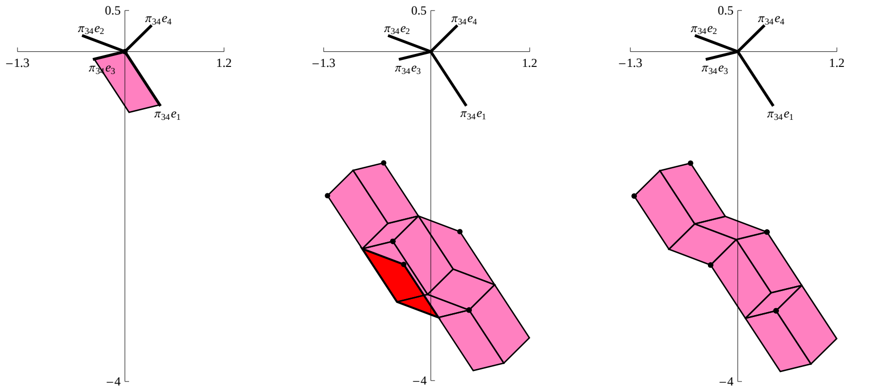
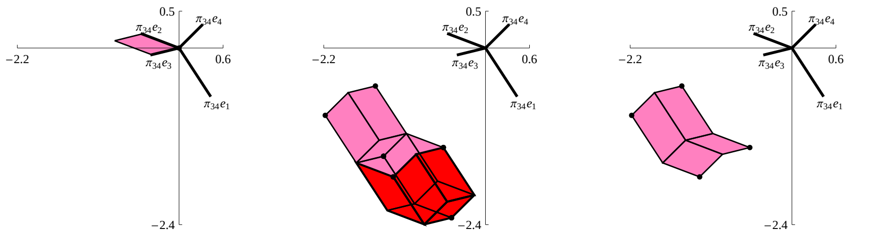


Figure 15: $E_2(\theta)\left(\mathbf{0}, \alpha \wedge \beta\right)$.

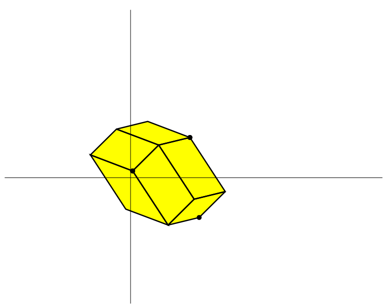


$\pi_{34}\left(\mathbf{0}, f(C) \wedge f(A)\right)$ $\pi_{34}E_2(\theta)\left(\mathbf{0}, f(C) \wedge f(A)\right)$ $\pi_{34}\widehat{E_2}(\theta)\left(\mathbf{0}, f(C) \wedge f(A)\right)$

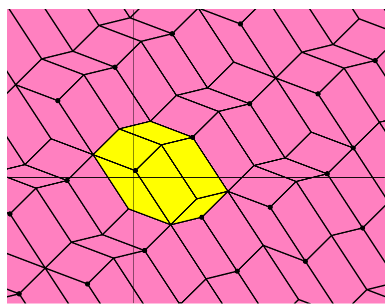


$\pi_{34}\left(\mathbf{0}, f(B) \wedge f(C)\right)$ $\pi_{34}E_2(\theta)\left(\mathbf{0}, f(B) \wedge f(C)\right)$ $\pi_{34}\widehat{E_2}(\theta)\left(\mathbf{0}, f(B) \wedge f(C)\right)$

Figure 16: The retiling method from $E_2(\theta)$ to $\widehat{E_2}(\theta)$.



$\pi_{34}\mathcal{U}_c$



A part of $\pi_{34}\widehat{E_2}(\theta)^3(\mathcal{U}_c)$

Figure 17: $\pi_{34}\mathcal{U}_c$ and a part of $\pi_{34}\widehat{E_2}(\theta)^3(\mathcal{U}_c)$.

(1) $\mathcal{T}_{c,1} := \left\{ \pi_{34}\left(\boldsymbol{x}, f(\alpha)\wedge f(\beta)\right) \;\middle|\; \widehat{E_2}(\theta)^{3n}(\mathcal{U}_c) \ni \left(\boldsymbol{x}, f(\alpha)\wedge f(\beta)\right) \right\}$ *is a quasi-periodic polygonal tiling of* $\mathcal{L}(\boldsymbol{v}_3, \boldsymbol{v}_4)$ *(see Figure 18);*

(2) Put

$$X_{\alpha\wedge\beta} := \lim_{n\to\infty} \pi_{34} M_\sigma^{3n} \widehat{E_2}(\theta)^{3n} \left(\mathbf{0}, f(\alpha)\wedge f(\beta)\right).$$

Then, $\{X_{\alpha\wedge\beta}\}$ *satisfies the set equations:*

$$M_\sigma\left(\pi_{34}\boldsymbol{x}_i + X_{\gamma_i}\right) = \sum_k \left(\pi_{34}\boldsymbol{x}_k^{(i)} + X_{\gamma_k^{(i)}}\right)$$

where $\mathcal{U}_c = \sum_{i=1}^{6}(\boldsymbol{x}_i, \gamma_i)$ *and* $\widehat{E_2}(\theta)(\boldsymbol{x}_i, \gamma_i) = \sum_k \left(\boldsymbol{x}_k^{(i)} + \gamma_k^{(i)}\right)$*;*

(3) $\mathcal{T}_{c,2} := \{\pi_{34}\boldsymbol{x} + X_{\alpha\wedge\beta} \mid \pi_{34}(\boldsymbol{x}, f(\alpha)\wedge f(\beta)) \in \mathcal{T}_{c,1}\}$. *Then,* $\mathcal{T}_{c,2}$ *is a quasi-periodic self-affine tiling of* $\mathcal{L}(\boldsymbol{v}_3, \boldsymbol{v}_4)$ *(see Figure 20).*

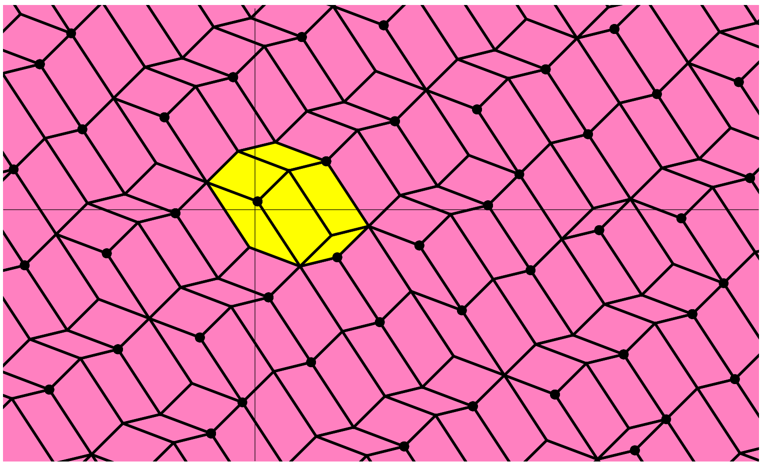


Figure 18: The quasi-periodic polygonal tiling $\mathcal{T}_{c,1}$ of $\mathcal{L}(\boldsymbol{v}_3, \boldsymbol{v}_4)$.

By the analogous discussion, we can construct the quasi-periodic polygonal/self-affine tiling from the "tiling substitution $\widehat{E_2}(\sigma)$" on the expanding plane $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$.

Let us observe the figure $\{\pi_{12}\boldsymbol{e}_i\}_{i=1,2,3,4}$ (see Figure 21) and we consider the *proto tiles* of parallelograms on $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ (see Figure 22).

Using the automorphism σ:

$$\begin{array}{rcl} \sigma(A) & = & ABD \\ \sigma(B) & = & ABBD \\ \sigma(C) & = & ABDCCD \\ \sigma(D) & = & ABDCD \end{array},$$

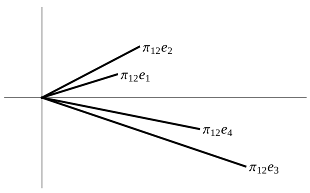
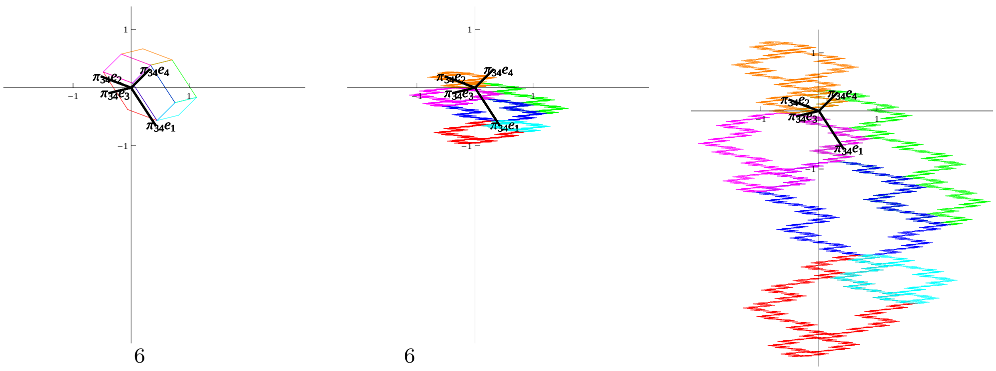


Figure 21: $\{\pi_{34}\boldsymbol{e}_i\}_{i=1,2,3,4}$.



$$\pi_{34}\mathcal{U}_c = \sum_{i=1}^{6} \pi_{34}\left(\boldsymbol{x}_i, \gamma_i\right) \qquad \sum_{i=1}^{6}\left(\pi_{34}\boldsymbol{x}_i + X_{\gamma_i}\right) \qquad M_\sigma\left(\pi_{34}\boldsymbol{x}_i + X_{\gamma_i}\right)$$

Figure 19: $\pi_{34}\mathcal{U}_c$ and the proto-tiles of the quasi-periodic self-affine tiling $\mathcal{T}_{c,2}$.

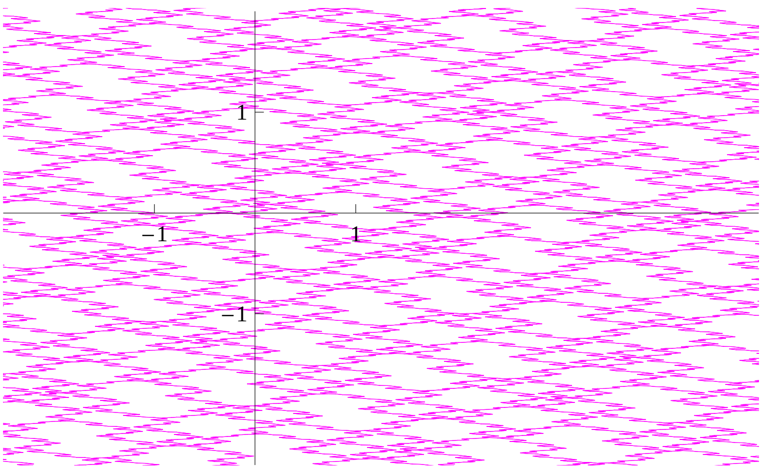


Figure 20: The quasi-periodic self-affine tiling $\mathcal{T}_{c,2}$ of $\mathcal{L}\left(\boldsymbol{v}_3, \boldsymbol{v}_4\right)$.

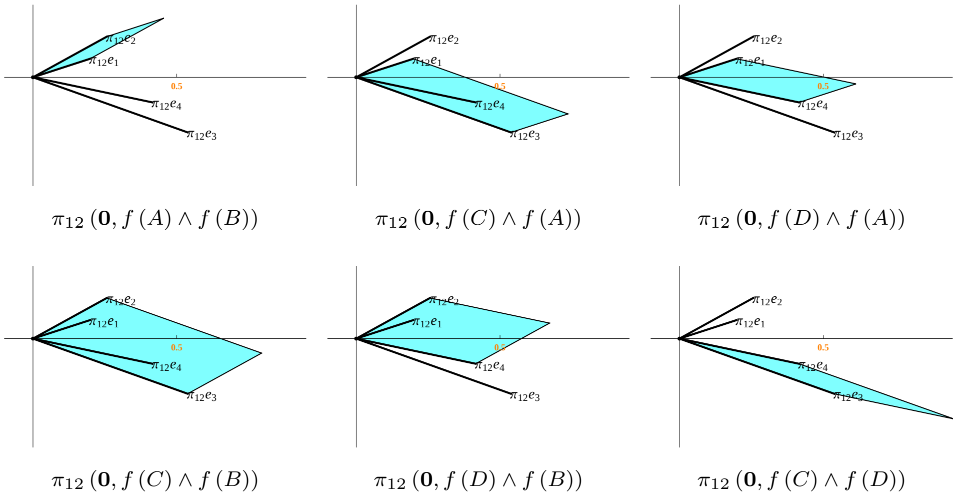


Figure 22: The proto tiles on $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ generated by $f(A) = \boldsymbol{e}_1$, $f(B) = \boldsymbol{e}_2$, $f(C) = \boldsymbol{e}_3$, $f(D) = \boldsymbol{e}_4$.

we try to consider the 2-dimensional extension of the automorphisms (substitutions) σ as follows:

$$\begin{aligned} E_2(\sigma)(\mathbf{0}, \alpha \wedge \beta) &:= (\mathbf{0}, \sigma(\alpha) \wedge \sigma(\beta)) \\ &= \sum_{\substack{1 \leq i \leq l_\alpha \\ 1 \leq j \leq l_\beta}} \left(f\left(P_i^{(\alpha)}\right) + f\left(P_j^{(\beta)}\right), W_i^{(\alpha)} \wedge W_j^{(\beta)} \right) \end{aligned}$$

(see Figure 23):

On our example, we know that A^* is positive, that is,

$$A^* = \begin{matrix} A \wedge B \\ C \wedge A \\ D \wedge A \\ C \wedge B \\ D \wedge B \\ C \wedge D \end{matrix} \left[\quad a^*_{i \wedge j, k \wedge l} \quad \right] = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 4 & 2 & 1 \\ 1 & 1 & 1 & 3 & 3 & 0 \\ 0 & 2 & 1 & 2 & 1 & 2 \end{bmatrix} \geq O$$

where $a^*_{i \wedge j, k \wedge l} = \det \begin{bmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{bmatrix}$ for $M_\sigma = [a_{ij}]_{1 \leq i,j \leq 4}$.

From this fact, we try to find the tiling substitution $\widehat{E_2}(\sigma)$ by the retiling method in [F-I-Rob] analogously (see Figure 24):

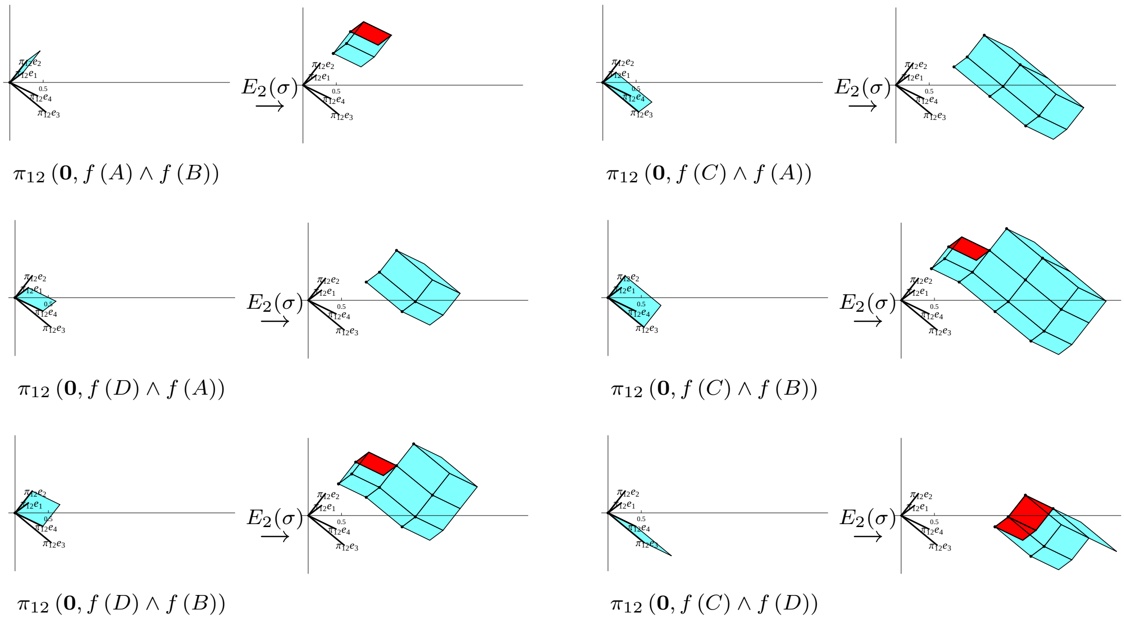


Figure 23: $E_2(\sigma)(\mathbf{0}, \alpha \wedge \beta)$.

Theorem 3.2. *Let $\mathcal{U}_e$ be a patch generated by the following proto tiles:*

$$\begin{aligned}\mathcal{U}_e &= (\boldsymbol{y}+\boldsymbol{e}_4, f(A) \wedge f(B)) + (\boldsymbol{y}+\boldsymbol{e}_4, f(C) \wedge f(A)) + (\boldsymbol{y}+\boldsymbol{e}_2, f(D) \wedge f(A)) \\ &\quad + (\boldsymbol{y}+\boldsymbol{e}_4, +\boldsymbol{e}_1, f(C) \wedge f(B)) + (\boldsymbol{y}, f(D) \wedge f(B)) + (\boldsymbol{y}, f(C) \wedge f(D)),\end{aligned}$$

where $\boldsymbol{y} = \left(\frac{8}{19}, -\frac{14}{19}, -\frac{2}{19}, -\frac{18}{19}\right)$. Then, we see that $\mathcal{U}_e$ is the seed, that is, $\widehat{E_2}(\sigma)^3(\mathcal{U}_e) \succ \mathcal{U}_e$ (see Figure 25).

Moreover,

(1) $\mathcal{T}_{e,1} := \left\{ \pi_{12}(\boldsymbol{x}, f(\alpha) \wedge f(\beta)) \;\middle|\; \widehat{E_2}(\sigma)^{3n}(\mathcal{U}_e) \ni (\boldsymbol{x}, f(\alpha) \wedge f(\beta)) \right\}$ is a quasi-periodic polygonal tiling of $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ (see Figure 26);

(2) Put

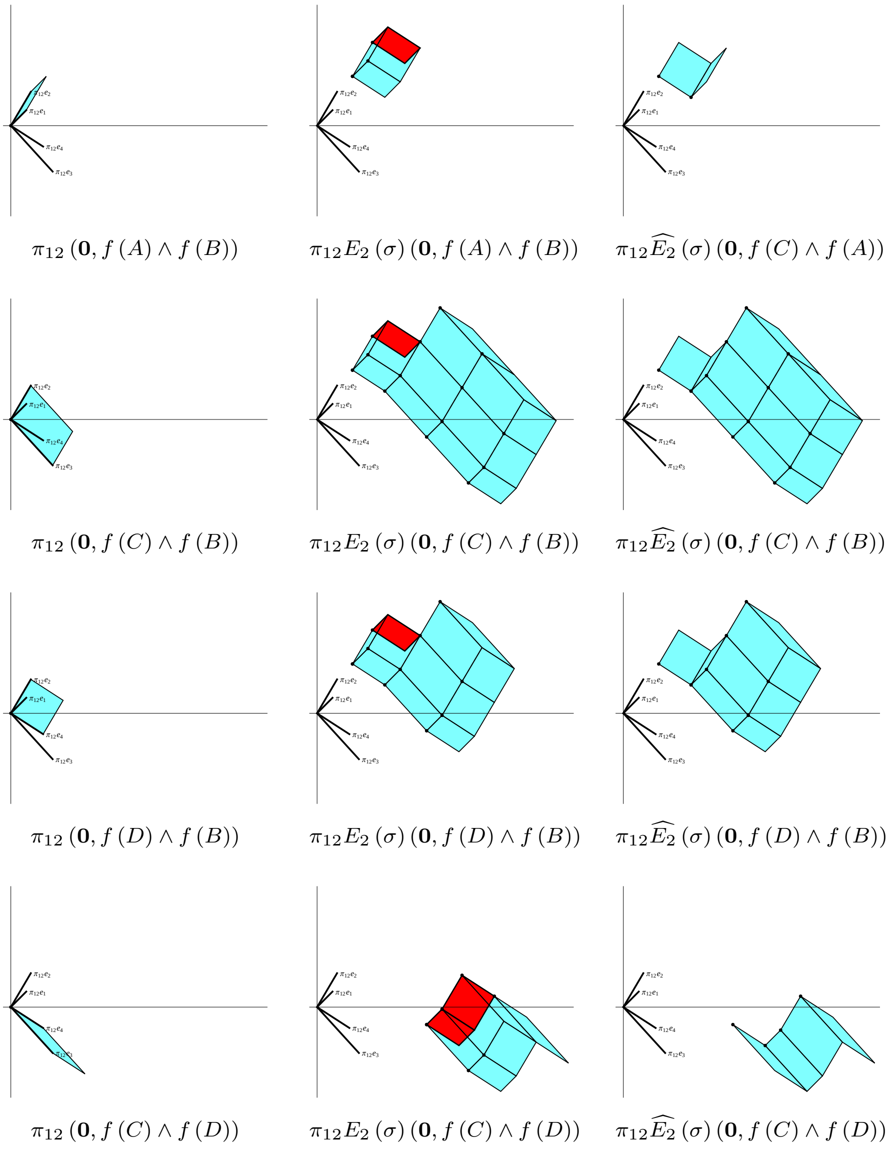
$$X_{\alpha\wedge\beta} := \lim_{n\to\infty} \pi_{12} M_\sigma^{-3n} \widehat{E_2}(\sigma)^{3n}(\mathbf{0}, f(\alpha) \wedge f(\beta)).$$

Then, $\{X_{\alpha\wedge\beta}\}$ satisfies the set equations:

$$M_\sigma^{-1}(\pi_{12}\boldsymbol{x}_i + X_{\delta_i}) = \sum_k \left(\pi_{12}\boldsymbol{x}_k^{(i)} + X_{\delta_k^{(i)}}\right)$$

where $\mathcal{U}_e = \sum_{i=1}^{6}(\boldsymbol{x}_i, \delta_i)$ and $\widehat{E_2}(\sigma)(\boldsymbol{x}_i, \delta_i) = \sum_k \left(\boldsymbol{x}_k^{(i)} + \delta_k^{(i)}\right)$;

(3) $\mathcal{T}_{e,2} := \{\pi_{12}\boldsymbol{x} + X_{\alpha\wedge\beta} \mid \pi_{12}(\boldsymbol{x}, f(\alpha) \wedge f(\beta)) \in \mathcal{T}_{e,1}\}$. Then, $\mathcal{T}_{e,2}$ is a quasi-periodic self-affine tiling of $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ (see Figure 28).



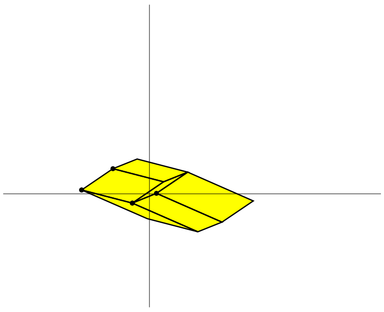
$\pi_{12}(\mathbf{0}, f(A) \wedge f(B))$ $\pi_{12}E_2(\sigma)(\mathbf{0}, f(A) \wedge f(B))$ $\pi_{12}\widehat{E_2}(\sigma)(\mathbf{0}, f(C) \wedge f(A))$

$\pi_{12}(\mathbf{0}, f(C) \wedge f(B))$ $\pi_{12}E_2(\sigma)(\mathbf{0}, f(C) \wedge f(B))$ $\pi_{12}\widehat{E_2}(\sigma)(\mathbf{0}, f(C) \wedge f(B))$

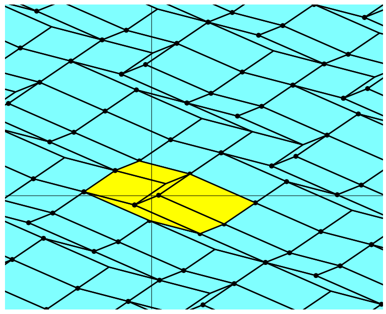
$\pi_{12}(\mathbf{0}, f(D) \wedge f(B))$ $\pi_{12}E_2(\sigma)(\mathbf{0}, f(D) \wedge f(B))$ $\pi_{12}\widehat{E_2}(\sigma)(\mathbf{0}, f(D) \wedge f(B))$

$\pi_{12}(\mathbf{0}, f(C) \wedge f(D))$ $\pi_{12}E_2(\sigma)(\mathbf{0}, f(C) \wedge f(D))$ $\pi_{12}\widehat{E_2}(\sigma)(\mathbf{0}, f(C) \wedge f(D))$

Figure 24: The retiling method from $E_2(\sigma)$ to $\widehat{E_2}(\sigma)$.



$\pi_{12}\mathcal{U}_e$



A part of $\pi_{12}\widehat{E_2}(\sigma)^3(\mathcal{U}_c)$

Figure 25: $\pi_{12}\mathcal{U}_e$ and a part of $\pi_{12}\widehat{E_2}(\sigma)^3(\mathcal{U}_e)$.

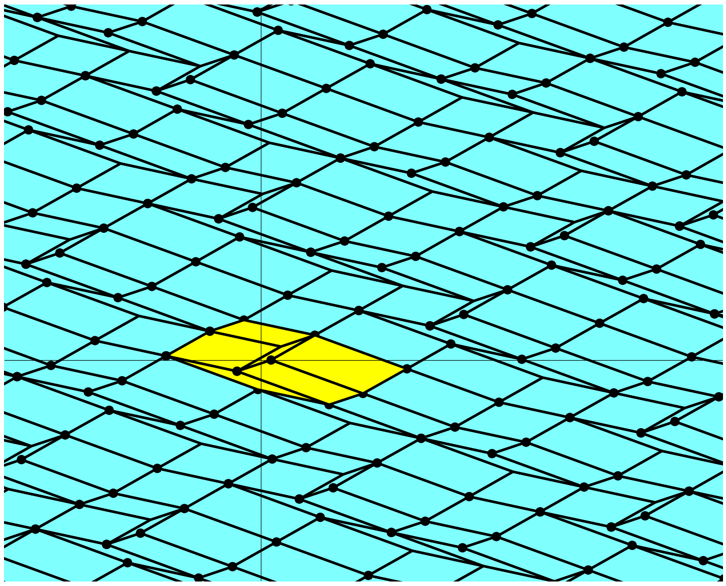
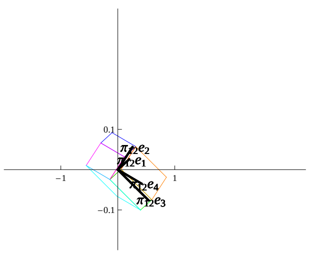
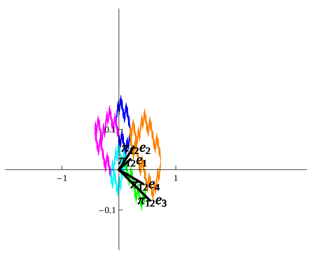


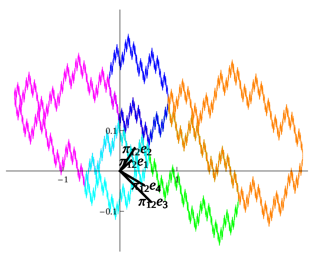
Figure 26: The quasi-periodic polygonal tiling $\mathcal{T}_{e,1}$ of $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$.



$\pi_{12}\mathcal{U}_e = \sum_{i=1}^{6} \pi_{12}(\boldsymbol{x}_i, \delta_i)$



$\sum_{i=1}^{6} (\pi_{12}\boldsymbol{x}_i + X_{\delta_i})$



$M_\sigma^{-1}(\pi_{12}\boldsymbol{x}_i + X_{\delta_i})$

Figure 27: $\pi_{12}\mathcal{U}_e$ and the proto-tiles of the quasi-periodic self-affine tiling $\mathcal{T}_{e,2}$.

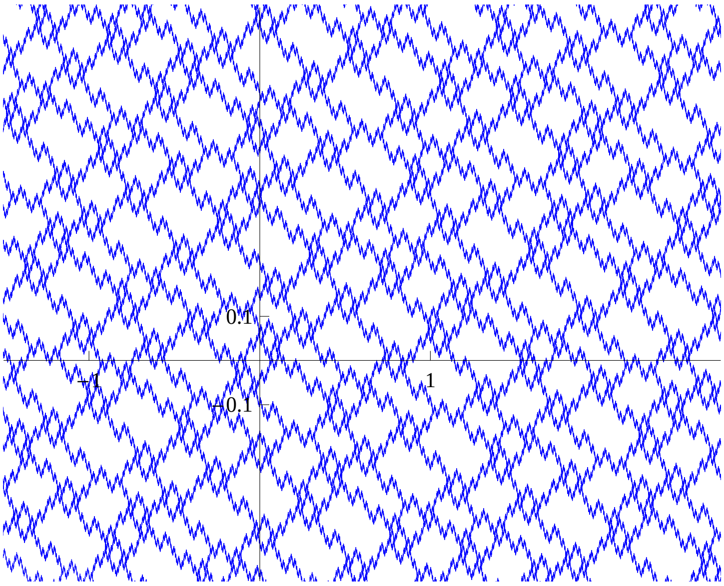


Figure 28: The quasi-periodic self-affine tiling $\mathcal{T}_{e,2}$ of $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$.

References

[A-F-H-I] P.Arnoux, M.Furukado, E.Harriss and S.Ito, *Algebraic numbers and automorphisms of free groups*, to appear in Transactions of the AMS.

[E] H.Ei, *Some properties of invertible substitutions of rank d, and higher dimensional substitutions*, Osaka J. Math. **40** (2003), 543–562.

[F-I-Rao] M.Furukado, S.Ito and H.Rao, *Geometric realizations of hyperbolic unimodular substitutions*, Progress in Probability, **61** (2009), 251–268.

[F-I-Rob] M.Furukado, S.Ito and E.A.Robinson Jr., *Tilings associated with non-Pisot matrices*, Ann. Inst. Fourier, Grenoble **56** (2006), 2391–2435.

[H-F-I] M.Hama, M.Furukado and S.Ito, *Complex Pisot Numeration Systems*, Comment. Math. Univ. St. Pauli, **58** (2009), 9–49.

[Y] J. C. Yoccoz, *Continued fraction algorithms for interval exchange maps: an introduction*, Frontiers in Number Theory, Physics, and Geometry I, Springer Verlag (2006), 403–437.