Substitutions from Rauzy Induction on 4-interval exchange transformations and Quasi-periodic tilings

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This is a survey that focuses on the 2-dimensional quasi-periodic tilings by using the non-Pisot hyperbolic substitution generated by the Rauzy induction on exchanges of four intervals.

Key word: Rauzy induction, 4-interval exchange transformation, Quasi-periodic tiling

1 A Rauzy induction on 4-interval exchange transformations

Let \( \mathcal{A} \) be the alphabet given by \( \mathcal{A} = \{A, B, C, D\} \) and let us consider seven \( 2 \times 4 \) matrices as follows:

\[
\begin{align*}
I &= \begin{bmatrix} A & B & C & D \\ D & C & B & A \end{bmatrix}, & 
II &= \begin{bmatrix} A & C & D & B \\ D & C & B & A \end{bmatrix}, & 
III &= \begin{bmatrix} A & D & B & C \\ D & C & B & A \end{bmatrix}, \\
IV &= \begin{bmatrix} A & D & B & C \\ D & C & A & B \end{bmatrix}, & 
V &= \begin{bmatrix} A & B & C & D \\ D & B & A & C \end{bmatrix}, & 
VI &= \begin{bmatrix} A & B & C & D \\ D & A & C & B \end{bmatrix}, \\
VII &= \begin{bmatrix} A & B & D & C \\ D & A & C & B \end{bmatrix}.
\end{align*}
\]

For each \( J \in \{I, II, \cdots, VII\} \), let us define the two bijections \( j\pi_0 : \mathcal{A} \to \{1, 2, 3, 4\} \) and \( j\pi_1 : \mathcal{A} \to \{1, 2, 3, 4\} \) by

\[
\begin{align*}
j\pi_0 &= \text{the location of } \alpha \in \mathcal{A} \text{ in the first row vector of } J, \\
j\pi_1 &= \text{the location of } \alpha \in \mathcal{A} \text{ in the second row vector of } J.
\end{align*}
\]

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For example, if $J = \Pi = \begin{bmatrix} A & C & D & B \\ D & C & B & A \end{bmatrix}$, then we obtain
\[
\begin{align*}
( J \pi_0 (A) , J \pi_0 (B) , J \pi_0 (C) , J \pi_0 (D) ) &= (1, 4, 2, 3), \\
( J \pi_1 (A) , J \pi_1 (B) , J \pi_1 (C) , J \pi_1 (D) ) &= (4, 3, 2, 1), \\
( J \pi_1^{-1} (1) , J \pi_1^{-1} (2) , J \pi_0^{-1} (3) , J \pi_0^{-1} (4) ) &= (A, C, D, B), \\
( J \pi_1^{-1} (1) , J \pi_1^{-1} (2) , J \pi_1^{-1} (3) , J \pi_1^{-1} (4) ) &= (D, C, B, A).
\end{align*}
\]

For each $J$, let us consider the 4-interval exchange transformation $R_J$, $J \in \{I, \Pi, \ldots, VII\}$ with the subintervals $\{I_\alpha\}_{\alpha \in A}$ of $[0, 1)$ as follows [Y] (see Figure 1).

[Diagram of 4-interval exchange transformations $R_J$.]

Figure 1: The 4-interval exchange transformations $R_J$.

Let $(\lambda_\alpha)_{\alpha \in A}$ be the length data of the intervals $I_\alpha$ satisfying $\sum_{\alpha \in A} \lambda_\alpha = 1$. Then, the transformation $R_J : [0, 1) \to [0, 1)$ is explicitly given by
\[
R_J (x) := x - \sum_{\beta : \pi_0 (\beta) < \pi_0 (\alpha)} \lambda_{J \pi_0 (\beta)} + \sum_{\beta : \pi_1 (\beta) < \pi_1 (\alpha)} \lambda_{J \pi_1 (\beta)} \quad \text{if } x \in I_\alpha.
\]

For example, if $J = I$,
\[
R_I (x) := \begin{cases} 
  x + \lambda_D + \lambda_C + \lambda_B & \text{if } x \in I_A \\
  x - \lambda_A + \lambda_D + \lambda_C & \text{if } x \in I_B \\
  x - (\lambda_A + \lambda_B) + \lambda_D & \text{if } x \in I_C \\
  x - (\lambda_A + \lambda_B + \lambda_C) & \text{if } x \in I_D 
\end{cases}
\]
(see Figure 2).

Figure 2: The 4-interval exchange transformation $R_I$ given by the length data $(\lambda_\alpha)_{\alpha \in A'}$. For each $J \in \{I, II, \ldots, VII\}$, let us consider the induced transformation $(R_J)_{[0, \lambda^*_\varepsilon(J)]}$ of $R_J$ where $\varepsilon$ is given by

$$\varepsilon := \begin{cases} 0 & \text{if } \lambda_J^{-1}(4) > \lambda_1^{-1}(4) \\ 1 & \text{if } \lambda_J^{-1}(4) < \lambda_1^{-1}(4) \end{cases}$$

and $\lambda^*_\varepsilon(J)$ is given by

$$\lambda^*_\varepsilon(J) := 1 - \min \left\{ \lambda_J^{-1}(4), \lambda_1^{-1}(4) \right\}.$$ 

Then, for $J$ and $\varepsilon$, there exists $J'$ such that the induced transformation $(R_J)_{[0, \lambda^*_\varepsilon(J)]}$ is isomorphic to $R_{J'}$ by the isomorphism $\varphi(J)(x) = \frac{x}{\lambda^*_\varepsilon(J)}$ from $[0, \lambda^*_\varepsilon(J)]$ to $[0, 1]$. For example, if $J = I$, the induced transformations $(R_I)_{[0, \lambda^*_\varepsilon(J)]}$, $\varepsilon \in \{0, 1\}$ are following (see Figure 3):

Figure 3: The induced transformations $(R_I)_{[0, \lambda^*_\varepsilon(J)]}$ of $R_I$, $\varepsilon = 0, 1$ and the renormalized transformations $R_{VI}$ and $R_{III}$ of $(R_I)_{[0, \lambda^*_\varepsilon(J)]}$, $\varepsilon = 0, 1$. 
The other cases of J = II, III, . . . , VII are defined analogously. By the length \( \lambda_{J_1} \) and \( \lambda_{J_1}^{-1} \) of the subintervals \( I_{J_1} \) and \( I_{J_1}^{-1} \) respectively, we have a part of the directed graph with the vertices \{I, II, . . . , VII\} and the labels \( \varepsilon \in \{0, 1\} \). For example, if J = I, see Figure 4.

\[
\begin{align*}
\text{I} = & \begin{bmatrix} A & B & C & D \\ D & C & B & A \end{bmatrix} \\
\varepsilon = 0 \rightarrow & \begin{bmatrix} A & B & C & D \\ D & A & C & B \end{bmatrix} \\
\text{II} = & \begin{bmatrix} A & C & D & B \\ D & C & B & A \end{bmatrix} \\
\varepsilon = 1 \rightarrow & \begin{bmatrix} A & D & B & C \\ D & C & B & A \end{bmatrix} \\
\text{III} = & \begin{bmatrix} A & D & B & C \\ D & C & B & A \end{bmatrix}
\end{align*}
\]

Figure 4: The directed graph that starting vertex is I.

The other cases are defined analogously.

Then we have the following Rauzy induction diagram from the 4-interval exchange transformations (see [Y]).

**Proposition 1.1** (The Rauzy induction diagram). We have the following Rauzy induction diagram (see Figure 5):

\[
\begin{align*}
\text{I} = & \begin{bmatrix} A & B & C & D \\ D & C & B & A \end{bmatrix} \\
\varepsilon = 0 \rightarrow & \begin{bmatrix} A & B & C & D \\ D & A & C & B \end{bmatrix} \\
\text{II} = & \begin{bmatrix} A & C & D & B \\ D & C & B & A \end{bmatrix} \\
\varepsilon = 1 \rightarrow & \begin{bmatrix} A & D & B & C \\ D & C & B & A \end{bmatrix} \\
\text{III} = & \begin{bmatrix} A & D & B & C \\ D & C & B & A \end{bmatrix} \\
\text{IV} = & \begin{bmatrix} A & B & C & D \\ D & C & B & A \end{bmatrix} \\
\varepsilon = 0 \rightarrow & \begin{bmatrix} A & B & C & D \\ D & A & C & B \end{bmatrix} \\
\text{V} = & \begin{bmatrix} A & B & C & D \\ D & A & C & B \end{bmatrix} \\
\varepsilon = 1 \rightarrow & \begin{bmatrix} A & B & C & D \\ D & A & C & B \end{bmatrix} \\
\text{VI} = & \begin{bmatrix} A & B & C & D \\ D & A & C & B \end{bmatrix} \\
\text{VII} = & \begin{bmatrix} A & B & C & D \\ D & A & C & B \end{bmatrix}
\end{align*}
\]

Figure 5: The Rauzy induction diagram (RID).

Using the Rauzy induction diagram (RID), we obtain the RID-admissible path \( (\varepsilon_1) (\varepsilon_2) \cdots (\varepsilon_i) \cdots \) of \( (\varepsilon_i) \in \{I, II, . . . , VII\} \times \{0, 1\} \).

Now let us introduce the family of the substitutions \( \sigma_{(\varepsilon_i)} \) on \( \mathcal{A}^* \) related to the induced transformation \( (R_1)_{(\theta, \lambda_\varepsilon(J))} \) as follows:

\[
\begin{align*}
\sigma_{(\varepsilon_1)} : A & \rightarrow AD & \sigma_{(\varepsilon_1)} : A & \rightarrow A & \sigma_{(\varepsilon_1)} : A & \rightarrow AB & \sigma_{(\varepsilon_1)} : A & \rightarrow A \\
B & \rightarrow B & B & \rightarrow B & B & \rightarrow B & B & \rightarrow AB \\
C & \rightarrow C & C & \rightarrow C & C & \rightarrow C & C & \rightarrow C \\
D & \rightarrow D & D & \rightarrow AD & D & \rightarrow D & D & \rightarrow D
\end{align*}
\]
We write the incidence matrices of the above substitutions \( \sigma_{(i_j)} \) as \( M_i \).

Then, we have the following RID with the substitutions.

**Proposition 1.2** (The RID with the substitutions). We have the following RID with the substitutions (see Figure 6):

![Figure 6: The RID with the substitutions.](image-url)
For any RID-admissible periodic path 
\[
\left( \begin{array}{c}
(\varepsilon_0) \\
(\varepsilon_1) \\
\vdots \\
(\varepsilon_{k-1})
\end{array} \right)
\] with period \(k\), we have the substitution \(\sigma_i\) as follows:
\[
\sigma_i = \sigma_{(\varepsilon_i)^0} \circ \sigma_{(\varepsilon_{i+1})} \circ \cdots \circ \sigma_{(\varepsilon_{k-1})} \circ \sigma_{(\varepsilon_0)} \circ \cdots \circ \sigma_{(\varepsilon_{i-1})}
\]
on \(A^*\). In this survey, we only consider the following RID-admissible periodic path:
\[
\left( \begin{array}{c}
(\varepsilon_0) \\
(\varepsilon_1) \\
\vdots \\
(\varepsilon_{k-1})
\end{array} \right) = \left( \begin{array}{c}
(II) \\
(III) \\
\vdots \\
(III)
\end{array} \right)
\] with period 8.

The substitution \(\sigma\) will be sometimes written by
\[
\sigma(\alpha) = W_1^{(\alpha)} W_2^{(\alpha)} \cdots W_{l_{\alpha}}^{(\alpha)} = P_{k}^{(\alpha)} S_{k}^{(\alpha)}
\] where \(P_{k}^{(\alpha)}\) (resp. \(S_{k}^{(\alpha)}\)) is the prefix (resp. suffix) of the letter \(W_{k}^{(\alpha)}\).

2 On an example

Let us consider the following substitution \(\sigma\) as an example:
\[
\sigma = \sigma_{(0)} \circ \sigma_{(1)} \circ \sigma_{(0)} \circ \sigma_{(1)} \circ \sigma_{(0)} \circ \sigma_{(1)} \circ \sigma_{(0)} \circ \sigma_{(1)}
\]
generated by a RID-admissible periodic path with period 8 (see Fig. 7). The substitution \(\sigma\) is explicitly given by
\[
\sigma : \begin{array}{c}
A \rightarrow ABD \\
B \rightarrow ABBBD \\
C \rightarrow ABDCCD \\
D \rightarrow ABDCD
\end{array}
\]
and its incidence matrix \(M_{\sigma}\) and its characteristic polynomial \(\Phi_{\sigma}(x)\) are given by
\[
M_{\sigma} = \left[ \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
0 & 0 & 3 & 1 \\
1 & 1 & 2 & 2
\end{array} \right], \quad \Phi_{\sigma}(x) = x^4 - 7x^3 + 13x^2 - 7x + 1
\]
respectively. Then, we see that the root of \(\Phi_{\sigma}(x)\) is distributed by Figure 8. Therefore we have the Perron-Frobenius eigenvector \(v_1\) satisfying
\[
v_1 = ^t[\lambda_A, \lambda_B, \lambda_C, \lambda_D], \quad \lambda_{\alpha} > 0, \quad \text{and} \quad \sum_{\alpha \in A} \lambda_{\alpha} = 1
\]
Figure 7: An example of a RID-admissible periodic path with period 8.

Figure 8: The distribution of the roots of $\Phi_\sigma(x)$. 
where \(^t M\) means the transpose of the matrix \(M\).

Starting from \(\sigma\), we obtain the following 4-interval exchange transformation. Let us define the partition \(\{ I_\alpha \mid \alpha \in A \} \) of \([0, 1)\) by

\[
I_A = [0, \lambda_A), \quad I_B = [\lambda_A, \lambda_A + \lambda_C), \quad I_C = [\lambda_A + \lambda_C, \lambda_A + \lambda_C + \lambda_D), \\
I_D = [\lambda_A + \lambda_C + \lambda_D, 1)
\]

(see Figure 9).

![Figure 9: The partition of [0, 1).](image)

From the definition, \(R_{II}\) is explicitly given by

\[
R_{II}(x) = \begin{cases} 
  x + \lambda_D + \lambda_C + \lambda_B & \text{if } x \in I_A \\
  x - \lambda_A - \lambda_D & \text{if } x \in I_C \\
  x - (\lambda_A + \lambda_C) & \text{if } x \in I_D \\
  x - \lambda_A & \text{if } x \in I_B
\end{cases}
\]

Then, \(R_{II}(x)\) by \(\lambda_B > \lambda_A\) and the induced transformation \((R_{II})\)\([0, \lambda_0(II))\) of \(R_{II}\) is isomorphic to \(R_{II}\) by the isomorphism \(\varphi_{II}(x) = \frac{x}{\lambda_0(II)}\) from \([0, \lambda_0(II))\) to \([0, 1)\) (see Figure 10).

Let \(W\) be the fixed point of \(\sigma\), that is,

\[
W = s_1s_2 \ldots s_k \ldots = \lim_{n \to \infty} \sigma^n (A).
\]

Let

\[
\mathcal{L}(v_1, v_2, v_3, v_4) := \mathcal{L}(v_1) \oplus \mathcal{L}(v_2) \oplus \mathcal{L}(v_3) \oplus \mathcal{L}(v_4)
\]

and let us define the projection \(\pi_i\) and \(\pi_{ij}\) by

\[
\pi_i : \mathcal{L}(v_1, v_2, v_3, v_4) \to \mathcal{L}(v_i) \\
\pi_{ij} : \mathcal{L}(v_1, v_2, v_3, v_4) \to \mathcal{L}(v_i, v_j)
\]

where \(v_i, i = 1, 2, 3, 4\) are the eigenvectors associated to the eigenvalues \(\lambda_i, i = 1, 2, 3, 4\) of \(M_\sigma\) satisfying \(\lambda_1 > \lambda_2 > 1 > \lambda_3 > \lambda_4 > 0\) respectively.

Moreover, let us define the homomorphism \(f : A^* \to \mathbb{Z}^4\) by

\[
f(A) := e_1, \quad f(B) := e_2, \quad f(C) := e_3, \quad f(D) := e_4, \quad f(\emptyset) := \emptyset
\]

\[
f(W_1 W_2 \ldots W_k) := f(W_1) + f(W_2) + \ldots + f(W_k).
\]

On the above notation, we have firstly the following proposition.
Figure 10: The induced transformation \((R_\Pi)_{[0, \lambda_0^*]}\) of \(R_\Pi\).

Proposition 2.1. Let us define the set \(X_\alpha, X'_\alpha, X\) as follows:

\[
X_\alpha := \text{the closure of } \pi_4 \{ f(s_1s_2\ldots s_{k-1}) \mid s_k = \alpha, k = 1, 2, \ldots \}, \alpha \in \mathcal{A}
\]

\[
X'_\alpha := \text{the closure of } \pi_4 \{ f(s_1s_2\ldots s_k) \mid s_k = \alpha, k = 1, 2, \ldots \}, \alpha \in \mathcal{A}
\]

\[
X := \text{the closure of } \pi_4 \{ f(s_1s_2\ldots s_{k-1}) \mid k = 1, 2, \ldots \}.
\]

Then, we have the following properties:

(1) \(X_\alpha\) is the interval of the line \(L(v_4)\);

(2) \(X = \bigcup_{\alpha \in \mathcal{A}} X_\alpha = \bigcup_{\alpha \in \mathcal{A}} X'_\alpha\);

(3) \(X_\alpha \cap X_\beta (\alpha \neq \beta), \alpha, \beta \in \mathcal{A}\) are not overlapped;

(4) \(\{X_\alpha\}_{\alpha \in \mathcal{A}}\) satisfies the set equation:

\[
\lambda_1 X_\alpha \left(= M_{\sigma^{-1}} X_\alpha \right) = \bigcup_{\beta \in \mathcal{A} W^{(\beta)}_{k} = \alpha} \left( \pi_4 f \left( P^{(\beta)} \right) + X_\beta \right);
\]

(5) The interval exchange transformation \(D : X \to X\) such that \(D(X_\alpha) = X'_\alpha\) is isomorphic to \(R_\Pi_{[0]}\) where \(D : X \to X\) such that \(D(X_\alpha) = X'_\alpha\) is isomorphic to \(R_\Pi = \begin{bmatrix} A & C & D & B \\ D & C & B & A \end{bmatrix}\) (see Figure 11).
Moreover, we have the following theorem.

**Theorem 2.2.** (cf. [F-I-Rao]) Let us define

\[
\hat{X}_\alpha := \text{the closure of } \pi_{34} \{ f(s_1 s_2 \ldots s_{k-1}) \mid s_k = \alpha, k = 1, 2, \ldots \} \\
\hat{X}'_\alpha := \text{the closure of } \pi_{34} \{ f(s_1 s_2 \ldots s_k) \mid s_k = \alpha, k = 1, 2, \ldots \} \\
\hat{X} := \text{the closure of } \pi_{34} \{ f(s_1 s_2 \ldots s_{k-1}) \mid k = 1, 2, \ldots \}.
\]

Then,

1. \( \hat{X} = \bigcup_{\alpha \in \mathcal{A}} \hat{X}_\alpha = \bigcup_{\alpha \in \mathcal{A}} \hat{X}_\alpha' \) (non-overlapping);

2. \( \{ \hat{X}_\alpha \}_{\alpha \in \mathcal{A}} \) satisfies the set set equation:

\[
M_\sigma^{-1} \hat{X}_\alpha = \bigcup_{\beta \in \mathcal{A}} \bigcup_{W_{k}^{(\beta)} = \alpha} \left( \pi_{34} \left( f \left( P_k^{(\beta)} \right) + \hat{X}_\beta \right) \right);
\]

3. The above set equation satisfies open set condition, that is, there exist a family of open set \( U_\alpha, \alpha \in \mathcal{A} \) such that

\[
M_\sigma^{-1} U_\alpha \supset \bigcup_{\beta \in \mathcal{A}} \bigcup_{W_{k}^{(\beta)} = \alpha} \left( \pi_{34} \left( f \left( P_k^{(\beta)} \right) + U_\beta \right) \right)
\]

where the right-hand side is non-overlapping union;

4. The domain exchange transformation \( \hat{D} : \hat{X} \rightarrow \hat{X} \) satisfying \( \hat{D} \left( \hat{X}_\alpha \right) = \hat{X}_\alpha' \) is well-defined (see Figure 12).
3 Quasi-periodic tiling

Starting from the hyperbolic and non-Pisot substitutions (automorphism) \( \sigma \) of degree 4, the generating method of quasi-periodic tiling on \( \mathcal{L}(v_1, v_2) \) and \( \mathcal{L}(v_3, v_4) \) were discussed in [A-F-H-I], [F-I-Rob], [H-F-I]. In this section, we show the existence of the quasi-periodic polygonal/self-affine tilings generated by substitution \( \sigma \) analogously.

Let us observe the figure of \( \{\pi_{34}e_i\}_{i=1,2,3,4} \) (see Figure 13). Using the projected basis \( \{\pi_{34}e_i\}_{i=1,2,3,4} \), we consider the proto tiles of parallelograms on \( \mathcal{L}(v_3, v_4) \) (see Figure 14).

Using the automorphism \( \theta := \sigma^{-1} \) (see [E]):

\[
\begin{align*}
\theta(A) &= AD^{-1}CD^{-1}AB^{-1}A \\
\theta(B) &= AD^{-1}CD^{-1}BA^{-1}DC^{-1}DA^{-1} \\
\theta(C) &= AD^{-1}CD^{-1}DA^{-1} \\
\theta(D) &= DC^{-1}DA^{-1}
\end{align*}
\]

we try to consider the 2-dim extension of the automorphisms \( \theta \) as follows:

\[
E_2(\theta)(0, \alpha \wedge \beta) := (0, \theta(\alpha) \wedge \theta(\beta))
= \sum_{1 \leq i \leq l_\alpha} \sum_{1 \leq j \leq l_\beta} \left( f(P^{(\alpha)}_i) + f(P^{(\beta)}_j) \right), W^{(\alpha)}_i \wedge W^{(\beta)}_j
\]

(see Figure 15). Attention that we find the negative oriented parallelograms in \( E_2(\theta)(0, \alpha \wedge \beta) \) which is characterized as the strong colored parallelograms in Figure 15.
Figure 14: The proto tiles on $\mathcal{L}(v_3, v_4)$ generated by $f(A) = e_1$, $f(B) = e_2$, $f(C) = e_3$, $f(D) = e_4$.

On the example, we know that $A^{*}$ is positive, that is,

$$A^{*} = \begin{bmatrix} A \land B & \sqrt{m_{i,k}}^* \\ C \land A & m_{k,l}^* \\ D \land A & m_{l,k}^* \\ B \land C & m_{j,k}^* \\ B \land D & m_{j,l}^* \\ D \land C & \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 3 & 2 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \succeq O$$

where $m_{i,k}^* = \det \begin{bmatrix} m_{ik} & m_{il} \\ m_{jk} & m_{jl} \end{bmatrix}$ for $M_{\sigma}^{-1}[m_{ij}]_{1 \leq i,j \leq 4}$.

From this fact, we try to find the tiling substitution $E_2(\theta)$ by the retiling method in [F-I-Rob] (see Figure 16).

**Theorem 3.1.** Let $\mathcal{U}_c$ be a patch generated by the following proto tiles:

$$\mathcal{U}_c = (z, f(B) \land f(A)) + (z + e_4 - e_3, f(C) \land f(A)) + (z, f(A) \land f(D)) + (z + e_4 - e_3, f(B) \land f(C)) + (z, f(D) \land f(B)) + (z + e_1 - e_3, f(D) \land f(C))$$

where $z = \left(-\frac{4}{19}, -\frac{5}{19}, -\frac{1}{19}, 0\right)$. Then, we see that $\mathcal{U}_c$ is the seed, that is, $E_2(\theta)^3(\mathcal{U}_c) \succ \mathcal{U}_c$ (see Figure 17).

Moreover,
Figure 15: $E_2(\theta)(0, \alpha \land \beta)$. 
Figure 16: The retiling method from $E_2 (\theta)$ to $\widehat{E}_2 (\theta)$.

Figure 17: $\pi_{34} \mathcal{U}_c$ and a part of $\pi_{34} \widehat{E}_2 (\theta)^3 (\mathcal{U}_c)$. 
(1) $T_{c,1} := \left\{ \pi_{34}(x, f(\alpha) \wedge f(\beta)) \mid \widehat{E}_2(\theta)^{3n}(U_c) \ni (x, f(\alpha) \wedge f(\beta)) \right\}$ is a quasi-periodic polygonal tiling of $\mathcal{L}(v_3, v_4)$ (see Figure 18);

(2) Put

$$X_{\alpha \wedge \beta} := \lim_{n \to \infty} \pi_{34}M_\sigma^3\widehat{E}_2(\theta)^{3n}(0, f(\alpha) \wedge f(\beta)).$$

Then, $\{X_{\alpha \wedge \beta}\}$ satisfies the set equations:

$$M_\sigma(\pi_{34}x_i + X_{\gamma_i}) = \sum_k \left( \pi_{34}x_k^{(i)} + X_{\gamma_k^{(i)}} \right)$$

where $U_c = \sum_{i=1}^6 (x_i, \gamma_i)$ and $\widehat{E}_2(\theta)(x_i, \gamma_i) = \sum_k \left( x_k^{(i)} + \gamma_k^{(i)} \right)$;

(3) $T_{c,2} := \{ \pi_{34}x + X_{\alpha \wedge \beta} \mid \pi_{34}(x, f(\alpha) \wedge f(\beta)) \in T_{c,1} \}$. Then, $T_{c,2}$ is a quasi-periodic self-affine tiling of $\mathcal{L}(v_3, v_4)$ (see Figure 20).

By the analogous discussion, we can construct the quasi-periodic polygonal/self-affine tiling from the "tiling substitution $\widehat{E}_2(\sigma)$" on the expanding plane $\mathcal{L}(v_1, v_2)$.

Let us observe the figure $\{\pi_{12}e_i\}_{i=1,2,3,4}$ (see Figure 21) and we consider the proto tiles of parallelograms on $\mathcal{L}(v_1, v_2)$ (see Figure 22).

Using the automorphism $\sigma$:

$$\sigma(A) = ABD \quad \sigma(B) = ABB \quad \sigma(C) = ABDC \quad \sigma(D) = ABDC$$

Figure 18: The quasi-periodic polygonal tiling $T_{c,1}$ of $\mathcal{L}(v_3, v_4)$. 

Figure 21: $\{\pi_{34}e_i\}_{i=1,2,3,4}$.
\[ \pi_{34} U_c = \sum_{i=1}^{6} \pi_{34} (x_i, \gamma_i) \]

\[ \sum_{i=1}^{6} (\pi_{34} x_i + X_{\gamma_i}) \]

\[ M_\sigma (\pi_{34} x_i + X_{\gamma_i}) \]

Figure 19: \( \pi_{34} U_c \) and the proto-tiles of the quasi-periodic self-affine tiling \( T_{c,2} \).

Figure 20: The quasi-periodic self-affine tiling \( T_{c,2} \) of \( \mathcal{L}(v_3, v_4) \).
we try to consider the 2-dimensional extension of the automorphisms (substitutions) \( \sigma \) as follows:

\[
E_2 (\sigma) (0, \alpha \land \beta) := (0, \sigma (\alpha) \land \sigma (\beta)) = \sum_{1 \leq i \leq l_\alpha \atop 1 \leq j \leq l_\beta} \left( f \left( P_i^{(\alpha)} \right) + f \left( P_j^{(\beta)} \right) , W_i^{(\alpha)} \land W_j^{(\beta)} \right)
\]

(see Figure 23):

On our example, we know that \( A^* \) is positive, that is,

\[
A^* = \begin{bmatrix}
A \land B \\
C \land A \\
D \land A \\
C \land B \\
D \land B \\
C \land D
\end{bmatrix}
\]

\[
a^*_{i\land,j\land,k\land,l} = \det \begin{bmatrix}
a_{ik} & a_{il} \\
a_{jk} & a_{jl}
\end{bmatrix}
\]

for \( M_\sigma = [a_{ij}]_{1 \leq i,j \leq 4} \).

From this fact, we try to find the tiling substitution \( \widehat{E}_2 (\sigma) \) by the retiling method in [F-I-Rob] analogously (see Figure 24):
Theorem 3.2. Let $U_e$ be a patch generated by the following proto tiles:

\[
U_e = (y + e_4, f(A) \land f(B)) + (y + e_4, f(C) \land f(A)) + (y + e_2, f(D) \land f(A)) + (y + e_1, f(C) \land f(B)) + (y, f(D) \land f(B)) + (y, f(C) \land f(D)),
\]

where $y = \left(\frac{8}{19}, -\frac{14}{19}, -\frac{2}{19}, -\frac{18}{19}\right)$. Then, we see that $U_e$ is the seed, that is, $\hat{E}_2(\sigma)^3(U_e) \supseteq U_e$ (see Figure 25).

Moreover,

1. $T_{e,1} := \left\{ \pi_{12}(x, f(\alpha) \land f(\beta)) \mid \hat{E}_2(\sigma)^3n(U_e) \ni (x, f(\alpha) \land f(\beta)) \right\}$ is a quasi-periodic polygonal tiling of $\mathcal{L}(v_1, v_2)$ (see Figure 26);

2. Put

\[
X_{\alpha \land \beta} := \lim_{n \to \infty} \pi_{12}M_{3n}^{-1}\hat{E}_2(\sigma)^3n(0, f(\alpha) \land f(\beta)).
\]

Then, $\{X_{\alpha \land \beta}\}$ satisfies the set equations:

\[
M_{3n}^{-1}(\pi_{12}x_i + X_{\delta_i}) = \sum_k \left(\pi_{12}x_k^{(i)} + X_{\delta_k^{(i)}}\right)
\]

where $U_e = \sum_{i=1}^{6}(x_i, \delta_i)$ and $\hat{E}_2(\sigma)(x_i, \delta_i) = \sum_k \left(x_k^{(i)} + \delta_k^{(i)}\right)$;

3. $T_{e,2} := \left\{ \pi_{12}(x + X_{\alpha \land \beta}) \mid \pi_{12}(x, f(\alpha) \land f(\beta)) \in T_{e,1} \right\}$. Then, $T_{e,2}$ is a quasi-periodic self-affine tiling of $\mathcal{L}(v_1, v_2)$ (see Figure 28).
Figure 24: The retiling method from $E_2(\sigma)$ to $\tilde{E}_2(\sigma)$. 
Figure 25: $\pi_{12} U_e$ and a part of $\pi_{12} E_2(\sigma)^3 (U_e)$.

Figure 26: The quasi-periodic polygonal tiling $T_{e,1}$ of $L(v_1, v_2)$.

Figure 27: $\pi_{12} U_e$ and the proto-tiles of the quasi-periodic self-affine tiling $T_{e,2}$. 
Figure 28: The quasi-periodic self-affine tiling $\mathcal{T}_{e,2}$ of $\mathcal{L}(v_1, v_2)$.

References


