WHEN DO SELF-AFFINE TILINGS HAVE THE MEYER PROPERTY?

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Abstract. Meyer sets have played important roles in the study of aperiodic systems. We present various properties on the Meyer sets. We consider self-affine tilings and determine when the corresponding point sets representing the tilings are the Meyer sets.

1. Preliminary

A discrete set Y is called *Delone set* if it is uniformly discrete and relatively dense. A Delone set $Y \subset \mathbb{R}^d$ is *Meyer* if it is relatively dense and $Y - Y$ is uniformly discrete. A *cluster* P of Λ is a finite subset of Λ.

Example 1.1. The examples of Meyer sets are

(i) $\Lambda = (1 + 2\mathbb{Z}) \cup S$ for any subset $S \subset 2\mathbb{Z}$.

(ii) $\Lambda = \{a + b\tau \in \mathbb{Z}[\tau] \mid a - b\frac{1}{\tau} \in [0, 1)\}$, where $\tau^2 - \tau - 1 = 0$.

An example of non-Meyer set is $\Lambda = \{n + \frac{1}{n} \mid n \in \mathbb{Z}\backslash\{0\}\}$.

The following are various equivalent properties of the Meyer sets.

Theorem 1.2. [10, 6, 11] *Let Λ be a Delone set. The following are equivalent;*

(i) Λ *is a Meyer set.*

(ii) $\Lambda - \Lambda \subset \Lambda + F$ *for some finite set* $F \subset \mathbb{R}^d$ *(almost lattice).*

(iii) *For each* $\epsilon > 0$, *there is a dual set* Λ^ϵ *in* $\widehat{\mathbb{R}^d}$,

$$\Lambda^\epsilon = \{\chi \in \widehat{\mathbb{R}^d} : |\chi(x) - 1| < \epsilon \text{ for all } x \in \Lambda\}$$

is relatively dense.

Let Λ be a Delone set in $\mathbb{R}^d$. We consider a measure of the form $\nu = a \cdot \delta_\Lambda = \sum_{x \in \Lambda} a \cdot \delta_x$ and $a \in \mathbb{C}$. The autocorrelation of ν is

$$\gamma(\nu) = \lim_{n \to \infty} \frac{1}{\mathrm{Vol}(B_n)} (\nu|_{B_n} * \widetilde{\nu}|_{B_n}),$$

where $\nu|_{B_n}$ is a measure of ν restricted on the ball B_n of radius n and $\widetilde{\nu}$ is the measure, defined by $\widetilde{\nu}(f) = \overline{\nu(\widetilde{f})}$, where f is a continuous function with compact support and $\widetilde{f}(x) = \overline{f(-x)}$. The diffraction measure of ν is the Fourier transform $\widehat{\gamma(\nu)}$ of the autocorrelation

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(see [4]). When the diffraction measure $\widehat{\gamma(\nu)}$ is a pure point measure, we say that Λ has pure point diffraction spectrum.

The following theorem characterizes the pure point diffractive sets.

Theorem 1.3. [2] *If Λ is a Meyer set admitting autocorrelation, then Λ is pure point diffractive if and only if for any $\epsilon > 0$, $\{t \in \mathbb{R}^d : density(\Lambda \backslash (\Lambda - t)) < \epsilon\}$ is relatively dense.*

We say that a Delone set Λ has *finite local complexity (FLC)* if for each radius $R > 0$ there are only finitely many translational classes of clusters whose support lies in some ball of radius R. A Delone set Λ is said to be *repetitive* if the translations of any given patch occur uniformly dense in $\mathbb{R}^d$; more precisely, for any Λ-cluster P, there exists $R > 0$ such that every ball of radius R contains a translated copy of P.

Given a Delone set Λ, we define the *space of Delone sets* as the orbit closure of Λ under the translation action: $X_\Lambda = \overline{\{-g + \Lambda \mid g \in \mathbb{R}^d\}}$, in the well-known "local topology": for a small $\epsilon > 0$ two tilings Γ_1, Γ_2 are ϵ-close if Γ_1 and Γ_2 agree on the ball of radius ϵ^{-1} around the origin, after a translation of size less than ϵ. It is known that X_Λ is compact whenever Λ has FLC. Thus we get a topological dynamical system $(X_\Lambda, \mathbb{R}^d)$ where $\mathbb{R}^d$ acts by translations. This system is minimal (i.e. every orbit is dense) whenever Λ is repetitive. Let μ be an invariant Borel probability measure for the action; then we get a measure-preserving system $(X_\Lambda, \mathbb{R}^d, \mu)$. Such a measure always exists; under the natural assumption of *uniform patch frequencies*, it is unique, see [7]. Tiling dynamical systems have been investigated in a large number of papers (e.g. [12, 3]).

Definition 1.4. A vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ is said to be an *eigenvalue* for the $\mathbb{R}^d$-action if there exists an eigenfunction $f \in L^2(X_\Lambda, \mu)$, that is, $f \not\equiv 0$ and for all $g \in \mathbb{R}^d$ and μ-almost all $\Gamma \in X_\Lambda$,

$$f(\Gamma - g) = e^{2\pi i \langle g, \boldsymbol{\alpha} \rangle} f(\Gamma). \tag{1.1}$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^d$.

The following gives an important criterion on Meyer sets.

Theorem 1.5. [15] *If Λ is a Meyer set with uniform cluster frequencies, then the Bragg peaks in the diffraction pattern of Λ are relatively dense. It implies that the set of eigenvalues for the dynamical system $(X_\Lambda, \mathbb{R}^d, \mu)$ is relatively dense.*

2. Substitution tilings

From now on, we consider substitution tilings. Note that whenever substitution tilings are given, we can get the corresponding substitution Delone sets taking representative points of tiles at the relatively same positions for the same type of tiles. So most of the properties on substitution tilings can be stated on substitution point sets.

We say that a linear map $Q : \mathbb{R}^d \to \mathbb{R}^d$ is *expansive* if all its eigenvalues lie outside the closed unit disk in $\mathbb{C}$.

Definition 2.1. Let $\mathcal{A} = \{T_1, \ldots, T_m\}$ be a finite set of tiles in $\mathbb{R}^d$ such that $T_i = (A_i, i)$; we will call them *prototiles*. Denote by $\mathcal{P}_{\mathcal{A}}$ the set of non empty patches. We say that $\Omega : \mathcal{A} \to \mathcal{P}_{\mathcal{A}}$ is a *tile-substitution* (or simply *substitution*) with an expansive map Q if there exist finite sets $\mathcal{D}_{ij} \subset \mathbb{R}^d$ for $i, j \le m$ such that

$$\Omega(T_j) = \{u + T_i : \ u \in \mathcal{D}_{ij}, \ i = 1, \ldots, m\} \tag{2.1}$$

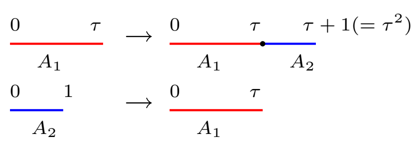
with

$$QA_j = \bigcup_{i=1}^{m} (\mathcal{D}_{ij} + A_i) \quad \text{for } j \le m. \tag{2.2}$$

Here all sets in the right-hand side must have disjoint interiors; it is possible for some of the $\mathcal{D}_{ij}$ to be empty.

We say that $\mathcal{T}$ is a *substitution tiling* if $\mathcal{T}$ is a tiling and $\Omega(\mathcal{T}) = \mathcal{T}$ with some substitution Ω. We say that substitution tiling is *primitive* if the corresponding substitution matrix S, with $S_{ij} = \sharp(\mathcal{D}_{ij})$, is primitive, i.e. S^ℓ is a matrix whose each entry is positive for some $\ell \in \mathbb{Z}_+$. A repetitive primitive substitution tiling with FLC is called a *self-affine tiling*. If ϕ is a similarity map, we can that the tiling is a *self-similar tiling*. Let $\mathbf{\Lambda}_{\mathcal{T}} = (\Lambda_i)_{i \le m}$ be the substitution point set representing $\mathcal{T}$.

Example 2.2. The Fibonacci substitution tiling is defined by the following substitution rule



where $\tau^2 - \tau - 1 = 0$. The tiles A_1 and A_2 satisfy the following tile-equations

$$\begin{aligned} \tau A_1 &= A_1 \cup (A_2 + \tau) \\ \tau A_2 &= A_1 \end{aligned}$$

Continuously iterating the tiles and subdividing them, we can construct a tiling.

3. Meyer property on self-affine tilings

Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ with an expansion map ϕ and $\mathbf{\Lambda}_{\mathcal{T}} = (\Lambda_i)_{i \le m}$ be a substitution point set representing $\mathcal{T}$. Suppose that all the eigenvalues of ϕ are algebraic conjugates with the same multiplicity. Let

$$\Xi = \{x \in \mathbb{R} \mid \exists\, T, T - x \in \mathcal{T}\} \ \text{ and } \mathcal{K} = \{x \in \mathbb{R}^d \mid \mathcal{T} = \mathcal{T} - x\}.$$

Before we talk about how to determine the Meyer property on substitution tilings, we present some preliminary results.

Theorem 3.1. [1] *If a substitution point set is a Meyer set, then one can determine pure point spectrum using a computational algorithm.*

Theorem 3.2. [8] *The set of eigenvalues for the dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ is relatively dense if and only if the corresponding substitution point set $\mathbf{\Lambda}_{\mathcal{T}}$ is a Meyer set.*

Theorem 3.3. [14] *γ is an eigenvalue for the dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ if and only if*

$$\lim_{n\to\infty} e^{2\pi i\langle \phi^n x, \gamma\rangle} = 1 \quad \text{for all } x \in \Xi\,,$$

$$e^{2\pi i\langle x, \gamma\rangle} = 1 \quad \text{for all } x \in \mathcal{K} = \{x \in \mathbb{R}^d \mid \mathcal{T} - x = \mathcal{T}\}.$$

Theorem 3.4. [5, 14] *Let $\mathcal{T}$ be a self-similar tiling in $\mathbb{R}^d$ with a similarity λ, where $|\lambda| > 1$. Then*

$$\Xi \subset \mathbb{Z}[\lambda]\alpha_1 + \cdots \mathbb{Z}[\lambda]\alpha_d$$

for some basis $\{\alpha_1, \ldots, \alpha_d\}$ of $\mathbb{R}^d$.

Question What can we say in the case of self-affine tilings?

Theorem 3.5. [9] *Suppose that ϕ is diagonalizable and all the eigenvalues of ϕ are algebraically conjugate with the same multiplicity m. Then $\exists$ an isomorphism $\rho : \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$\rho\phi = \phi\rho \quad \text{and } \Xi \subset \rho(\mathbb{Z}[\phi]\alpha_1 + \cdots + \mathbb{Z}[\phi]\alpha_J),$$

where $Jm = d$ and

$$(\alpha_j)_n = \begin{cases} 1 & \text{if } (j-1)m + 1 \le n \le jm \\ 0 & \text{else} \end{cases}\,.$$

We show now how this theorem is used to get the Meyer property of Ξ. To be simple, we consider the case that all the eigenvalues of ϕ are real. However the main result of Theorem 3.11 is not restricted on this case.

An algebraic integer λ is a *Pisot number* if $|\lambda| > 1$ and all other algebraic conjugates are less than 1 in modulus. A set $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$ of algebraic integers is a *Pisot family* if for every $\lambda_i \in \Lambda$, if γ is an algebraic conjugate of λ_i and $\gamma \notin \Lambda$, then $|\gamma| < 1$. Let $\mathrm{dist}(x, \mathbb{Z})$ be the minimal distance from x to $\mathbb{Z}$.

Lemma 3.6. *Let λ be a Pisot number. Then $dist(\lambda^n, \mathbb{Z}) \to 0$ as $n \to \infty$.*

Proof. Let $\lambda_2, \ldots, \lambda_s$ be all the algebraic conjugates of λ. For any $n \in \mathbb{Z}_+$,

$$\lambda^n + \sum_{j=2}^{s} (\lambda_j)^n \in \mathbb{Z}.$$

Note that

$$\sum_{j=2}^{s} (\lambda_j)^n \le (s-1) \sup_{2\le j\le m} |\lambda_j|^n \to 0 \quad \text{as } n \to \infty\,.$$

Thus the claim follows. □

Lemma 3.7. *Let $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$ be a Pisot family. Then*

$$dist(\sum_{k=1}^{m} (\lambda_k)^n, \mathbb{Z}) \to 0 \quad as\ n \to \infty\,.$$

Proposition 3.8. *If the set of eigenvalues of ϕ is a Pisot family, then the set of eigenvalues for $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ is relatively dense.*

Proof. For any $n \in \mathbb{Z}_{\geq 0}$ and $0 \leq \ell < m$,

$$\langle \phi^n \alpha_j, (\phi^T)^\ell \alpha_j \rangle = \langle \phi^{n+\ell} \alpha_j, \alpha_j \rangle = \sum_{k=1}^{m} \lambda_k^{n+\ell}\,.$$

Since $\{\lambda_1, \ldots, \lambda_m\}$ is a Pisot family,

$$\mathrm{dist}(\sum_{k=1}^{m} \lambda_k^{n+\ell}, \mathbb{Z}) \to 0 \quad \text{as } n \to \infty.$$

Note

$$\langle \phi^n \alpha_i, (\phi^T)^\ell \alpha_j \rangle = 0 \ \text{ if } i \neq j.$$

Hence

$$\lim_{n\to\infty} e^{2\pi i \langle \phi^n y, (\phi^T)^\ell \alpha_j \rangle} = 1 \quad \text{for all } y \in \mathbb{Z}[\phi]\alpha_1 + \cdots + \mathbb{Z}[\phi]\alpha_J.$$

Thus

$$\lim_{n\to\infty} e^{2\pi i \langle \phi^n x, (\rho^T)^{-1} (\phi^T)^\ell \alpha_j \rangle} = 1 \quad \text{for all } x \in \Xi.$$

From the uniform convergence of the limit in $x \in \Xi$,

$$e^{2\pi i \langle x, (\rho^T)^{-1} (\phi^T)^{k+\ell} \alpha_j \rangle} = 1 \quad \text{for all } x \in \mathcal{K} \text{ and some big } k \in \mathbb{Z}_+\,.$$

So $(\rho^T)^{-1}(\phi^T)^{k+\ell}\alpha_j$ is an eigenvalue for $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ for $\ell = 0, \ldots, m-1$. Since

$$\{\alpha_1, \ldots, (\phi^T)^{m-1}\alpha_1, \ldots, \alpha_J, \ldots, (\phi^T)^{m-1}\alpha_J\}$$

is a basis of $\mathbb{R}^d$, the claim follows.

Theorem 3.9. [16] *Let $U_1, U_2, \ldots$ be a sequence of real numbers, where*

$$U_n = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_m\lambda_m^n, \quad c_1c_2\cdots c_m \neq 0,$$

$\lambda_1, \ldots, \lambda_m$ are distinct algebraic numbers, and $|\lambda_k| > 1$ $(k = 1, \ldots, m)$. If $dist(U_n, \mathbb{Z}) \to 0$ as $n \to \infty$, then $\{\lambda_1, \ldots, \lambda_m\}$ is a Pisot family.

Proposition 3.10. *If γ is a non-zero eigenvalue for $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$, then the set of eigenvalues of ϕ is a Pisot family.*

Proof. For any $x \in \Xi$, $x \in \rho(\sum_{j=1}^{J} g_j(\phi)\alpha_j)$ for some polynomials $g_j(x) \in \mathbb{Z}[x]$. Then

$$\begin{aligned} \langle \phi^n x, \gamma \rangle &= \sum_{j=1}^{J} \langle \phi^n g_j(\phi)\alpha_j, \rho^T\gamma \rangle \\ &= \sum_{k=1}^{m} c_k\lambda_k^n \quad \text{for some } c_k \in \mathbb{C}. \end{aligned}$$

Since γ is an eigenvalue, $\mathrm{dist}(\langle \phi^n x, \gamma\rangle, \mathbb{Z}) \to 0$ as $n \to \infty$. By Vijayaraghavan's theorem, the set of eigenvalues of ϕ is a Pisot family.

Theorem 3.11. *Let $\mathcal{T}$ be a self-affine tiling of $\mathbb{R}^d$ with a diagonalizable expansion map ϕ. Suppose that all the eigenvalues of ϕ are algebraic conjugates with the same multiplicity. Then the following are equivalent:*

(i) *$Spec(\phi)$ is a Pisot family.*

(ii) *The set of eigenvalues of $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ is relatively dense in $\mathbb{R}^d$.*

(iii) *The system $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ is not weakly mixing (i.e., it has eigenvalues other than $\mathbf{0}$).*

(iv) *$\Xi(\mathcal{T})$ is a Meyer set.*

References

[1] S. Akiyama and J.-Y. Lee, Algorithm for determining pure pointedness of self-affine tilings, *Adv. Math.*, in press.

[2] M. Baake and R. V. Moody, Weighted Dirac combs with pure point diffraction, *J. Reine Angew. Math.* **573** (2004), 61–94.

[3] A. Clark and L. Sadun, *When shape matters: deformations of tiling spaces*, Ergodic Theory Dynam. Systems **26** (2006), no. 1, 69–86.

[4] A. Hof, Diffraction by aperiodic structures, in *The Mathematics of Long-Range Aperiodic Order*, (R. V. Moody, ed.), 239–268, Kluwer, 1997.

[5] R. Kenyon, Inflationary tilings with a similarity structure, *Comment. Math. Helv.* **69** (1994), 169–198.

[6] J. C. Lagarias, Meyer's concept of quasicrystal and quasiregular sets, *Comm. Math. Phys.*, 179 (1996), no. 2, 365–376

[7] J.-Y. Lee, R. V. Moody, and B. Solomyak, Pure Point Dynamical and Diffraction Spectra, *Ann. Henri Poincaré* **3** (2002), 1003–1018.

[8] J.-Y. Lee and B. Solomyak, Pure point diffractive substitution Delone sets have the Meyer property, *Discrete Comput. Geom.* **39** (2008), 319–338.

[9] J.-Y. Lee and B. Solomyak, Pisot family substitution tilings, discrete spectrum and the Meyer property, *Discrete Contin. Dyn. Syst.*, Accepted.

[10] Y. Meyer *Algebraic Numbers and Harmonic Analysis*, North Holland, New York. (1972).

[11] R. V. Moody, Meyer sets and their duals, in *The Mathematics of Long-Range Aperiodic Order (Waterloo, ON, 1995)*, R. V. Moody, ed., NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Vol. 489, Kluwer Acad. Publ., Dordrecht, 1997, 403–441.

[12] K. Petersen, *Factor maps between tiling dynamical systems*, Forum Math. **11** (1999), 503–512.

[13] B. Solomyak, Dynamics of self-similar tilings, *Ergodic Th. Dynam. Sys.* **17** (1997), 695–738. Corrections to ' Dynamics of self-similar tilings', *ibid.* **19** (1999), 1685.

[14] B. Solomyak, Eigenfunctions for substitution tiling systems, *Adv. Stud. Pure Math.* **43** (2006), 1–22.

[15] N. Strungaru, Almost periodic measures and long-range order in Meyer sets, *Discrete Comput. Geom.* **33**(3) (2005), 483–505.

[16] T. Vijayaraghavan, On the fractional parts of the powers of a number (II), *Math. Proc. Cambridge Philos. Soc.* **37** (1941), 349–357.