# $(-\beta)$- 展開の性質について (On some aspects of $(-\beta)$-expansions)

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**Abstract**

We review some properties of well-known $\beta$-expansions and $(-\beta)$-expansions by Ito-Sadahiro, and consider another $(-\beta)$-expansions constructed by using the structure of Sturmian sequences.

## 1 Introduction

$\beta$-expansion was first introduced by Renyi[10], and since then many studies have been done on several aspects of that, for instance, characterization of admissible sequences and shift space, necessary and/or sufficient conditions of periodic or finite expansions, properties of corresponding dynamical system, and self-similar tilings associated to that (e.g., [10, 9, 6, 11, 2, 1]). Recently, Ito-Sadahiro[5] constructed a theory of beta expansions with negative bases, and studied its basic properties. In this manuscript, we review the well-known facts about those, and consider another notion of $(-\beta)$-expansion. The main feature of this expansion is that, it uses the structure of Sturmian sequences, and the Shift space is always of finite type(SFT). However, apart from particular $\beta$ satifying $\beta^2 = k\beta + 1$, $k \in \mathbf{N}$, we have to introduce an extended notion of expansions. In Section 2, 3, we review some of known facts on $\beta$-expansions and $(-\beta)$-expanpansions respectively, and discuss some open problems. In Section 4 we define a $(-\beta)$-expansions related to the Sturmian sequences and state some of its properties.

## 2 $\beta$-expansion

In this section we review the theory of $\beta$-expansion for the sake of comparison with $(-\beta)$-expansions. Many of them are due to [9] unless stated otherwise.

## 2.1 Definition

Let $\beta > 1$. We introduce the $\beta$-transformation $T_\beta : [0,1) \to [0,1)$ as follows.

$$T_\beta(x) := \beta x - [\beta x] = (\beta x)$$

where $(x) := x - [x]$ is the fractional part of $x$. The power series expansion of $x \in [0,1)$ in terms of $\beta^{-1}$ given by

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \cdots, \quad x_n = [\beta T_\beta^{n-1} x] \in \{0, 1, \cdots, [\beta]\}, \quad n = 1, 2, \cdots$$

is called the $\beta$**-expansion** of $x$.

## 2.2 Admissible sequence and Shift space

In this subsection we characterize the sequences which appear in the $\beta$-expansion of some $x$.

**Definition**
A integer-valued sequence $\{x_n\}_{n\geq 1}$ ($x_n \in \{0, 1, \cdots, [\beta]\}$) is called $\beta$-**admissible** iff $\{x_n\}_{n\geq 1}$ is the $\beta$-expansion of some $x \in [0,1)$.

Let

$$d(x, \beta) := (x_1, x_2, \cdots)$$

be the $\beta$-expansion of $x \in [0,1)$. By letting $T_\beta^n(1) := T_\beta^{n-1}((\beta))$ we can naturally define the $\beta$-expansion $d(1, \beta)$ of 1. We further define

$$d^*(1, \beta) := \lim_{\epsilon \downarrow 0} d(1 - \epsilon, \beta)$$

where the limit is taken under the topology of pointwise convergence. Then we have the following simple and natural characterization of the $\beta$-admissibility.

**Theorem 2.1** *$\{x_n\}_{n\geq 1}$ is $\beta$-admissible iff*

$$(x_n, x_{n+1}, \cdots) \prec_{lex} d^*(1, \beta)$$

*for all $n \geq 1$ where $\prec_{lex}$ is the lexicographic order.*

**Remark 2.1** *$d(1,\beta) \neq d^*(1,\beta)$ happens iff the orbit $\{T_\beta^n(1)\}_{n=0}^\infty$ meets the discontinuity point of $T_\beta$, that is, iff $d(1,\beta)$ is eventually $0$. Such $\beta$ is called the* **simple Parry number**, *the set of which is dense in $(1,\infty)$.*

We next consider the closure of the set of translations of $\beta$-admissible sequences which is the shift-invariant set of double-sided sequences.

**Definition**

$S_\beta := \{\{x_n\}_{n\in\mathbf{Z}} \mid$ any finite subword of $\{x_n\}$ appears in $\beta$-admissible seq. $\}$

**Theorem 2.2**
*$\{x_n\}_{n\in\mathbf{Z}} \in S_\beta$ iff for any $n \in \mathbf{Z}$, $(x_n, x_{n+1}, \cdots) \preceq_{lex} d^*(1,\beta)$.*

$d(1,\beta)$ determines the nature of $S_\beta$ in some sense. To be precise,

**Theorem 2.3**
*(1) $S_\beta$ is SFT iff $d(1,\beta)$ is finite [6].*
*(2) $S_\beta$ is sofic iff $d(1,\beta)$ is eventually periodic [2].*

**Example** Let $\beta$ is the root of $\beta^2 = k\beta + 1$ $(k \in \mathbf{N})$. Then

$$d(1,\beta) = (k,1,0,0\cdots), \quad d^*(1,\beta) = (k,0,k,0,\cdots).$$

Therefore
$\{x_n\}_{n\geq 1}$ is $(\beta)$-admissible

$\iff (x_n, x_{n+1}, \cdots) \prec_{lex} (k,0,k,0,\cdots), \quad \forall n \geq 1$
$\iff$ $k$ is isolated and followed by 0 and its tail does not coincide with $(\overline{k0})$.

$d^*(1,\beta)$ is purely perodic and $S_\beta$ is SFT whose set of forbidden words is $\{k1, k2, \cdots, k(k-1), kk\}$.

For the conditions of finite or periodic expansions, the following facts are known.

**Theorem 2.4** [4]
*If $\beta$ is a quadratic Pisot number, then the elements of $\mathbf{Z}[\beta^{-1}]$ have finite $\beta$-expansions.*

**Theorem 2.5** [11]
*If $\beta$ is a Pisot-number, the set $Per(\beta)$ of numbers with eventually periodic $\beta$-expansion satisfies $Per(\beta) = \mathbf{Q}(\beta) \cap (0, \infty)$.*

### 2.3 Invariant measure

By Renyi[10], $T_\beta$ has unique invariant measure $\nu_\beta$ and is equivalent to Lebesgue measure.

**Theorem 2.6** *$d\nu_\beta = h_\beta dx$ where $h_\beta$ is given by*

$$h_\beta(x) = \sum_{n=0}^{\infty} \frac{1}{\beta^n} 1_{\{x < T_\beta^n(1)\}}.$$

**Example** When $\beta$ is the root of $\beta^2 = k\beta + 1$, we have

$$h_\beta(x) = \begin{cases} 1 + \frac{1}{\beta} & (0 < x < \frac{1}{\beta}) \\ 1 & (\frac{1}{\beta} < x < 1) \end{cases}$$

Let

$$H(\beta) := \int_0^1 h_\beta(x) dx, \quad \beta > 1$$

be the normalization constant of $h_\beta$ which has the following interesting properties.

**Theorem 2.7**
*(1) $H$ is right continuous,*
*(2) $H$ is discontinuous at simple Parry numbers, and continuous elsewhere,*
*(3) $\lim_{\beta \downarrow 1} H(\beta) = \infty$.*

In particular, $H$ is discontinuous almost everywhere.

## 3 $(-\beta)$-expansion by Ito-Sadahiro

The results stated in this section are mostly due to [5].

## 3.1 Definition

Let $\beta > 1$ and let

$$I_\beta := [l_\beta, r_\beta), \quad l_\beta := -\frac{\beta}{\beta+1}, \quad r_\beta := \frac{1}{\beta+1}.$$

We introduce the $(-\beta)$-transformation $T_{-\beta} : I_\beta \to I_\beta$ as follows.

$$T_{-\beta}(x) := -\beta x - [-\beta x - l_\beta].$$

The power series expansion of $x \in I_\beta$ in terms of $(-\beta)^{-1}$ given by

$$x = \sum_{n=1}^{\infty} \frac{x_n}{(-\beta)^n}, \quad x_n = [-\beta T_{-\beta}^{n-1}(x) - l_\beta], \quad n = 1, 2, \cdots$$

is called the $(-\beta)$**- expansion** of $x \in I_\beta$. For $x \notin I_\beta$, we define the $(-\beta)$-expansion of $x$ by taking $k \geq 1$ such that $\frac{x}{(-\beta)^k} \in I_\beta$, and then multiplying $(-\beta)^k$ with the $(-\beta)$-expansion of $\frac{x}{(-\beta)^k}$.

## 3.2 Admissible sequence and Shift space

Let

$$\begin{aligned} d(l_\beta, -\beta) &:= (b_1, b_2, \cdots), \\ d(r_\beta, -\beta) &:= (0, b_1, b_2, \cdots), \end{aligned}$$

be the $(-\beta)$- expansions of $l_\beta$, $r_\beta$ (the $(-\beta)$-expansion of $r_\beta$ can naturally be defined as above). We further set

$$d^*(r_\beta, -\beta) := \begin{cases} (\overline{0, b_1, \cdots, b_{q-1}, b_q - 1}) & (d(l_\beta, -\beta) = (\overline{b_1, \cdots, b_q}),\ q : \text{odd}) \\ d(r_\beta, -\beta) & (\text{otherwise}) \end{cases}$$

We divided into two cases above because, if $d(l_\beta, -\beta) = \overline{(b_1, \cdots, b_q)}$, the orbit $\{T_{-\beta}^n(l_\beta)\}_{n=0}^\infty$ meets the discontinuity points of $T_{-\beta}$.

**Definition**
We define $\prec_{IS}$ to be an order on two integer-valued sequences $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1} \in \{0, 1, \cdots, [\beta]\}^{\mathbf{N}}$ as follows.

(1) $\{a_n\}_{n\geq 1} \prec_{IS} \{b_n\}_{n\geq 1} \overset{def}{\Longleftrightarrow}$ setting $k := \min\{n \in \mathbf{N} : a_n \neq b_n\}$, we have $(-1)^k(a_k - b_k) < 0$.
(2) $\{a_n\}_{n\geq 1} \preceq_{IS} \{b_n\}_{n\geq 1} \overset{def}{\Longleftrightarrow} \{a_n\}_{n\geq 1} \prec_{IS} \{b_n\}_{n\geq 1}$ or $\{a_n\}_{n\geq 1} = \{b_n\}_{n\geq 1}$.

The $(-\beta)$-admissibility of a sequence $\{x_n\}$ is defined similarly as in the $\beta$-expansions, which has the characterization similar to that in $\beta$-expansions.

**Theorem 3.1**
*$\{x_n\}_{n\geq 1}$ is $(-\beta)$-admissible iff*

$$d(l_\beta, -\beta) \preceq_{IS} (x_n, x_{n+1}, \cdots) \prec_{IS} d^*(r_\beta, -\beta)$$

*for any $n \geq 1$.*

**Example** Let $\beta = \tau$ be the golden number. Then

$$d(l_\beta, -\beta) = (10\cdots) = (1\overline{0})$$
$$d(r_\beta, -\beta) = (0100\cdots)$$

Therefore $(x_n)_{n\geq 1}$ is $(-\tau)$-admissible iff, once 1 appears, then 0 always appears even times consecutively and its tail does not concide with $01\overline{0}$.

**Example** Let $\beta$ be the root of $\beta^2 = k\beta + 1$, $k \geq 2$. Then

$$d(l_\beta, -\beta) = (k\,(k-1)\cdots) = (k\,\overline{(k-1)})$$
$$d(r_\beta, -\beta) = (0100\cdots) = (0, k\,\overline{(k-1)})$$

Therefore $\{x_n\}_{n\geq 1}$ is $(-\beta)$-admissible iff, when $k$ appears, its neighborhood is equal to $\overbrace{k\cdots k}^{n}\overbrace{(k-1)\cdots(k-1)}^{j}\,a$ such that $n, j \geq 1$, and $a = \begin{cases} 0, 1, \cdots, (k-1) & (j : \text{odd}) \\ (k-1), k & (j : \text{even}) \end{cases}$, and moreover its tail does not coincide with $0k\overline{(k-1)}$.

The shift space $S_{-\beta}$ is defined similarly as in the $\beta$-expansion.

**Theorem 3.2**
*$\{x_n\}_{n\in\mathbf{Z}} \in S_{-\beta}$ iff for any $n \in \mathbf{Z}$, $d(l_\beta, -\beta) \preceq_{IS} (x_n, x_{n+1}, \cdots) \preceq_{IS} d^*(r_\beta, -\beta)$.*

We have the analogue of Theorem 2.3(2).

**Theorem 3.3** *$S_{-\beta}$ is Sofic iff $d(l_\beta, -\beta)$ is eventually periodic.*

In the examples above, $S_{-\beta}$ is Sofic.

**Remark 3.1** *(1) If $\beta$ is a Pisot number, $d(l_\beta, -\beta)$ is eventually periodic [3].*
*(2) It is not known under which conditions $S_{-\beta}$ is SFT.*
*(3) It is not known under which conditions $x \in \mathbf{Z}[\frac{1}{\beta}]$ has finite expansion.*
*(4) Suppose $\beta$ be a quadratic Pisot number. Letting $\beta^2 - a\beta - b$, $a, b \in \mathbf{Z}$ be the minimal polynomial of that, we have the following two cases [4] : Case (i) $a \geq b > 0$, and Case (ii) $-a+1 < b < 0$. Some numerical experiments imply the following conjecture.*
*(i) In Case (i), the negative integers do not have finite expansions.*
*(ii) In Case (ii), the elements of $\mathbf{Z}[\frac{1}{\beta}]$ have finite expansions.*

**Remark 3.2** *Let $\beta$ be a Pisot number. Then it is possible to consider a dual numeration system and a map from $S_{-\beta}$ onto a torus $\mathbf{T}^3$ such that the shift map on $S_{-\beta}$ is conjugate to the torus automorphism. Then by projecting the cylinder set in $S_{-\beta}$ we have self-similar tilings on the plane(Figure 1, 2). Numerical experiments seems to imply that the origin usually lies in the interior(Figure 1), with rare exceptions(Figure 2). In $\beta$-expansions, if the set Fin $(\beta)$ of numbers with finite $\beta$-expansion satisfies the condition (F) Fin $(\beta) = \mathbf{Z}[\beta^{-1}]$, then the origin is in the interior [1]. The $\beta$'s corresponding to the tilings in Figures 1,2 satisfy the condition (F).*

## 3.3 Invariant measure

By the theorem of Li-Yorke[7], $T_{-\beta}$ has unique invariant measure $\nu_{-\beta}$.

**Theorem 3.4** *$d\nu_{-\beta} = h_{-\beta}dx$ where*

$$h_{-\beta}(x) = \sum_{n=0}^{\infty} \frac{1}{(-\beta)^n} 1_{\{x > T^n_{-\beta}(l_\beta)\}}.$$

On the contrary to the case of $\beta$-expansions, there is some $\beta$'s with supp $\nu_{-\beta} \neq [l_\beta, r_\beta)$.

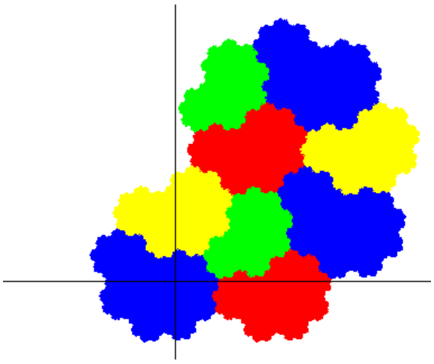

Figure 1: $\beta:\ \beta^3-\beta-1=0$

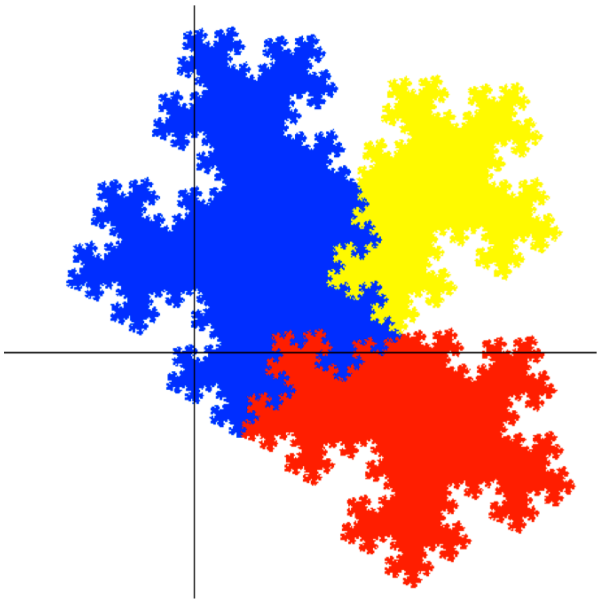

Figure 2: $\beta:\ \beta^3-\beta^2-1=0$

**Example :** Let $\beta$ be the root of $\beta^2 = k\beta + 1$. Then

$$h_{-\beta}(x) = \begin{cases} 1 & (l_\beta < x < -\frac{k-1}{\beta+1}) \\ \frac{\beta}{\beta+1} & (-\frac{k-1}{\beta+1} < x < r_\beta) \end{cases}$$

Let

$$H_{IS}(\beta) := \int_{l_\beta}^{r_\beta} h_{-\beta}(x)dx$$

be the normalization constant of $h_{-\beta}$. Let

$$\begin{aligned} Q_{even} &:= \{\beta > 1 \mid d(l_\beta, -\beta) = (\overline{b_1, b_2, \cdots, b_q}),\ q:\ \text{even}\ \} \\ Q_{odd} &:= \{\beta > 1 \mid d(l_\beta, -\beta) = (\overline{b_1, b_2, \cdots, b_q}),\ q:\ \text{odd}\ \} \end{aligned}$$

be the set of $\beta$'s such that the orbit $\{T_{-\beta}^n(l_\beta)\}_{n=0}^\infty$ meets the discontinuity points of $T_{-\beta}$. For $\beta \in Q_{odd} \cup Q_{even}$, we denote by $q(\beta) := q$ the period of $d(l_\beta, -\beta) = (\overline{b_1, b_2, \cdots, b_q})$.

**Theorem 3.5**
*(1) $H_{IS}$ is right continuous but not left continuous at $\beta \in Q_{odd}$, $q(\beta) \geq 2$,*
*(2) $H_{IS}$ is left continuous but not right continuous at $\beta \in Q_{even}$,*
*(3) $H_{IS}$ is neither right continuous nor left continuous at $\beta \in \mathbf{N}$, $\beta \geq 2$ (which corresponds to $\beta \in Q_{odd}, q(\beta) = 1$)*
*(4) $H_{IS}$ is continuous at $\beta \notin Q_{even} \cup Q_{odd}$.*

Some questions remain unsolved : (i) whether or not $Q_{even} \cup Q_{odd}$ is dense on $(1, \infty)$, (ii) the behavior of $H_{IS}(\beta)$ as $\beta \downarrow 1$.

# 4 Another $(-\beta)$-expansion associated to the Sturmian sequences

In this section we consider another $(-\beta)$-expansion obtained by using the structure of the corresponding Sturmian sequences. We set $\alpha = \frac{1}{\beta}$ throughout this section.

## 4.1 Definition

This expansion is properly defined only when $\beta$ is the root of $\beta^2 = k\beta + 1$, $k \in \mathbf{N}$. For general $\beta > 1$, we have to consider $(-\beta)$-expansion in an extended form(Remark 4.2).

(1) To begin with, we consider the case of $k = 1$, that is, $\beta = \tau$ is the golden number. We introduce $(-\tau)_S$-transformation $T_{-\tau,S} : [0,1] \to [0,1]$ as follows.

$$T_{-\tau,S}(x) = \begin{cases} -\tau x + 1 & (x \in [0, 1 - \frac{1}{\tau})) \\ -\tau x + \tau & (x \in [1 - \frac{1}{\tau}, 1]) \end{cases}$$

Let $1_A$ be the characteristic function of a set $A$.

**Definition**
The power series expansion of $x \in [0,1]$ in terms of $(-\tau)^{-1}$ given by

$$x = 1 + \frac{1}{(-\tau)} + \sum_{n \geq 2} \frac{x_n}{(-\tau)^n}, \quad x_n := 1_{[1-\frac{1}{\tau},1]}(T^{n-2}_{-\tau,S}(x)),\ n \geq 2 \tag{4.1}$$

is called the $(-\tau)_S$**-expansion** of $x$ (subscript S is for the Sturmian sequences).

**Remark 4.1** *(1) Since $T_{-\tau,S}$ is defined on the closed interval $[0,1]$, it is not a torus automorphism on $[0,1)$ as they are in $\beta$- and $(-\beta)$- expansions.*
*(2) It is possible to regard (4.1) as the expansion of $x - \frac{1}{\tau^2} \in [-\frac{1}{\tau}^2, \frac{1}{\tau}]$ : $x - \frac{1}{\tau^2} = \sum_{n\geq 2} \frac{x_n}{(-\tau)^n}$. If we do that, we translate $T_{-\tau,S}$ for $(-\frac{1}{\tau^2}, -\frac{1}{\tau^2})$ and consider*

$$\hat{T}_{-\tau,S}(x) = -\tau x + \frac{1_A(x)}{\tau}, \quad A = \left[0, \frac{1}{\tau}\right], \quad x \in [-\frac{1}{\tau^2}, \frac{1}{\tau}].$$

*We then have the following expansion of $x \in [-\frac{1}{\tau^2}, \frac{1}{\tau}]$.*

$$x = \sum_{n \geq 2} \frac{x_n}{(-\tau)^n}, \quad x_n := 1_A(\hat{T}^{n-2}_{-\tau,S}(x)), \quad n \geq 2.$$

*As in the Ito-Sadahiro expansion, we can also define $(-\beta)_S$-expansion of any $x \notin [-\frac{1}{\tau^2}, \frac{1}{\tau}]$.*

This notion of expansion is related to the Sturmian sequences in the following sense. Let

$$v_\theta(n) := 1_{[1-\frac{1}{\tau},1)}\left(\frac{1}{\tau}n + \theta, \quad (\text{mod } 1)\right), \quad \theta \in \mathbf{T} := [0,1)$$

be the Sturmian sequence of rotation number $\frac{1}{\tau}$ and let $\{s_n\}_{n=0}^{\infty}$ be the associated sequence of words defined inductively by

$$s_0 := 0, \quad s_1 = 1,$$
$$s_{n+1} = s_n \, s_{n-1}, \quad n \geq 1.$$

Then the right(resp. left) limit of $s_n$ (resp. $s_{2n}$) coincides with $\{v_0(n)\}_{n\geq 1}$ (resp. $\{v_0(n)\}_{n\leq 0}$). Let $R : s_n \mapsto s_n s_{n-1} = s_{n+1}$ be the embedding map of $s_n$ into $s_{n+1}$, and let $L : s_n \mapsto s_{n+1}s_n = s_{n+2}$ be the embedding map of $s_n$ into $s_{n+2}$. Then for any $\theta \in \mathbf{T}$ we can find $(O_1, O_2, \cdots) \in \{R, L\}^{\mathbf{N}}$ uniquely which corresponds to $v_\theta$ [8] [1]. Let $\{x_n\}_{n\geq 1}$ be the sequence derived by the substitution $R \mapsto 1$, $L \mapsto 10$. Then $x = 1 + \sum_{n\geq 1} \frac{x_n}{(-\tau)^n}$ is the $(-\tau)_S$-expansion of $x$ [2]. In fact, derinving the sequence of embedding maps $(O_1, O_2, \cdots)$ corresponding to $v_x$ is equivalent to approximating $x \in [0, 1]$ by a sequence of divisions of $[0, 1]$, and $(-\tau)_S$-expansion is derived using this approximation.

Since $\beta$-expansion is in some sense similar to the Euclidian algorithm, and since the Sturmian sequence has the same nature, it seems not to be absolutely nonsense to consider such expansions.

(2) When $\beta$ is the root of $\beta^2 = k\beta + 1$, $k \geq 2$, the definition of $(-\beta)_S$-expansion is similar but its expression becomes slightly involved. Let $\alpha := \frac{1}{\beta}$ and let $x_2 : [0, \alpha^2 + \alpha] \to \{0, 1, \cdots, k\}$ given by

$$x_2(x) = \begin{cases} j & (j\alpha^2 \leq x < (j+1)\alpha^2,\ j = 0, 1, \cdots, k) \\ k & ((k+1)\alpha^2 \leq x \leq \alpha^2 + \alpha) \end{cases}$$

We introduce $(-\beta)_S$-transformation $T_{-\beta,S} : [0, \alpha^2+\alpha] \to [0, \alpha^2+\alpha]$ as follows.

$$T_{-\beta,S}(x) = -\beta x + 1 + \frac{(x_2(x) - k + 1)}{\beta}.$$

**Definition**

The power series expansion of $x \in [0, \alpha^2 + \alpha]$ in terms of $(-\beta)^{-1}$ given by

$$x = 1 + \frac{k}{(-\beta)} + \sum_{n\geq 2} \frac{x_n}{(-\beta)^n}, \quad x_n = x_2(T^{j-2}(x)), \quad n = 2, 3, \cdots \qquad (4.2)$$

[1] $O_1$ specifies whether we start from $s_0$ or $s_1$.

[2] we need $x \notin D_- := \{-n\alpha \pmod 1 \,|\, n \geq 0\}$ here.

is called $(-\beta)_S$**-expansion** of $x$. As is done in $(-\tau)_S$-expansion, we can regard it as an expansion of $x-(1+\frac{k}{(-\beta)}) \in [-\alpha^2, \alpha]$. In this case $T_{-\beta,S}$ is replaced by its translation of $(-\alpha^2, -\alpha^2)$ :

$$\hat{T}_{-\beta,S} = -\beta x + \frac{x_2(x)}{\beta}, \quad x \in [-\alpha^2, \alpha]$$

**Remark 4.2** *For general $\beta > 1$, let $\alpha = [a_1, a_2, \cdots]$ be the fraction expansion of $\alpha$. Let $\{\gamma_n\}_{n\geq 1}$ be the sequence defined by*

$$\gamma_{n-1} = \frac{1}{a_n + \gamma_n}, \quad n \geq 1, \quad \gamma_0 = \alpha$$

*and set*

$$\begin{aligned} \alpha_n &:= \gamma_{n-1} \cdot \gamma_{n-2} \cdots \gamma_1 \cdot \alpha \\ \beta_n &:= \frac{1}{(-1)^n \alpha_n}. \end{aligned}$$

*Then we can define an expansion of $x$ in an extended sense by replacing $(-\beta)^n$ in (4.2) by $\beta_n$ where $x_n \in \{0, 1, \cdots, a_n\}$.*

## 4.2 Admissible sequence and Shift space

In what follows we discuss both the cases of $k = 1$ and $k \geq 2$ together. The notion of $(-\beta)_S$- admissibility is defined similarly as before, but is defined for $n \geq 2$ part $\{x_n\}_{n\geq 2}$ of the sequence $\{x_n\}$, since we always have $x_1 = 1$.

**Theorem 4.1** *The set of $(-\beta)_S$-admissible sequences has the following expression.*

$$\{\{x_n\}_{n\geq 2} \,|\, \{x_n\}_{n\geq 2} \text{ is } (-\beta)_S\text{-admissible } \} = X \setminus Y$$

*where*

$$\begin{aligned} X &= \{\{x_n\}_{n\geq 2} \,|\; 00,\ 10,\ \cdots, (k-1)0 \text{ are prohibited}\} \\ Y &= \{\{x_n\}_{n\geq 2} \,|\; \text{eventually } (\cdots, (j-1), 1, \overline{(k\,0)}),\ j = 1, 2, \cdots, k\ \} \end{aligned}$$

**Theorem 4.2** *$S_{-\beta,S}$ is SFT whose set of forbidden words is $\{00, 10, \cdots, (k-1)0\}$.*

## 4.3 Invariant Measure

**Theorem 4.3** *Let $\nu_{-\beta,S}$ be the invariant measure of $T_{-\beta,S}$. Then $d\nu_{-\beta,S} = h_{-\beta,S}dx$ where*

$$h_{-\beta,S}(x) = \begin{cases} \alpha & (0 < x < \alpha^2) \\ 1 & (\alpha^2 < x < \alpha^2 + \alpha) \end{cases}$$

There seems not to be a power series expression as there are in $\beta$-expansions and $(-\beta)$-expansions.

## 4.4 Local Flip Connectedness

In this subsection we regard $(-\beta)_S$-expansion as an expansion of $x \in \mathbf{R}$ using $\hat{T}_{-\beta,S}$, as is explained in subsection 4.1. We adopt the following terminology. Let $x \in \mathbf{R}$ and let $\{x_n\}_{n \geq n_0}$ ($x_n \in \{0, 1, \cdots, [\beta]\}$) be an integer-valued sequence. We call $\{x_n\}_{n \geq n_0}$ **$(-\beta)$-expansion** of $x$ if it satisfies

$$x = \sum_{n \geq n_0} \frac{x_n}{(-\beta)^n}, \quad x_n \in \{0, 1, \cdots, [\beta]\}, \quad n_0 \in \mathbf{Z}. \tag{4.3}$$

For clarity we denote by $(-\beta)_{IS}$-expansion, the $(-\beta)$-expansion in the sense of Ito-Sadahiro. We do not consider the sequences corresponding to finite expansions (that is, whose tail coincides with $\overline{0}$), for it can be fit to infinite expansions by replacing its tail $\overline{0}$ by $\overline{k\,0}$.

Since $\beta^2 = k\beta + 1$ implies

$$\frac{1}{(-\beta)^n} + \frac{k}{(-\beta)^{n+1}} = \frac{1}{(-\beta)^{n+2}}, \quad n \in \mathbf{Z},$$

we can locally flip $00k \leftrightarrow 1k0$ in $(-\beta)$-expansions, so that we define the following operations on sequences $\{x_n\}_{n \geq n_0} \in \{0, 1, \cdots, k\}^{\mathbf{N}}$.

$(A)_{l,j}$ $\quad (l+1)\,k\,(j-1) \leftrightarrow l\,0\,j \quad (l = 0, 1, \cdots, k-1, \quad j = 1, 2, \cdots, k)$

$(B)_j$ $\quad k\,(j-1)\,1\,\overline{(k\,0)} \leftrightarrow k\,j\,\overline{(k\,0)}, \quad j = 1, 2, \cdots, k$

$(C)$ $\quad 0\,k\,\overline{(k-1)} \leftrightarrow 1\,k\,\overline{(k-1)}$

$(B)_j$ (resp. $(C)$) convert non-admissible sequences into admissible ones in $(-\beta)_S$-expansions (resp. $(-\beta)_{IS}$-expansions). Then any $(-\beta)$-expansions are connected each other via these local flips.

**Theorem 4.4** *Any $(-\beta)$-expansion can be transformed into $(-\beta)_{IS}$-expansion and $(-\beta)_S$-expansion by successive applications of those local operations $(A)_{l,j}$, $(B)_j$ and $(C)$.*

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# References

[1] Akiyama, S., Self affine tiling and Pisot numeration system, In: Number theory and its Applications, (eds. K. Gyory and S. Kanemitsu), Kluwer Acad. Publ., (1999), p.7-17.

[2] A.Bertrand-Mathis. Développement en base $\theta$, répartition modulo un de la suite $(x\theta^n)$, n$\geq$ 0, langages codés et $\theta$-shift. *Bulletin de la Société Mathématique de France*, **114**(1986) p.271-323.

[3] Frougny, C., Lai., A.C., On negative bases, Proceedings of DLT 09, Lectures Notes in Computer Science, **5583** (2009) p.252-263.

[4] Frougny, C., Solomyak, B., Finite beta-expansions, Ergod. Th. Dynam. Sys., **12**(1992), p.713-723.

[5] Ito, S., Sadahiro, T., Beta-expansions with negative bases. Integers **9** (2009) p.239-259.

[6] Ito, S., Takahashi, Y., Markov subshifts and realization of $\beta$-expansions Journal of Mathematical Society of Japan Vol. **26** No.1(1974) p.33-55.

[7] Li, T.Y., Yorke, J., Ergodic transformations from an interval into itself., Trans. Amer. Math. Soc., **235**(1978) p.183-192.

[8] Nakano, F., Elementary properties of circle map sequences, Nihonkai Journal of Mathematics, Vol. **19** No.2, (2008), p.85-104.

[9] Parry, W., On the $\beta$-expansions of real numbers, Acta Math. Acad. Sci. Hungar. **11** (1960) p.401-416.

[10] Renyi, A., Representations for real numbers and their ergodic properties, Acta. Math. Acad. Sci. Hungar., **8**(1957) p.477-493.

[11] Schmidt, K., On periodic expansions of pisot numbers and salem numbers, Bull. London Math. Soc.**12**(1980) p.269-278.